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Expression of the Pollaczeck-Khintchine Fuzzy Formulas for a Fuzzy Retrial Queuing System FM/FG/1-FR

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ABSTRACT

The Pollaczek-Khintchine formulas are one of the best and most widely used strategies in the analysis of non-markovian standard or retrial queuing systems with a single server and a general service law. This is particularly the case for classical M/G/1 or M/G/1-R queuing systems because these formulas establish a direct link between the mean number of customers in the system and two first moments of the general service law. The Pollaczek-Khintchine formulas generally allow to evaluate any performance measure Ψ of classical M/G/1-R queuing system by a formula such as: $\Psi = f(\lambda, \theta, m_1, m_2)$, where $\lambda, \theta, m_1, m_2$ are respectively the operating parameters of the system and the two first moments mentioned above. In a fuzzy environment, the literature shows that researchers simply resort to Zadeh's extension principle to obtain fuzzy formula from the classical version above. Instead of doing this to evaluate the performance measures of a non-Markovian fuzzy queuing system denoted FM/FG/1-FR, we have shown in this text that it is possible to derive fuzzy formulas of the kind: $\tilde{\Psi} = \tilde{f}(\tilde{\lambda}, \tilde{\theta} \tilde{m}_1, \tilde{m}_2)$, which are an emanation of the fuzzy generating functions of stationary distributions of the number of customers in orbit and in the system; and in which the fuzzy moments of order 1 and 2 follow directly from the fuzzy distribution function of the general service law. This is the originality of this paper and its contribution is to show how Pollakzek-Khintchine fuzzy formulas can be constructed from these two generating functions. The formulas thus obtained are the same as those obtained from the classical versions by extension according to Zadeh's extension principle. So, they can be validly applied in the evaluation of performance measures of the fuzzy retrial queuing system FM/FG/1-FR.

1. INTRODUCTION

The Pollaczeck-Khinchine formulas are one of the most important mathematical tools for queuing theory. They are widely used in the analysis of performance measures of classical non-markovian standard queuing systems M/G/1 or retrial queuing systems M/G/1-R with a single server and a general service law (Boussaha et al., 2023; 2022; Santhi & Epsya, 2022).

There are also several studies in the literature on the analysis of fuzzy non-markovian standard queuing systems FM/FG/1 or retrial queues FM/FG/1-FR that have used these Pollaczeck-Khintchine formulas (Saritha et al., 2018; Pramela & Kumar, 2019; Narayanamoorthy et al., 2020; Merlyn et al., 2021;

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Adia et al., 2022; Ritha & Rajeswari, 2022: Kannadasan & Padmavathi, 2022; Lakshmi et al., 2022).These queuing systems are characterised by a fuzzy markovian arrival process, a fuzzy general service law and a fuzzy call back process, symbolised by the letter F for "Fuzzy" in English.

Some of the works cited have studied these queuing systems using the α-cuts method (Saritha et al., 2018; Pramela & Kumar, 2019; Merlyn et al., 2021; Kannadasan & Padmavathi, 2022), others by the L-R method L-R (Narayanamoorthy et al., 2020; Adia et al., 2022; Ritha & Rajeswari, 2022; Lakshmi et al., 2022), others once more using the DSW algorithmic approach (Shanmugasundaram & Venkatesh, 2016).

Essentially used to evaluate the mean number of units in orbit and the mean number of customers in the system, the Pollaczeck-Khintchine formulas establish a direct link between these performance measures and the first two moments of the law of the general service law (Adia et al., 2022; Lakshmi et al., 2022).

In practice, in order to evaluate a given performance measure, most researchers use the fuzzy version of the classical Pollaczeck-Khintchine formula of the targeted characteristic by a simple extension according to the Zadeh principle. Indeed, the Pollaczeck-Khintchine formulas for the mean number of customers in orbit and the mean number of customers in an M/G/1-R system are given respectively by the relations (Boussaha, 2023):

$$\mathrm{N_O} = \frac{\lambda^2 m_2}{2(1-\rho)} + \frac{\lambda\rho}{\theta(1-\rho)} \tag{1}$$

$$\mathrm{N} = \rho + \frac{\lambda^2 m_2}{2(1-\rho)} + \frac{\lambda\rho}{\theta(1-\rho)} \tag{2}$$

which are of the form:

$$\mathrm{N} = \mathrm{f}(\lambda, \theta, \mathrm{m}_1, \mathrm{m}_2),$$

where $\lambda, \theta, \mathrm{m}_1, \mathrm{m}_2$ are respectively the operating parameters, the traffic rate in the system and the first two moments of the general service law (with $\rho = \lambda m_1$). Researchers have then tendency to use directly Zadeh's extension principle to write the corresponding fuzzy version:

