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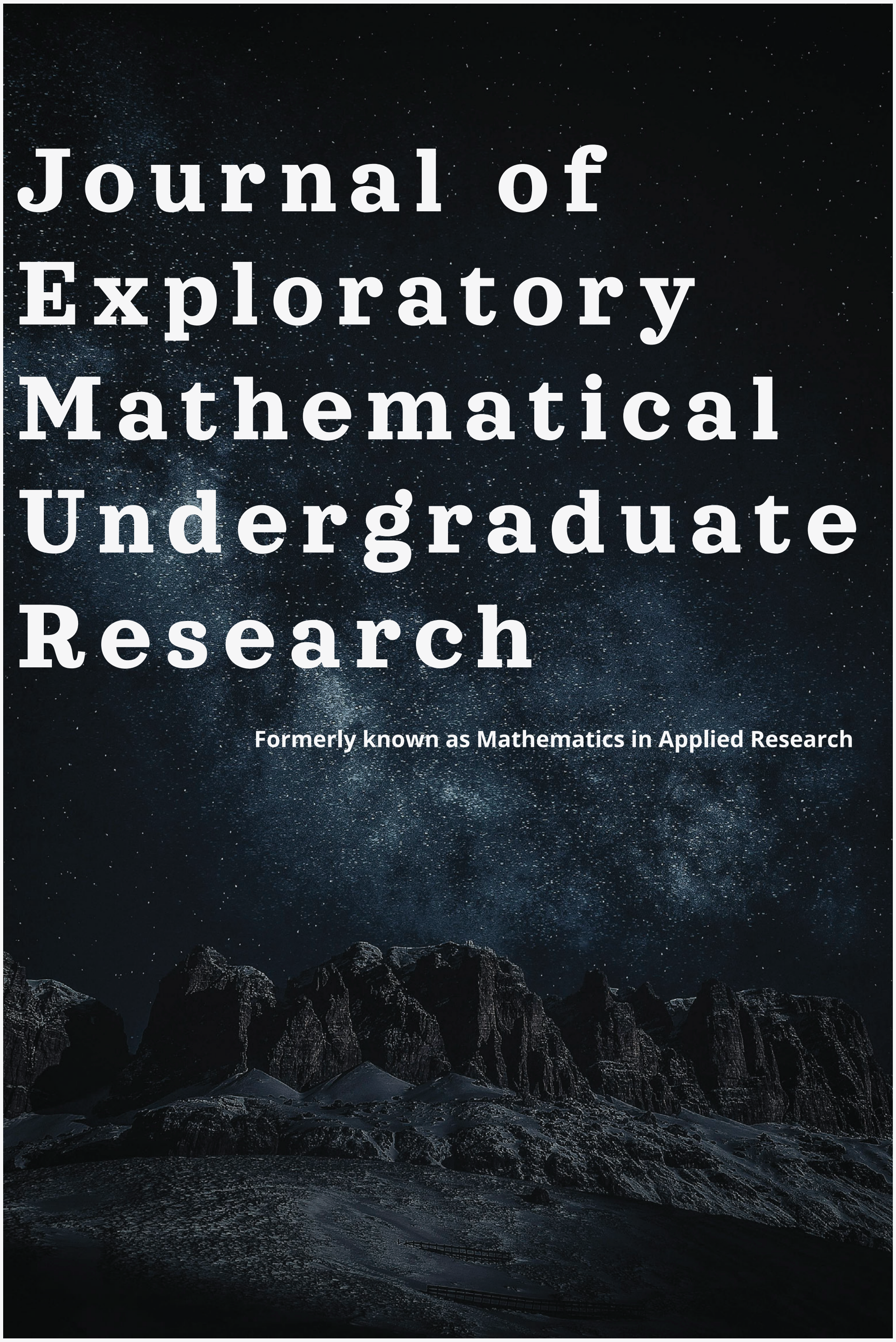

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UPPER BOUND OF SECOND HANKEL DETERMINANT FOR SUBCLASSES OF CLOSE-TO-CONVEX FUNCTIONS

