Journal of Mathematics and Computing Science, 2023,
Volume 9, No 2, 67-75

The Solution of Third Order Ordinary Differential Equations using Adomian Decomposition Method and Variational Iteration Method

W. Khairiyah Hulaini Wan Ramli[1]*, Nur Iffah Zulaikha Hamdan[2], Nur Ain Dayana Ruslan[3], Nur Aisyah Mohamad Asri [4], Zati Iwani Abdul Manaf[5] , Farahanie Fauzi[6] and Norlaila Md Nor[7]

[1,2,3,4,5,6,7]College of Computing, Informatics and Mathematics, Universiti Teknologi MARA Cawangan Kelantan, Kampus Machang, 18500 Machang, Kelantan, Malaysia

Authors' email: wkriyah@uitm.edu.my*, iffah3676@gmail.com, aindayana00@gmail.com, nuraisyah6214@gmail.com, zati431@uitm.edu.my, farah244@uitm.edu.my and nlaila907@uitm.edu.my

*Corresponding author

Received 15 October 2023; Received in revised 2 November 2023; Accepted 25 November 2023
Available online 18 December 2023

Abstract: Solving ordinary differential equations (ODE) is crucial in physics, engineering, and mathematics. Adomian decomposition method (ADM) and variational iteration method (VIM) are two of many methods in solving ODE. Both techniques involve constructing the required iterative or recurrence formulas based on the equation under consideration and additional requirements, allowing the identification of succeeding iterations of a series or sequence approximately matching to solve the third order ODE. In this study, ADM and VIM will be used in solving homogenous and nonhomogeneous third order ordinary differential equations and analysis on method of solutions will be discussed. Overall, it is identified that both methods are efficient in solving linear third order ordinary differential equations.

Keywords: Adomian Decomposition Method, Approximate Solution, Third Order Ordinary Differential Equation, Variational Iteration Method

1 Introduction

In many branches of science and technology, nonlinear phenomena are fundamentally significant. Theoretically and numerically solving nonlinear models of practical issues is still challenging. The search for new and more effective approximation or exact, analytical or numerical approaches for solving nonlinear models has received a lot of attention recently. The Adomian decomposition method (ADM) and the variational iteration method (VIM) are two of the most used semi-analytical techniques for solving linear and nonlinear partial or ordinary differential equations.

According to Zill [1], an equation can be classified as an ordinary differential equation if the equation contains the derivative of one or more dependent variables with respect to a single independent variable. The order of a differential equation is determined by the order of the highest derivative in the equation.

ADM, which was first developed by George Adomian, entails providing the function sought as a series of functions and deriving an iterative formula that allows one to compute the subsequent parts of the series using the initial and boundary conditions provided [2,3]. Wazwaz [4] introduced some modifications to ADM which produced the solution with minimum iterations, but problems arose in splitting the function into two components. Hence, Wazwaz and El-Sayed [5] enhanced the method with a new modification of ADM. The method has a significant advantage in that it provides the solution in a rapidly convergent series with elegantly computable components [6-9].

He [10] and He and Wu [11] developed VIM, which entails building the proper correction functional linked to the equation under consideration. A Lagrange multiplier is present in the correction functional, which results in a recurrence formula. VIM uses the iteration of the correction functional to produce a number of subsequent approximations [12-14]. Recently, Lanlege et al. [15] have used VIM in solving Fredholm Integro-differetial Equations and they identified that the approximate solution given by VIM is close to the exact solution. VIM can also be implemented to solve partial differential equations. Shihab et al. [16] applied VIM in their study where they solved various types of linear and nonlinear partial differential equations.

VIM and ADM have been used in numerous studies to solve ordinary differential equations. Research has also been conducted comparing these two methods. Recently, Bello et al. [17] conducted a study to solve fourth order ordinary differential equations using VIM and ADM and they identified that both methods are powerful. However, they were only focusing on solving homogeneous fourth-order ordinary differential equations.

In this study, both Adomian decomposition method (ADM) and variational iteration method (VIM) will be used in solving both homogenous and nonhomogeneous third order ordinary differential equations. Analysis of the method of solutions and analysis of errors will be made to help summarize the final conclusion.