$$\tilde{\mathrm{N}} = \tilde{\mathrm{f}}(\tilde{\lambda}, \tilde{\theta}, \tilde{\mathrm{m}}_1, \tilde{\mathrm{m}}_2), \tag{3}$$

where these $\tilde{\lambda}, \tilde{\theta}, \tilde{\mathrm{m}}_1, \tilde{\mathrm{m}}_2$ are respectively the fuzzy descriptor parameters of the system and the fuzzy moments of the general service law.

In this paper, our step is original by the fact that the relation (3) above can be elaborated by the generating functions approach instead of simply extending the classical formulas (1) and (2).

In concrete terms, we want to show how we can construct the two Pollaczeck-Khintchine formulas below from the generating functions of the stationary distributions of the number of customers in orbit and in the system:

$$\tilde{\mathrm{N}}_{\mathrm{O}} = \frac{\tilde{\lambda}^2 \odot \tilde{\mathrm{m}}_2}{2(1-\tilde{\rho})} \oplus \frac{\tilde{\lambda} \odot \tilde{\rho}}{\tilde{\theta} \odot (1-\tilde{\rho})} \tag{4}$$

$$\tilde{\mathrm{N}} = \tilde{\rho} \oplus \frac{\tilde{\lambda}^2 \odot \tilde{\mathrm{m}}_2}{2(1-\tilde{\rho})} \oplus \frac{\tilde{\lambda} \odot \tilde{\rho}}{\tilde{\theta} \odot (1-\tilde{\rho})} \tag{5}$$

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Being the same as those derived from the classical formulas by extension of Zadeh extension priciple, these formulas allow us to evaluate respectively the mean number of clients in orbit and the mean number of clients in the FM/FG/1- FR queuing system respectively.

This paper is organised as follows: section 2 presents some preliminary notions. Section 3 discusses mainly the Pollaczek-Khintchine formulas. Section 4 closes the article with a conclusion.

2. PRELIMINARIES

2.1 Definitions

Definition 1: A fuzzy set $\tilde{A}$ of a universe $X \subset \mathbb{R}$ is given by a membership function $\mu_{\tilde{A}}$ defined on X into [0, 1] by:

$$\mu_{\tilde{A}}(x) = \begin{cases} 0 & \text{if } x \notin A \text{ (not at all)} \\ r \in]0,1[& \text{if } x \in A \text{ (partially)} \\ 1 & \text{if } x \in A \text{ (totally)} \end{cases}, \tag{6}$$

where $\mu_{\tilde{A}}(x)$ indicates the degree to which the element x belongs to the set $\tilde{A}$.

The main elements that characterise a fuzzy set $\tilde{A}$ are:

- Its α-cuts or parametic representations:
$$\tilde{A}_\alpha = \{x \in X, \mu_{\tilde{A}}(x) \geq \alpha\};$$
- Its support:
$$\text{supp}(\tilde{A}) = \{x \in X, \mu_{\tilde{A}}(x) > 0\};$$
- Its height:
$$h(\tilde{A}) = \max\{\mu_{\tilde{A}}(x), x \in X\};$$
- Its kernel:
$$\text{kernel}(\tilde{A}) = \{x \in X, \mu_{\tilde{A}}(x) = 1\}.$$

Definition 2: A fuzzy number is a fuzzy subset $\tilde{A}$ that is:

- Normal of universe $\mathbb{R}$ ($h(\tilde{A}) = 1$) ;
- Convex ($\forall x, y, \forall \lambda \in [0, 1], \mu_{\tilde{A}}(\lambda x + (1 - \lambda y)) \geq \min\{\mu_{\tilde{A}}(x), \mu_{\tilde{A}}(y)\}$) ;
- Such that $\text{kernel}(\tilde{A}) \neq \emptyset$, $\text{supp}(\tilde{A})$ and its $\tilde{A}_\alpha$ are bound intervals of the set $\mathbb{R}$. The set of all fuzzy numbers is generally denoted by $\mathbb{F}(\mathbb{R})$.

Definition 3: Any $m \in \text{supp}(\tilde{A})$ such that $\mu_{\tilde{A}}(m) = 1$ is called the modal value or mode of the fuzzy number $\tilde{A}$. It is the element of the support of $\tilde{A}$ with higher possibility.