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Abstract

Let S denote the subclass of univalent functions f in an open unit disc, $U=\{z\in\mathbb{C}:|z|<1\}$, given by $f(z)=z+\sum_{n=2}^{\infty}a_nz^n$. We define $L(\alpha,\delta,t,s)$ as a class of close-to-convex functions which satisfy $\operatorname{Re}\left[e^{i\alpha}\frac{zf'(z)}{g'(z)}\right]>\delta$ where $|\alpha|\le\pi,\ \cos\alpha>\delta,\ 0\le\delta\le1,\ g'(z)=\frac{1}{(1+tz)(1-sz)}$, $s>t$, $-1\le t<1$ and $-1<s\le1$. In this paper, the upper bounds for second Hankel determinant, $|a_2a_4-a_3^2|$ for the class $L(\alpha,\delta,t,s)$ are obtained by using Toeplitz determinant.

Keywords: Upper bound, Second Hankel inequality, close-to-convex functions, Toeplitz determinant

1. Introduction

An analytic function assumes that every complex number, with possibly by one exception, infinitely often in any neighborhood of an essential singularity. That is an example of complex analysis from Picard's great theorem. An analytic function, Ą, is one-to-one mapping of one region onto another region in the complex plane.

The normalized analytic function is defined if function is regular and univalent in Ą (Goodman, 1983). Taylor series represent the normalized analytic that denote as

$$f(z)=z+a_2z^2+a_3z^3+...+a_nz^n+...=z+\sum_{n=2}^{\infty}a_nz^n\ . \tag{1.1}$$

The domain D of univalent functions can be in any shapes and can be defined as indeterminate. We denote the unit disc $U=\{z\in\mathbb{C}:|z|<1\}$ to narrow and focus on scope of domain. For the class of functions with positive real part, P it can be form

$$p(z)=1+p_1z+p_2z^2+\cdots+p_nz^n+\cdots=1+\sum_{n=1}^{\infty}p_nz^n \tag{1.2}$$

that is regular at U, and (1.2) satisfies $\operatorname{Re}(p(z)) > 0$ when z in U. For any function of P known as a function for positive real part in U. Let denote the close-to-convex function as Ct in U given by (1.1), a function f is said to be in Ct if

$$\operatorname{Re}\left(\frac{f'(z)}{g'(z)}\right) > 0, \quad z \in U.$$

The univalent functions in the class A, it is well known that nth coefficient is bounded by n. Ehrenborg (2000) stated that determinant of the corresponding Hankel matrix is the Hankel determinant of order $(m+1)$, whereby

$$\det(a_{i+j})_{0\le i,j\le m} = \begin{bmatrix} a_0 & a_1 & \cdots & a_m \\ a_1 & a_2 & \cdots & a_{m+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_m & a_{m+1} & \cdots & a_{2m} \end{bmatrix}.$$

Then, the q^{th} from Hankel determinant of f for q was defined by Noonan and Thomas (1976), that is

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}. \tag{1.3}$$

Based on (1.3), the Hankel determinant by $q = 2$ and $n = 2$ is form the second Hankel determinant, whereby

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = \left|a_2a_4 - a_3^2\right|.$$

One of the findings by Janteng et al. (2006) is on second Hankel determinant of positive real part satisfies $\operatorname{Re}(f'(z)) > 0$ which is

$$\left|a_2a_4 - a_3^2\right| \le \frac{4}{9}.$$

Janteng et al. (2007) attained the result second Hankel for another class of functions which are starlike functions and convex functions. For starlike functions the result for it second Hankel was

$$\left|a_2a_4 - a_3^2\right| \le 1$$

and for convex functions was

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{8}.$$

Furthermore, many researchers obtained new result of second Hankel determinant for different classes of functions based on method of classical analysis devised Libera and Zlotkiewicz (1982, 1983). The same method has been employed by many authors in similar works which are Janteng et al. (2006) and Kaharudin et al. (2011). In the present paper, we introduce $L(\alpha,\delta,t,s)$ as the class of close-to-convex functions, the functions of $L(\alpha,\delta,t,s)$ are normalized functions $f \in A$ given by (1.1) that satisfy the conditions

$$\operatorname{Re}\left\{e^{i\alpha}\frac{f'(z)}{g'(z)}\right\} > \delta \tag{1.4}$$

where $|\alpha| \le \pi,\ \cos\alpha > \delta,\ 0 \le \delta \le 1,\ g'(z) = \dfrac{1}{(1+tz)(1-sz)}$, $-1 \le t < 1,\ \ -1 < s \le 1$ and $s > t$.