2 Methodology

A Adomian Decomposition Method

Consider the general ODE,

$$Lu + Nu = g \tag{1}$$

where Lu is taken as the highest order derivative, Nu is nonlinear term and g is the source inhomogeneous term. Solving Lu from Eq. (1) yields

$$Lu = g - Nu \tag{2}$$

Since L is invertible, the inverse operator L^{-1} is assumed to exist. If L is third-order operator, L^{-1} is triple integral operator from 0 to x denoted by

$$L^{-1} = \int_0^x \int_0^x \int_0^x (\cdot)\, dx\, dx\, dx \tag{3}$$

Apply L^{-1} on both sides of Eq. (2) gives

$$L^{-1}Lu = L^{-1}g - L^{-1}Nu \tag{4}$$

Then, the Eq. (4) for u can be simplified by using the initial conditions yields

$$u(x) = A + Bx + Cx^2 + L^{-1}g - L^{-1}Nu \tag{5}$$

Therefore, u can be represented as a series

$$u(x) = \sum_{n=0}^{\infty} u_n \tag{6}$$

with u_0 identified as $A + Bx + Cx^2 + L^{-1}g$ and u_n for $n > 0$ is to be determined.

B Variational Iteration Method

Consider the differential equation in Eq. (1) where g is the source inhomogeneous term, Lu and Nu defined as linear and nonlinear operators, respectively. VIM contains corrective action functional for Eq. (1) as follows:

$$u_{n+1}(x) = u_n(x) + \left(\int_0^x \lambda(t)\left[Lu_n(t) + Nu_n(t) - g(t)\right] dt \right) \tag{7}$$

where u_n is a restricted variation, which means $\delta u_n = 0$, and is a generic Lagrange's multiplier that may be discovered optimum using the variational theory. For the third order ordinary differential equation,

$$\lambda = -\frac{(t-x)^2}{2} \tag{8}$$

Once the Lagrangian multiplier has been established, any selective function u_0 may be used to easily derive the subsequent approximations u_{n+1} , $n \geqslant 0$, of the solution u. Hence,

$$u = \lim_{n\to\infty} u_n \tag{9}$$

In other words, the correction functional Eq. (7) will provide a number of approximations, and the precise solution is consequently found at the upper bound of the next series of approximations.

3 Numerical Examples

A Example 1

Consider the homogeneous differential equation

$$u''' + 2u'' - 5u' - 6u = 0 \text{ , } u(0) = 0, u'(0) = 0, u''(0) = 1 \tag{10}$$

i. Adomian Decomposition Method

Proceeding according to the method explained earlier, we obtained the following successive approximations:

$$u_0 = \frac{x^2}{2}$$

$$u_1 = -\frac{1}{3}x^3 + \frac{5}{24}x^4 + \frac{1}{20}x^5$$

$$u_2 = \frac{1}{6}x^4 - \frac{1}{6}x^5 + \frac{1}{720}x^6 + \frac{1}{84}x^7 + \frac{1}{1120}x^8$$

$$u_3 = -\frac{1}{15}x^5 + \frac{1}{12}x^6 - \frac{13}{840}x^7 - \frac{47}{8064}x^8 + \frac{13}{20160}x^9 + \frac{1}{6720}x^{10} + \frac{1}{184800}x^{11}$$

The summation of $u_0, u_1, u_2, \ldots u_n$ will give the solution of $u(x)$.

ii. Variational Iteration Method

Using Eq. (9) and $u_0 = \frac{1}{2}x^2$ from the initial condition, the iteration formula shows :

$$u_1 = \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{5}{24}x^4 + \frac{1}{20}x^5$$

$$u_2 = \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{3}{8}x^4 - \frac{7}{60}x^5 + \frac{1}{720}x^6 + \cdots$$

$$u_3 = \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{3}{8}x^4 - \frac{11}{60}x^5 + \frac{61}{720}x^6 - \frac{1}{280}x^7 + \cdots$$

$$\vdots$$

$$u_n = \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{3}{8}x^4 - \frac{11}{60}x^5 + \frac{77}{720}x^6 - \frac{1}{24}x^7 + \frac{673}{40320}x^8 + \cdots$$