Definition 4: Let X, Y be two universes and $\tilde{\mathcal{P}}(Y)$ the set of all fuzzy subsets on Y. The application $\tilde{f} : X \longrightarrow \tilde{\mathcal{P}}(Y),\ x \mapsto \tilde{B} = \tilde{f}\,(x)$ is a fuzzy function if,

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$$\forall\ (x,y) \in X{\times}Y, \qquad \mu_{\tilde{B}}(y) = \mu_{\tilde{R}}(x,y) \tag{7}$$

($\tilde{R}$ being a fuzzy relation between the elements of $X{\times}Y$).

The Zadeh's extension principle

This principle extends any classical binary operation $*$ in $\mathbb{R}$ to a fuzzy binary operation $\circledast$ in $\mathbb{F}(\mathbb{R})$ such that $\forall \widetilde{A}, \widetilde{B} \in \mathbb{F}(\mathbb{R})$, $\forall z \in \mathbb{R}$:

$$\mu_{\widetilde{A} \circledast \widetilde{B}}(z) = \sup\{\min\{\mu_{\widetilde{A}}(x), \mu_{\widetilde{B}}(y)\} \,/\, x, y \in \mathbb{R}, x * y = z\} \tag{8}$$

Definition 5: Let $E = E_1 \times \ldots \times E_n$ and F be two classical sets. Let f be also an application from E into F. Zadeh's extension principle creates another application $\tilde{f}$ de $\tilde{\mathcal{P}}(E)$ dans $\tilde{\mathcal{P}}(F)$ such that $\forall \widetilde{A} \in \tilde{\mathcal{P}}(E)$, $\exists \widetilde{B} \in \tilde{\mathcal{P}}(F)$: $\tilde{f}(\widetilde{A}) = \widetilde{B}$ and $\forall y \in F$, we have :

$$\begin{cases} \mu_{\widetilde{B}}(y) = \sup_{x \in E / f(x) = y}\left\{\min\left\{\mu_{\widetilde{A}_1}(x_1), \ldots, \mu_{\widetilde{A}_n}(x_n)\right\}\right\} \text{ si } f^{-1}(y) \neq \emptyset \\ \mu_{\widetilde{B}}(y) = 0 \qquad \text{otherwise} \end{cases} \tag{9}$$

where f^{-1} is the reciproque of f ; $\tilde{\mathcal{P}}(E)$ et $\tilde{\mathcal{P}}(F)$ are respectively the sets of all fuzzy subsets of E and F.

Definition 6: Let $f(x_1, \ldots, x_n)$ be a classical function of $\mathbb{R}^n$ into $\mathbb{R}$ and $\tilde{A}_1, \ldots, \tilde{A}_n$ $\boldsymbol{n}$ fuzzy subsets of $\mathbb{R}$. Zadeh's extension principle allows us to induce from $f(x_1, \ldots, x_n)$ a fuzzy function $\tilde{f} : \mathbb{F}^n(\mathbb{R}) \longrightarrow \mathbb{F}(\mathbb{R})$ such as $\tilde{f}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)$ be a fuzzy subset $\tilde{B}$ of $\mathbb{R}$ of which :

- The membership function is defined $\forall y \in \mathbb{R} | f(x_1, \ldots, x_n) = y$ by :

$$\mu_{\tilde{B}}(y) = \begin{cases} \sup\limits_{(x_1, \ldots, x_n)} \left\{\min\left\{\mu_{\tilde{A}_1}(x_1), \ldots, \mu_{\tilde{A}_n}(x_n)\right\}\right\} & \text{if } f^{-1}(y) \neq \emptyset \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases} \tag{10}$$

- The parametic representation is given $\forall \alpha \in [0, 1]$ by :

$$\tilde{B}(\alpha) = \left(\tilde{f}\left(\tilde{A}_1, \ldots, \tilde{A}_n\right)\right)_\alpha = f\left(\tilde{A}_1(\alpha), \ldots, \tilde{A}_n(\alpha)\right) \tag{11}$$

This definition establishes a compatibility between Zadeh's extension principle approach and the arithmetic of alpha-cuts by transforming the classical operations $(+, -, \times, \div)$ into fuzzy arithmetic operations $(\oplus, \ominus, \odot, \oslash)$ of α-cuts.

Definition 7: $\tilde{f}(x)$ is said to be a fuzzy function of a classical variable if $\tilde{f} : \mathbb{R} \longrightarrow \mathbb{F}(\mathbb{R})$.