The main objective of this paper is to obtain upper bound of $|a_2 a_4 - a_3^2|$ for $L(\alpha,\delta,t,s)$.

2. Preliminaries

In order to derive the main result, we will applied some of the following lemmas.

Lemma 1: Pommerenke (1975)

Let $p \in P$, that is p be analytic in U and be given by (1.2) and $\operatorname{Re}(p(z)) > 0$ for $z \in U$, then

$$|p_n| \le 2$$

and

$$\left|p_2 - \frac{p_1^{\,2}}{2}\right| \le 2 - \frac{|p_1^{\,2}|}{2}.$$

Lemma 2 : Toeplitz determinant

By referring Janteng et al. (2007), the power series for $p(z)$ in (1.2) converges in U to a function in P if and only if the Toeplitz determinants

$$D_n = \begin{pmatrix} 2 & p_1 & p_2 \cdots p_n \\ p_{-1} & 2 & p_1 \cdots p_{n-1} \\ \vdots & \vdots & \vdots \quad \vdots \\ p_{-n} & p_{-n+1} & p_{-n+2} \cdots 2 \end{pmatrix}, n = 1,2,3... \tag{2.1}$$

and $p_{-4} = \bar{p}_k$, are all nonnegative. They are strictly positive except for $p(z) = \sum_{k=1}^{m} \beta_k c_0(e^{if_k} z), \beta_k > 0, t_k$ real and $t_k \ne t_j$ for $k \ne j$;in this case $D_n < 0$ for $n < m-1$ and $D_n = 0$ for $n \ge m$.

Lemma 3: Libera and Zlotkiewicz (1982, 1983)

Based on Libera and Zlotkiewicz (1982, 1983), let the function $p(z) \in P$ and be given by (2.1). Assume without restrictions that $p_1 > 0$. By rewriting (2.1) for the cases $n = 2$ and $n = 3$

$$D_2 = \begin{pmatrix} 2 & p_1 & p_2 \\ p_1 & 2 & p_1 \\ \overline{p}_2 & p_1 & 2 \end{pmatrix} = 8 + 2\,\mathrm{Re}(p_1{}^2 p_2) - 2|p_2|^2 - 4p_1{}^2 \geq 0 \tag{2.2}$$

which is equivalent to

$$2p_2 = p_1{}^2 + x(4 - p_1{}^2)$$

for some $x, |x| \leq 1$. Then $D_3 \geq 0$ is equivalent to

$$\left|\left(4p_3 - 4p_1p_2 + p_1{}^3\right)\left(4 - p_1{}^2\right) + p_1\left(2p_2p_1{}^2\right)\right| \leq 2\left(4 - p_1{}^2\right)^2\left(2\left|2p_2 - p_1{}^2\right|^2\right).$$

By referred (2.2), provides the relation

$$4p_3 = p_1{}^3 + 2(4 - p_1{}^2)p_1x - p_1(4 - p_1{}^2)x^2 + 2(4 - p_1{}^2)(1 - |x|^2)z$$

for some value of $z, |z| \leq 1$.

Now, we shall prove our main results.

3. Main Results

Theorem 3.1

Let $f \in L(\alpha, \delta, t, s)$, $C = s - t$, $I = \left(t^2 - ts - s^2\right)$, $G = \left(-t^3 + t^2s - ts^2 + s^3\right)$ *based on functions* $g(z)$, $Q = -\left(2t^2 + 14ts + 2s^2 + 18\right)$, $R = 2s^3 + 5s^2t - 5t^2s - 2t^3 + 41(s - t)$, $p = \dfrac{Q + \sqrt{54CR + Q}}{27C}$, $J = 9(s - t)\left(-t^3 + t^2s - ts^2 + s^3\right) - 72\left(t^2 - ts - s^2\right)^2$ *and* $K = 8(s - t)^2 - 9\left(t^2 - ts - s^2\right)$. *If*

$$\left|a_2a_4 - a_3^2\right| \leq \frac{1}{72}J + \frac{1}{72}\left\{\left[-\frac{9}{2}(p)^3 C + (R - 14C)(p) + 18C^2 - 32I\right]A_{\alpha\delta} + \right.$$
$$\left. + \left[(p)^2 K - 9(p)^2 + 14(p)C + 32\right]A_{\alpha\delta}{}^2\right\}.$$

The functions of this inequality take holds for $g'(z) = \dfrac{1}{(1 + tz)(1 - sz)}$.