B Example 2

Consider the nonhomogeneous differential equation

$$u''' + 3u'' + 3u' + u = 1, \; u(0) = 0, u'(0) = 2, u''(0) = 2 \tag{11}$$

i. Adomian Decomposition Method

Proceeding according to the method explained earlier, we obtained the following successive approximations:

$$u_0 = 2x + x^2 + \frac{1}{6}x^3$$

$$u_1 = -2x^3 - \frac{11}{24}x^4 - \frac{1}{24}x^5 - \frac{1}{720}x^6$$

$$u_2 = \frac{3}{2}x^4 + \frac{23}{40}x^5 + \frac{1}{12}x^6 + \frac{29}{5040}x^7 + \frac{1}{5040}x^8 + \frac{1}{362880}x^9$$

$$u_3 = -\frac{9}{10}x^5 - \frac{7}{16}x^6 - \frac{47}{560}x^7 - \frac{1}{120}x^8 - \frac{19}{40320}x^9 - \frac{1}{64800}x^{10} - \frac{1}{3628800}x^{11} - \frac{1}{479001600}x^{12}$$

The summation of $u_0, u_1, u_2, \ldots u_n$ will give the solution of $u(x)$.

ii. Variational Iteration Method

Using Eq. (9) and $u_0 = 2x + x^2$ from the initial condition, the iteration formula shows:

$$u_1 = 2x + x^2 - \frac{11}{6}x^3 - \frac{1}{3}x^4 - \frac{1}{60}x^5$$

$$u_2 = 2x + x^2 - \frac{11}{6}x^3 + \frac{24}{25}x^4 + \frac{11}{24}x^5 + \frac{41}{720}x^6 + \cdots$$

$$u_3 = 2x + x^2 - \frac{11}{6}x^3 + \frac{24}{25}x^4 - \frac{11}{30}x^5 - \frac{229}{720}x^6 - \frac{313}{5040}x^7 + \cdots$$

$$\vdots$$

$$u_n = 2x + x^2 - \frac{11}{6}x^3 + \frac{24}{25}x^4 - \frac{11}{30}x^5 + \frac{17}{180}x^6 - \frac{97}{5040}x^7 + \frac{131}{40320}x^8 + \cdots$$

C Example 3

Consider the nonhomogeneous differential equation

$$u''' - u' = 4e^{-x} + 3e^{2x}\ ,\ u(0) = 0,\ u'(0) = -1,\ u''(0) = 2 \tag{12}$$

i. Adomian Decomposition Method

Proceeding according to the method explained earlier, we obtained the following successive approximations:

$$u_0 = \frac{9}{4}x^2 - \frac{23}{4}x + \frac{29}{8} - 4e^{-x} + \frac{3}{8}e^{2x}$$

$$u_1 = \frac{125}{32} - \frac{67}{16}x + \frac{29}{16}x^2 - \frac{23}{24}x^3 + \frac{3}{16}x^4 + \cdots$$

$$u_2 = \frac{509}{128} - \frac{259}{64}x + \frac{125}{64}x^2 - \frac{67}{96}x^3 + \frac{29}{192}x^4 - \frac{23}{480}x^5 + \cdots$$

$$u_3 = \frac{2045}{512} - \frac{1027}{256}x + \frac{509}{256}x^2 - \frac{259}{384}x^3 + \frac{125}{768}x^4 - \frac{67}{1920}x^5 + \frac{29}{5760}x^6 + \cdots$$

The summation of $u_0, u_1, u_2, \ldots u_n$ will give the solution of $u(x)$.

ii. Variational Iteration Method

Using Eq. (9) and $u_0 = -x + x^2$ from the initial condition, the iteration formula shows :

$$u_1 = \frac{29}{8} - \frac{23}{4}x + \frac{9}{4}x^2 - \frac{1}{6}x^3 + \cdots$$

$$u_2 = \frac{241}{32} - \frac{159}{16}x + \frac{65}{16}x^2 - \frac{23}{24}x^3 + \frac{3}{16}x^4 + \cdots$$

$$u_3 = \frac{1473}{128} - \frac{895}{64}x + \frac{385}{64}x^2 - \frac{53}{32}x^3 + \frac{65}{192}x^4 + \cdots$$
$$\vdots$$
$$u_n = \frac{82837505}{2097152} - \frac{44040191}{1048576}x + \frac{20971521}{1048576}x^2 + \cdots$$

4 Results and Discussion

ADM and VIM were successfully applied to the third-order ordinary differential equation with initial conditions until $n = 10$. The results from these two methods including the exact solutions are showed in the following tables.