2.2 Generating functions

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The generating functions of the stationary distributions of the number of customers in-orbit and in the classical M/G/1-R system with poissoniean arrival rates λ and exponential call back rate θ are given by (Boussaha, 2023):

$$P(z) = \frac{(1-\rho)(1-z)}{A(z)-z} exp\left\{\frac{\lambda}{\theta}\int_1^z \frac{1-A(u)}{A(u)-u}\, du\right\} \quad (12)$$

$$Q(z) = \frac{(1-\rho)(1-z)A(z)}{A(z)-z} exp\left\{\frac{\lambda}{\theta}\int_1^z \frac{1-A(u)}{A(u)-u}\, du\right\} \quad (13)$$

where $A(z) = B^*(\lambda - \lambda z)$ is the generating function of the number of primary customers arriving in the system during the service of a client in the server, $B^*(s)$ is the Laplace transform of the general service law.

3. POLLACZEK-KHINTCHINE FORMULAS

As announced above, we will work out the fuzzy Pollaczec-Khintchine formulas based on the generating functions of the stationary distributions of the number of customers in orbit and in the system.

It is a matter of writing formula (4) and (5) in the form:

$$\tilde{Z} = \tilde{f}(\tilde{\lambda}, \tilde{\theta}, \tilde{m}_1, \tilde{m}_2),$$

where $\tilde{\lambda}$ and $\tilde{\theta}$ are the parameters for the poissoniean arrival of primary clients in the system and exponential call back rate, $\tilde{m}_1$ and $\tilde{m}_2$ are the first moments of the general service law.

This strategy is justified by the fact that the operating parameters $\tilde{x}_1, \dots, \tilde{x}_n$ being fuzzy, the distribution function of the service law of the queuing system FM/FG/1-FR is a fuzzy function induced by the fuzziness of these parameters. By denoting it $\tilde{B}(t)$, it opens the way to our approach by the following definition:

Definition 8: We call *fuzzy distribution function of a general service law*, the function $\tilde{B}(t)$ of classical variable t, induced by the fuzzy character of the system's operating parameters.

Lemma1: *Let G be the general service law of a fuzzy non-markovian queuing system with distribution function $\tilde{B}(t)$. Then, the moments of order 1 and 2 of the law G are fuzzy quantities obtained by a classical derivative of the fuzzy Laplace transform of its distribution function at the point $\check{t} = 0$.*

Proof: Let T be the continuous random variable that measures the length of service of a client supported by the server. Its repartition function is nothing else than the function $\tilde{B}(t)$. The Laplace transform of this random variable T is a fuzzy function of classical variable s finite by:

$$\tilde{B}^*(s) = \int_0^{+\infty} e^{-st}\tilde{b}(t)d(t), \quad (14)$$

where the function $\tilde{b}(t)$ is nothing else than the classical derivative of the repartition function $\tilde{B}(t)$:

$$\tilde{b}(t) = \frac{d\tilde{B}(t)}{dt}. \quad (15)$$

The moments of order k of the random variable T are given (Baynat B., 2000) by :

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$$\widetilde{m}_k = (-1)^k \frac{d^k \tilde{B}^*(s)}{ds^k}(0) \tag{16}$$

where again, the expression $\frac{d\tilde{B}^*(s)}{ds}$ is a classical derivative of $\tilde{B}^*(s)$ with respect to variable s.

Hence the following results:

$$\widetilde{m}_1 = (-1)\frac{d\tilde{B}^*(s)}{ds}(0) \quad \text{et} \quad \widetilde{m}_2 de = (-1)^2 \frac{d^2\tilde{B}^*(s)}{ds^2}(0) \tag{17}$$

■

Corollary 1: *The traffic rate in the fuzzy queuing system is written as follows:*

$$\tilde{\rho} = \tilde{\lambda} \odot \widetilde{m}_1 \tag{18}$$

Lemma 2: L*et FM/FG/1-FR be a fuzzy retrial non-Markovian queuing system with poissoniean arrivals of rate $\tilde{\lambda}$ and general service law.*

Then the probability generating function of the number of customers arriving in the system during the service time interval of a client (handled by the server) is given by:

$$\tilde{A}(z) = \tilde{B}^*(\tilde{\lambda} - \tilde{\lambda} z) \tag{19}$$

where $\tilde{\lambda}$ and $\tilde{B}(t)$ are respectively the fuzzy rate of primary customers arrival and the distribution function of general service law.