Proof

From representation theorem of $L(\alpha,\delta,t,s)$ we have

$$f'(z)=\left(p(z)A_{\alpha\delta}+i\sin\alpha+\delta\right)g'(z)e^{-i\alpha} \tag{3.1}$$

Since $f(z)$ is in form (1.1), therefore

$$f'(z)=1+2a_2z+3a_3z^2+4a_4z^3+\cdots.$$

By define function in form of series, we got

$$\sum_{n=1}^{\infty}\frac{\left(g(z)\right)^n(z)^n}{n!}=1+(s-t)z+\left(t^2-ts-s^2\right)z^2+\left(-t^3+t^2s-ts^2+s^3\right)z^2 +\left(t^4-t^3s^1+t^2s^2-t^1s^3+s^4\right)z^4+\cdots. \tag{3.2}$$

From (3.1), by letting the coefficient, we have

$$C=(s-t),\mathrm{I}=\left(t^2-ts-s^2\right),\ G=\left(-t^3+t^2s-ts^2+s^3\right),H=\left(t^4-t^3s^1+t^2s^2-t^1s^3+s^4\right) \tag{3.3}$$

$$\sum_{n=1}^{\infty}\frac{\left(g(z)\right)^n(z)^n}{n!}=1+Cz+Iz^2+Gz^3+Hz^4+\cdots.$$

Therefore, by consider on (3.1) and (3.3)

$$\begin{aligned}&\left[p(z)A_{\alpha\delta}+i\sin\alpha+\delta\right]g'(z)e^{-i\alpha}\\&=1+\left(C+p_1A_{\alpha\delta}e^{-i\alpha}\right)z+\left[I+\left(Cp_1+p_2\right)A_{\alpha\delta}e^{-i\alpha}\right]z^2+\left[G+\left(Ip_1+Cp_2+p_3\right)\right]z^3+\\&\quad+\left[H+\left(Gp_1+Ip_2+Cp_3+p_4\right)A_{\alpha\delta}e^{-i\alpha}\right]+\cdots.\end{aligned}$$

Thus, we successfully achieved:

$$\begin{aligned}2a_2&=C+p_1A_{\alpha\delta}e^{-i\alpha}\\3a_3&=I+\left(Cp_1+p_2\right)A_{\alpha\delta}e^{-i\alpha}\\4a_4&=G+\left(Ip_1+Cp_2+p_3\right)A_{\alpha\delta}e^{-i\alpha}.\end{aligned}$$

Therefore, from a_2, a_3 and a_4. We have

$$\begin{aligned}a_2a_4-a_3{}^2=\frac{1}{72}\Big\{&9\left(CG-8I^2\right)+\\&+\left(9p_2C^2+9p_3C+9p_1G-7p_1CI-16p_2I\right)A_{\alpha\delta}e^{-i\alpha}+\\&+\left(9p_1{}^2I+9p_1p_3-8p_1{}^2C^2-7p_1p_2C-8p_2{}^2\right)(A_{\alpha\delta}e^{-i\alpha})^2\Big\}.\end{aligned}$$

Consequently, application of the triangle inequality shows

$$\left|a_2a_4-a_2^3\right|\leq\frac{1}{72}\left|\left[9\left(CG-8I^2\right)\right]\right|++\frac{1}{72}\left\{\left[\left|\left(-7pC-8p^2\right)I+9pG+\frac{9}{2}p^2C^2+\frac{9}{4}p^3C\right|+\right.\right.$$
$$+\left|\left[\frac{9}{2}C^2-8I+\frac{9}{2}pC\right]\left(4-p^2\right)\right||x|++\left|-\frac{9}{4}pC\left(4-p^2\right)p\right|\left|x^2\right|+\left|\frac{9}{2}C\left(4-p^2\right)\right|\left|\left(1-|x|^2\right)\right||z|\Big]A_{\alpha\delta}+$$
$$+\left[\left|\frac{p^4}{4}\right|+\left|\frac{7}{2}p^3C\right|+\left|p^2(9I-8C^2)\right|+\left|\left(\frac{p^2}{2}-\frac{7p}{2}C\right)\right|\left|(4-p^2)\right||x|+\right.$$
$$\left.\left.+\left|\left(\frac{p^2}{4}+8\right)\right|\left|(4-p^2)\right|\left|x^2\right|+\left|\frac{9p}{2}\right|\left|4-p^2\right|\left|1-|x|^2\right||z|\right](A_{\alpha\delta})^2\right\}.$$