Example 1 - $u''' + 2u'' - 5u' - 6u = 0$, $u(0) = 0, u'(0) = 0, u''(0) = 1$

Table 1: Exact solution and approximate solution using ADM for example 1

x	Exact Solution	Approximate Solution	Relative Error
0.1	0.00470243630	0.004702436273	0.000000005742
0.2	0.01788101800	0.017881017930	0.000000003915
0.3	0.03866184920	0.038661849230	0.000000000776
0.4	0.06676880870	0.066768808750	0.000000000749
0.5	0.10244336120	0.102443361200	0.000000000000
0.6	0.14640241100	0.146402411200	0.000000001366
0.7	0.19982809000	0.199828090500	0.000000002502
0.8	0.26438579620	0.264385798500	0.000000008699
0.9	0.34226877220	0.342268779800	0.000000022205
1.0	0.43626920650	0.436269226300	0.000000045385

Table 2: Exact solution and approximate solution using VIM for example 1

x	Exact Solution	Approximate Solution	Relative Error
0.1	0.00470243630	0.004702436273	0.000000005742
0.2	0.01788101800	0.017881017930	0.000000003915
0.3	0.03866184920	0.038661849220	0.000000000517
0.4	0.06676880870	0.066768808750	0.000000000749
0.5	0.10244336120	0.102443361200	0.000000000000
0.6	0.14640241100	0.146402411200	0.000000001366
0.7	0.19982809000	0.199828090600	0.000000003003
0.8	0.26438579620	0.264385798400	0.000000008321
0.9	0.34226877220	0.342268779600	0.000000021620
1.0	0.43626920650	0.436269226200	0.000000045156

Example 2 - $u''' + 3u'' + 3u' + u = 1$, $u(0) = 0, u'(0) = 2, u''(0) = 2$

Table 3: Exact solution and approximate solution using ADM for example 2

x	Exact Solution	Approximate Solution	Relative Error
0.1	0.2082672592	0.2082672592	0.00000000000
0.2	0.4268884728	0.4268884729	0.00000000023
0.3	0.6481113451	0.6481113452	0.00000000015
0.4	0.8659359909	0.8659359918	0.00000000104
0.5	1.0758163320	1.0758163510	0.00000001766
0.6	1.2744058180	1.2744060260	0.00000016321

0.7	1.4593414060	1.4593430350	0.00000111626
0.8	1.6290605500	1.6290703120	0.00000599241
0.9	1.7826465950	1.7826942740	0.00002674619
1.0	1.9196986030	1.9198966770	0.00010317974

Table 4: Exact solution and approximate solution using VIM for example 2

x	Exact Solution	Approximate Solution	Relative Error
0.1	0.2082672592	0.2082672592	0.00000000000
0.2	0.4268884728	0.4268884728	0.00000000000
0.3	0.6481113451	0.6481113452	0.00000000015
0.4	0.8659359909	0.8659359902	0.00000000081
0.5	1.0758163320	1.0758163220	0.00000000930
0.6	1.2744058180	1.2744057070	0.00000008710
0.7	1.4593414060	1.4593406320	0.00000053038
0.8	1.6290605500	1.6290565200	0.00000247382
0.9	1.7826465950	1.7826300710	0.00000926936
1.0	1.9196986030	1.9196438150	0.00002853990

Example 3 - $u''' - u' = 4e^{-x} + 3e^{2x}$, $u(0) = 0, u'(0) = -1, u''(0) = 2$

Table 5: Exact solution and approximate solution using ADM for example 3

x	Exact Solution	Approximate Solution	Relative Error
0.1	-0.0889814654	-0.0889814704	0.00000005653
0.2	-0.1516723378	-0.1516723351	0.00000001780
0.3	-0.1811767850	-0.1811767871	0.00000001159
0.4	-0.1696933150	-0.1696933056	0.00000005539
0.5	-0.1082057870	-0.1082057859	0.00000001017
0.6	0.0138789690	0.0138789655	0.00000025074
0.7	0.2091606240	0.2091606289	0.00000002343
0.8	0.4927584110	0.4927584095	0.00000000304
0.9	0.8829277580	0.8829277557	0.00000000260
1.0	1.4018046970	1.4018047060	0.00000000642