Proof: Let V be the random variable which measures the number i of primary customers arrives s in the system during the service time interval of $(n+1)$

ième customer. The probability distributions of that variable are defined by:

$$\mathbb{P}[V = i] = \tilde{a}_i = \int_0^{+\infty} e^{-\tilde{\lambda} t} \frac{(\tilde{\lambda} t)^i}{i!} \tilde{b}(t) d(t) \tag{20}$$

Consequently, the generating function of these probabilities is a fuzzy function of classical variable z defined by:

$$\tilde{A}(z) = \sum_{i=0}^{+\infty} \tilde{a}_i z^i = \sum_{i=0}^{+\infty} \left(\int_0^{+\infty} e^{-\tilde{\lambda} t} \frac{(\tilde{\lambda} t)^i}{i!} \tilde{b}(t) d(t) \right) z^i$$

$$= \int_0^{+\infty} e^{-\tilde{\lambda} t} \left(\sum_{i=0}^{+\infty} \frac{(\tilde{\lambda} z t)^i}{i!} \right) \tilde{b}(t) d(t) = \int_0^{+\infty} e^{-\tilde{\lambda} t} e^{\tilde{\lambda} z t} \tilde{b}(t) d(t)$$

$$= \int_0^{+\infty} e^{-\tilde{\lambda} t} e^{\tilde{\lambda} z t} \tilde{b}(t) d(t) = \int_0^{+\infty} e^{-(\tilde{\lambda} - \tilde{\lambda} z) t} \tilde{b}(t) d(t) = \tilde{B}^*(\tilde{\lambda} - \tilde{\lambda} z).$$

Hence the result. ■

Corollary 2: *This result gives the fuzzy expression of the generating functions given by (12) and (13). They thus become fuzzy functions of classical variable z defined by:*

$$\tilde{P}(z) = (1-\tilde{\rho})\frac{1-z}{\tilde{A}(z)-z}exp\left\{\frac{\tilde{\lambda}}{\tilde{\theta}}\int_1^z \frac{1-\tilde{A}(u)}{\tilde{A}(u)-u}du\right\} \quad (21)$$

$$\tilde{Q}(z) = (1-\tilde{\rho})\frac{(1-z)\tilde{A}(z)}{\tilde{A}(z)-z}exp\left\{\frac{\tilde{\lambda}}{\tilde{\theta}}\int_1^z \frac{1-\tilde{A}(u)}{\tilde{A}(u)-u}du\right\} \quad (22)$$

In a simpler way, these generating functions can be written down as follows:

$$\tilde{P}(z) = \frac{(1-\tilde{\rho})(1-z)}{\tilde{A}(z)-z}exp\left\{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)\right\} \quad (23)$$

$$\tilde{Q}(z) = \frac{(1-\tilde{\rho})(1-z)\tilde{A}(z)}{\tilde{A}(z)-z}exp\left\{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)\right\} \quad (24)$$

with:

$$\tilde{h}(z) = \int_1^z \frac{1-\tilde{A}(u)}{\tilde{A}(u)-u}du. \quad (25)$$

Note : It is implied that, for reasons of economy and efficiency , all arithmetic operations contained in (21) and (22) are fuzzy arithmetic operations, except for those concerning the classical variable z or on the classical numbers: e.g. $1 - z$, $2\tilde{\rho}$, $\tilde{\lambda}z$ etc. It is the same for every operation that flows from them.

These fuzzy arithmetic operations will only be restored in the final stages of the results generated by these functions.

Theorem: *Let $\tilde{P}(z)$ and $\tilde{Q}(z)$ be the generating functions for the number of clients in orbit and in the FM/FG/1-FR queuing system respectively. Then, the mean numbers of customers in orbit and in the system are given by formulas:*

$$\tilde{N}_O = \frac{\tilde{\lambda}^2 \odot \tilde{m}_2}{2(1-\tilde{\rho})} \oplus \frac{\tilde{\lambda}\odot\tilde{\rho}}{\tilde{\theta}\odot(1-\tilde{\rho})} \quad (26)$$

$$\tilde{N} = \tilde{\rho} \oplus \frac{\tilde{\lambda}^2 \odot \tilde{m}_2}{2(1-\tilde{\rho})} \oplus \frac{\tilde{\lambda}\odot\tilde{\rho}}{\tilde{\theta}\odot(1-\tilde{\rho})} \quad (27)$$

Proof: Let us first note the following observations before getting into the heart of the demonstration:

1. $\tilde{P}(z)$ and $\tilde{Q}(z)$are fuzzy functions, but of classical variable z. Any derivative of these functions with respect to the variable z is an operation of classical derivation where any fuzzy quantity contained therein is to be considered as a constant ;

2. As $\tilde{P}(z)$ and $\tilde{Q}(z)$ are generating functions of the number of customers in orbit and in the system, we have by definition

$$N_O = \frac{d\tilde{P}(z)}{dz}(1) \text{ and } N = \frac{d\tilde{Q}(z)}{dz}(1) \quad (28)$$

in classical sence of derivative with respect to the variable z;

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3. Let h be a classical function defined by $h(t) = \int_1^t f(u)du$.