Since $|z|\leq1$ and by letting $\gamma=|x|$, so we have

$$\left|a_2a_4-a_2^3\right|\leq\frac{1}{72}\left|\left[9\left(CG-8I^2\right)\right]\right|++\frac{1}{72}\left\{\left[\left(-7pC-8p^2\right)I+9pG+\frac{9}{2}p^2C^2+\frac{9}{4}p^3C+\right.\right.$$
$$+\left[\frac{9}{2}C^2-8I+\frac{9}{2}pC\right]\left(4-p^2\right)\gamma++\left[\left(\frac{9p-18|z|}{4}\right)C\left(4-p^2\right)p\right]\gamma^2+\frac{9}{2}C\left(4-p^2\right)\Big]A_{\alpha\delta}+$$
$$+\left[\frac{7}{2}p^3C-\frac{p^4}{4}+\left|p^2(8C^2-9I)\right|+\left|\frac{7p}{2}C-\frac{p^2}{2}\right|\left|(4-p^2)\right|\gamma\right.$$
$$\left.\left.+\frac{9p}{2}\left|(4-p^2)\right|+\left|\frac{p^2-18p+32}{4}\right|\left|(4-p^2)\right|\gamma^2\right](A_{\alpha\delta})^2\right\}. \tag{3.4}$$
$$=F\left(p,\gamma,t,s\right)$$

By differentiating (3.4), letting $A_{\alpha\delta}=1$, where $A_{\alpha\delta}=\cos\alpha-\delta$, we can attained

$$F'(p,\gamma,t,s)=\frac{1}{72}\left\{\left[\frac{9}{2}C^2-8I+\frac{9}{2}pC\right]\left(4-p^2\right)+\left[\left(\frac{9p-18|z|}{2}\right)C\left(4-p^2\right)p\right]\gamma+\right.$$
$$\left.+\left(\frac{7p}{2}C-\frac{p^2}{2}\right)(4-p^2)+\left(\frac{p^2-18+32}{2}\right)(4-p^2)\gamma\right\}.$$

Since $F'\left(p,\gamma,t,s\right)>0$, by considering $F\left(p,\gamma,t,s\right)$ is increasing and $\underset{\gamma\leq1}{Max}F\left(p,\gamma,t,s\right)=F\left(p,1,t,s\right)$ where $\gamma=[0,1]$. Since only focus on bound in the equation (3.4), we can let $J\left(p,t,s\right)=F\left(p,1,t,s\right)$ which it can be simplified to become

$$J\left(p,t,s\right)=-\frac{9}{2}p^3C+\left(8C^2-9I-9\right)p^2+\left[-7CI+9G+41C\right]p+18C^2-32I+32\ . \tag{3.5}$$

From (3.5), by setting $J'(p,t,s)=0$, we got

$$p=\frac{16C^2-18I-18\pm\sqrt{54C(-7CI+9G+41C)+(16C^2-18I-18)^2}}{27C} \tag{3.6}$$

By checking using Maple software, we found some restriction on value of s and t, whereby $-1<s\le 1$, $-1\le t<1$ and $s>t$ for satisfy $p=[0,2]$. Thus, the upper bound for (3.5) corresponds to $\gamma=1$, $p=\frac{Q+\sqrt{54CR+Q}}{27C}$, $Q=-(2t^2+14ts+2s^2+18)$ and $R=2s^3+5s^2t-5t^2s-2t^3+41(s-t)$, we attained

$$\begin{aligned}\left|a_2a_4-a_3^2\right|\le \frac{1}{72}(9CG-72I^2)+\frac{1}{72}\Bigg\{\Bigg[-\frac{9}{2}\left(\frac{Q+\sqrt{54CR+Q^2}}{27C}\right)^3C+\\ +(-7CI+9G+27C)\left(\frac{Q+\sqrt{54CR+Q^2}}{27C}\right)+18C^2-32I\Bigg]A_{\alpha\delta}+\\ +\Bigg[\left(\frac{Q+\sqrt{54CR+Q^2}}{27C}\right)^2(8C^2-9I)-9\left(\frac{Q+\sqrt{54CR+Q^2}}{27C}\right)^2+14\left(\frac{Q+\sqrt{54CR+Q^2}}{27C}\right)C+32\Bigg]A_{\alpha\delta}{}^2\Bigg\}.\end{aligned}$$