   By definition of integral's derivative, we have:

$$h(1) = 0 \text{ et } \frac{dh(t)}{dt} = h'(t) = f(t) \tag{28}$$

   Consequently, the function $\tilde{h}(z) = \int_1^z \frac{1-\tilde{A}(u)}{\tilde{A}(u)-u} du$ is such that:

$$\tilde{h}(1) = 0 \text{ et } \tilde{h}'(z) = \frac{1-\tilde{A}(z)}{\tilde{A}(z)-z} \tag{29}$$

4. From relation (19), we have:

   - $\tilde{A}(1) = \tilde{B}^*(0) = 1$ (Normalization condition of $\tilde{b}(t)$) (30)
   - $\tilde{A}'(1) = (-\tilde{\lambda}).(\tilde{B}^*)'(0) = \tilde{\lambda}\odot\tilde{m}_1 = \tilde{\rho}$ (31)
   - $\tilde{A}''(1) = \tilde{\lambda}^2.(\tilde{B}^*)''(0) = \tilde{\lambda}^2\odot\tilde{m}_2$ (32)
   - $\tilde{h}'(1) = \frac{\tilde{\rho}}{1-\tilde{\rho}}$ (By Hospital's rule) (33)

5. The functions $\tilde{P}(z)$ and $\tilde{Q}(z)$ in relation (21) and (22) are such that $\tilde{P}(1)$ and $\tilde{Q}(1)$ are indefinite forms $\frac{0}{0}$ which must be removed by the Hospital's rule as follows :

   - $$\tilde{P}(1) = (1-\tilde{\rho})\frac{\left((1-z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}\right)'}{(\tilde{A}(z)-z)'}(1)$$
     $$= (1-\tilde{\rho})\frac{\left(-1+(1-z)\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}'(z)\right)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}}{\tilde{A}'(z)-1}(1) = (1-\tilde{\rho})\frac{-1}{\tilde{A}'(1)-1}$$
     $$= (1-\tilde{\rho})\frac{-1}{\tilde{\rho}-1} = 1 \tag{34}$$
   - $$\tilde{Q}(1) = (1-\tilde{\rho})\frac{\left((1-z)\tilde{A}(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}\right)'}{(\tilde{A}(z)-z)'}(1)$$
     $$= (1-\tilde{\rho})\frac{\left(-\tilde{A}(z)+(1-z)\tilde{A}'(z)+(1-z)\tilde{A}(z)\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}'(z)\right)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}}{\tilde{A}'(z)-1}(1)$$
     $$= (1-\tilde{\rho})\frac{-\tilde{A}(1)}{\tilde{A}'(1)-1} = (1-\tilde{\rho})\frac{-1}{\tilde{\rho}-1} = 1 \tag{35}$$

(i) To better calculate the derivative $\frac{d\tilde{P}(z)}{dz}(1)$, let us write the expression $\tilde{P}(z)$ of relation (23) in the following form :

$$\tilde{P}(z)\left(\tilde{A}(z)-z\right) = (1-\tilde{\rho})(1-z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}.$$

Then, let's derive it member to member to get:

$$\left(\tilde{P}(z)\left(\tilde{A}(z)-z\right)\right)' = \left((1-\tilde{\rho})(1-z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}\right)'.$$

The derivative of the first member gives:

$$\widetilde{P}'(z)\big(\widetilde{A}(z)-z\big)+\widetilde{P}(z)\big(\widetilde{A}'(z)-1\big)\,;$$

That of the second member gives:

$$(1-\tilde{\rho})\left(-e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}+(1-z)\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}'(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}\right).$$

This allows us to pull:

$$\widetilde{P}'(z)=\frac{-\widetilde{P}(z)\big(\widetilde{A}'(z)-1\big)+(1-\tilde{\rho})\left(-e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}+(1-z)\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}'(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}\right)}{\widetilde{A}(z)-z}=\frac{N(z)}{D(z)}.$$

At the point $z = 1$, we obtain an indefinite form $\frac{0}{0}$ as the value of $\widetilde{P}'(1)$ and that we solve in a traditional way by the Hospital's rule:

- $N'(z)=-\widetilde{P}'(z)\big(\widetilde{A}'(z)-1\big)-\widetilde{P}(z)\widetilde{A}''(z)+(1-\tilde{\rho})\left(-\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}'(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}-\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}'(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}+(1-z)\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}''(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}+(1-z)\left(\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}'(z)\right)^2e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}\right).$

  This gives to point z = 1 the value:

$$N'(1)=-\widetilde{P}'(1).(\tilde{\rho}-1)-\tilde{\lambda}^2.\tilde{m}_2-2\tilde{\lambda}.\frac{\tilde{\rho}}{\tilde{\theta}}.$$

- $D'(z)=\widetilde{A}'(z)-1$ et $D'(1)=\tilde{\rho}-1.$

Consequently, we have:

$$\widetilde{P}'(1)=\frac{N'(1)}{D'(1)}=\frac{-\widetilde{P}'(1).(\tilde{\rho}-1)-\tilde{\lambda}^2.\tilde{m}_2-2\tilde{\lambda}.\frac{\tilde{\rho}}{\tilde{\theta}}}{\tilde{\rho}-1}=-\widetilde{P}'(1)-\frac{\tilde{\lambda}^2.\tilde{m}_2}{\tilde{\rho}-1}-2\frac{\tilde{\lambda}.\frac{\tilde{\rho}}{\tilde{\theta}}}{\tilde{\rho}-1}$$

or

$$2\widetilde{P}'(1)=\frac{\tilde{\lambda}^2.\tilde{m}_2}{(1-\tilde{\rho})}+2\frac{\tilde{\lambda}.\tilde{\rho}}{\tilde{\theta}.(1-\tilde{\rho})}.$$

Hence:

$$\frac{d\widetilde{P}(z)}{dz}(1)=\widetilde{P}'(1)=\frac{\tilde{\lambda}^2\odot\tilde{m}_2}{2(1-\tilde{\rho})}\oplus\frac{\tilde{\lambda}\odot\tilde{\rho}}{\tilde{\theta}\odot(1-\tilde{\rho})}. \quad (36)$$

(ii) Samelly, to better calculate the derivative of $\frac{d\widetilde{Q}(z)}{dz}(1)$, let us write $\widetilde{Q}(z)$ of relation (24) in the following form:

$$\widetilde{Q}(z)\big(\widetilde{A}(z)-z\big)=(1-\tilde{\rho})(1-z)\widetilde{A}(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}$$

Then, let's derive it member to member to say:

$$\left(\widetilde{Q}(z)\big(\widetilde{A}(z)-z\big)\right)'=\left((1-\tilde{\rho})(1-z)\widetilde{A}(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}\right)'.$$

The derivative of the first member gives:

$$\widetilde{Q}'(z)\big(\widetilde{A}(z)-z\big)+\widetilde{Q}(z)\big(\widetilde{A}'(z)-1\big)\,;$$

That of the second member gives:

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$$(1-\tilde{\rho})\left(-\tilde{A}(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}+(1-z)\tilde{A}'(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}+(1-z)\tilde{A}(z)\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}'(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}\right).$$

After equalization of these two derivatives, we can write the following relation:

$$\tilde{Q}'(z)\big(\tilde{A}(z)-z\big)=-\tilde{Q}(z)\big(\tilde{A}'(z)-1\big)+(1-\tilde{\rho})\left(-\tilde{A}(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}+(1-z)\tilde{A}'(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}+(1-z)\tilde{A}(z)\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}'(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}\right);$$

Let be,

$$\tilde{Q}'(z)=\frac{-\tilde{Q}(z)\big(\tilde{A}'(z)-1\big)+(1-\tilde{\rho})\left(-\tilde{A}(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}+(1-z)\tilde{A}'(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}+(1-z)\tilde{A}(z)\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}'(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}\right)}{\tilde{A}(z)-z},$$

thus giving an indefinite form $\frac{0}{0}$ as value of $\tilde{Q}'(1)$ and that we can solve once again by Hospital's rule:

Noting $\tilde{Q}'(z)$ as $\frac{N(z)}{D(z)}$, we have:

- $N'(z)=-\tilde{Q}'(z)\big(\tilde{A}'(z)-1\big)-\tilde{Q}(z)\tilde{A}''(z)+(1-\tilde{\rho})\Bigg(-2\tilde{A}'(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}-2\tilde{A}(z)\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}'(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}+(1-z)\tilde{A}''(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}+2(1-z)\tilde{A}'(z)\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}'(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}+(1-z)\tilde{A}(z)\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}''(z)e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}+(1-z)\tilde{A}(z)\left(\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}'(z)\right)^2e^{\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}(z)}\Bigg),$