From above, by simplifying the equations (3.6), $J=9(s-t)(-t^3+t^2s-ts^2+s^3)-72(t^2-ts-s^2)^2$, $K=8(s-t)^2-9(t^2-ts-s^2)$, we achieved

$$\begin{aligned}\left|a_2a_4-a_3^2\right|\le \frac{1}{72}J+\frac{1}{72}\Bigg\{\Bigg[-\frac{9}{2}\left(\frac{Q+\sqrt{54CR+Q^2}}{27C}\right)^3C++(R-14C)\left(\frac{Q+\sqrt{54CR+Q^2}}{27C}\right)+18C^2-32I\Bigg]A_{\alpha\delta}+\\ +\Bigg[\left(\frac{Q+\sqrt{54CR+Q^2}}{27C}\right)^2K-9\left(\frac{Q+\sqrt{54CR+Q^2}}{27C}\right)^2+14\left(\frac{Q+\sqrt{54CR+Q^2}}{27C}\right)C+32\Bigg]A_{\alpha\delta}{}^2\Bigg\}.\end{aligned}$$

and

$$\begin{aligned}\left|a_2a_4-a_3^2\right|=\frac{1}{72}J+\frac{1}{72}\Bigg\{\Bigg[-\frac{9}{2}(p)^3C+(R-14C)(p)+18C^2-32I\Bigg]A_{\alpha\delta}+\\ +\Big[(p)^2K-9(p)^2+14(p)C+32\Big]A_{\alpha\delta}{}^2\Bigg\}.\end{aligned}$$

This completes the proof of Theorem 3.1.

Corollary 1

For $L(\alpha,\delta,-1,1)$, by letting f for the functions (1.4) whereby $t=-1$ and $s=1$, we obtained

$$\left|a_2a_4-a_3^2\right| \le A_{\alpha\delta}\left(-\frac{215}{162}+\frac{4}{81}\sqrt{613}\right)+A_{\alpha\delta}{}^2\left(-\frac{400}{6561}+\frac{229}{6561}\sqrt{613}\right).$$

The inequality above for the functions $g'(z)=\dfrac{1}{(1-z)^2}$.

Corollary 2

For $L\left(\alpha,\delta,-\frac{1}{2},\frac{1}{2}\right)$, by letting f for the functions (1.4) whereby $t=-\frac{1}{2}$ and $s=\frac{1}{2}$, we obtained

$$\left|a_2a_4-a_3^2\right| \le A_{\alpha\delta}\left(-\frac{1841}{5184}+\frac{35}{5184}\sqrt{9655}\right)+A_{\alpha\delta}{}^2\left(-\frac{6199}{104976}+\frac{2473}{419904}\sqrt{9655}\right).$$

The inequality above for the functions $g'(z)=\dfrac{4}{(2-z)^2}$.

From the corollary, we can say that $L(\alpha,\delta,t,s)$ can be reduced into several types of new classes of functions for upper bound of second Hankel determinant. At the same time, certain $L(\alpha,\delta,t,s)$ can get the same result if different values of t and s can have same function of $g(z)$.

4. Conclusion

In conclusion, we have found a satisfactory solution for determining the upper bound from the coefficient inequalities called the second Hankel determinant for class of close-to-convex functions. There are two purposes of this paper, which are to introduce the class of function $L(\alpha,\delta,t,s)$ and to obtain the upper bound of second Hankel determinant for this class of functions. We believe that we have achieved all the objectives that we highlight and the aim as well as the result obtained. Based on the results, this is in line with previous findings by Kaharudin et al. (2011) and is in line with our research. Initially, we thought that our functions could be reduced to Janteng et al. (2007). However, a more careful analysis has been revealed that s cannot be equal to t. Therefore, we can reduce our results into Kaharudin et al. (2007).

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