  At the point z = 1, the expression N'(z) takes the value:

  $$N'(1)=-\tilde{Q}'(1)\big(\tilde{A}'(1)-1\big)-\tilde{Q}(1)\tilde{A}''(1)\oplus(1-\tilde{\rho})\left(-2\tilde{A}'(1)-2\tilde{A}(1)\frac{\tilde{\lambda}}{\tilde{\theta}}\tilde{h}'(1)\right);$$

  Let,

  $$N'(1)=-\tilde{Q}'(1)(\tilde{\rho}-1)-\tilde{\lambda}^2.\tilde{m}_2+(1-\tilde{\rho})\left(-2\tilde{\rho}-2\frac{\tilde{\lambda}}{\tilde{\theta}}\frac{\tilde{\rho}}{1-\tilde{\rho}}\right);$$
- $D'(z)=\tilde{A}'(z)-1$ et $D'(1)=\tilde{\rho}-1$.

  In definitive, the true value of $\tilde{Q}'(z)$ at the point z = 1 is:

$$\tilde{Q}'(1)=\frac{N'(1)}{D'(1)}=\frac{-\tilde{Q}'(1)(\tilde{\rho}-1)-\tilde{\lambda}^2.\tilde{m}_2+(1-\tilde{\rho})\left(-2\tilde{\rho}-2\frac{\tilde{\lambda}}{\tilde{\theta}}.\frac{\tilde{\rho}}{1-\tilde{\rho}}\right)}{\tilde{\rho}-1}$$

Let,

$$\tilde{Q}'(1)=-\tilde{Q}'(1)-\frac{\tilde{\lambda}^2.\tilde{m}_2}{\tilde{\rho}-1}+2\tilde{\rho}+2\frac{\tilde{\lambda}}{\tilde{\theta}}.\frac{\tilde{\rho}}{1-\tilde{\rho}}$$

Or again:

$$2\tilde{Q}'(1)=\frac{\tilde{\lambda}^2.\tilde{m}_2}{1-\tilde{\rho}}+2\tilde{\rho}+2\frac{\tilde{\lambda}}{\tilde{\theta}}.\frac{\tilde{\rho}}{1-\tilde{\rho}}.$$

Hence:

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$$\frac{d\tilde{Q}(z)}{dz}(1) = \tilde{Q}'(1) = \frac{\tilde{\lambda}^2 \odot \tilde{m}_2}{2(1-\tilde{\rho})} \oplus \tilde{\rho} \oplus \frac{\tilde{\lambda} \odot \tilde{\rho}}{\tilde{\theta} \odot (1-\tilde{\rho})}$$

or

$$\frac{d\tilde{Q}(z)}{dz}(1) = \tilde{\rho} \oplus \frac{\tilde{\lambda}^2 \odot \tilde{m}_2}{2(1-\tilde{\rho})} \oplus \frac{\tilde{\lambda} \odot \tilde{\rho}}{\tilde{\theta} \odot (1-\tilde{\rho})} \quad (37)$$

These two results (36) and (37) show that the PollaczekKhintchine fuzzy formulas can well be obtained otherwise than by a simple extension of the classical formulas (1) and (2).

4. CONCLUSION

In this article, we have proposed a more interesting way of obtaining the Pollaczek-Khintchine fuzzy formulas to evaluate the characteristics of a of a non-Markovian fuzzy queuing system of the FM/FG/1-FR type. The formulae (36) and (37) obtained at the end of this step are nothing other than the fuzzy extension of the classical formulas of performance measures of M/G/1-R model. These results have therefore shown that it is possible to construct fuzzy formulas rather than simply extending them using Zadeh's extension principle.

Wouldn't this approach provide a way of analysing the queuing system FM/FG/1-FR in a transient regime? Similarly, would this approach lead to the desired results if the recall law were general? Or what if the number of servers was doubled? All these questions will be the subject of our future research and may allow us to discover the limitations of this method.

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6. CONFLICT OF INTEREST STATEMENT

All authors declare that they have no conflicts of interest to disclose.

7. AUTHORS' CONTRIBUTIONS

Baudouin ADIA LETI MAWA conducted the research, drafted, conceptualized the central idea of the research, provided the theoretical framework and revised the article. Rostin MABELA MAKENGO MATENDO designed the research, supervised the progress of the research, anchored the review, revisions and approved the submission of the article.

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