Table 6: Exact solution and approximate solution using VIM for example 3

x	Exact Solution	Approximate Solution	Relative Error
0.1	-0.0889814654	-0.0889814634	0.00000002248
0.2	-0.1516723378	-0.1516723529	0.00000009956
0.3	-0.1811767850	-0.1811767689	0.00000008886
0.4	-0.1696933150	-0.1696933070	0.00000004714
0.5	-0.1082057870	-0.1082057720	0.00000013862
0.6	0.0138789690	0.0138789680	0.00000007205
0.7	0.2091606240	0.2091606300	0.00000002869
0.8	0.4927584110	0.4927584200	0.00000001826
0.9	0.8829277580	0.8829277660	0.00000000906
1.0	1.4018046970	1.4018046860	0.00000000785

Table 1 until Table 6 show the comparison between exact solution and approximate solution using Adomian decomposition and variational iteration method alongside relative errors on homogeneous and nonhomogeneous third order ordinary differential equation. From the tables, it can be shown that both methods produce approximate solutions which are nearly close to the exact solution for the third order ordinary differential equation.

5 Conclusion

The ADM and the VIM are both highly effective methods for obtaining accurate approximations that are close to the exact solution. However, the VIM makes iteration simple and direct by eliminating the need for the time-consuming Adomian polynomials. VIM simplifies the computational effort and provides the solution quickly for nonlinear equations that commonly occur to represent nonlinear phenomena [18-19]. But all in all, it can be concluded that both VIM and ADM methods are very effective in finding solutions for wide classes of ordinary differential equations [20].

In the future, researchers can explore the effectiveness of ADM and VIM for nonlinear ordinary differential equations. Furthermore, investigating the applicability of these methods for alternative types of differential equations, particularly partial differential equations, may broaden their usefulness in solving real-world problems.

Acknowledgements

This research received no specific grant from any funding agency in the public, commercial, or private sectors.

References

[1] D. G. Zill, *Differential Equations with Boundary-Value Problems,* 9th ed. Los Angeles: Cengage Learning, 2018.
[2] G. Adomian, "A Review of the Decomposition Method in Applied Mathematics," *Journal of Mathematical Analysis and Applications,* vol. 135, pp. 501–544, 1988.
[3] G. Adomian, "A review of the decomposition method and some recent results for nonlinear equations," *Journal of Computers Math. Applic.,* vol. 21, no. 5, pp. 101–127, 1991.
[4] A. M. Wazwaz, "A reliable modification of Adomian decomposition method," *Journal of Applied Mathematics and Computation*, vol. 102, pp. 77–86, 1999.
[5] A. M. Wazwaz and S. M. El-Sayed, "A new modification of Adomian decomposition method for linear and nonlinear operators," *Journal of Applied Mathematics and Computation*, vol. 122, pp. 393–405, 2001.
[6] A. M. Wazwaz, "A new algorithm for calculating Adomian polynomials for nonlinear operators," *Journal of Applied Mathematics and Computation*, vol. 111, pp. 53–69, 2000.
[7] J. Biazar, E. Babolian and R. Islam, "Solution of the system of ordinary differential equations by Adomian decomposition method," *Journal of Applied Mathematics and Computation*, vol. 147, no. 3, pp. 713–719, 2004.
[8] M. M. Hosseini and H. Nasabzadeh, "On the convergence of Adomian decomposition method," *Applied Mathematics and Computation*, vol. 182, pp. 536–543, 2006.
[9] H. Jafari and V. Daftardar-Gejji, "Solving a system of nonlinear fractional differential equations using Adomian decomposition," *Journal of Computational and Applied Mathematics*, vol. 196, pp. 644–651, 2006.
[10] J. H. He, "Variational iteration method - a kind of non-linear analytical technique: some examples," *International Journal of Non-Linear Mechanics,* vol. 34, pp. 699-708, 1999.
[11] J. H. He and X. H. Wu, "Variational iteration method: new development and applications," *Computers & Mathematics with Applications*, vol. 54, pp. 881–894, 2007.
[12] S. Momani, S. Abuasad and Z. Odibat, "Variational iteration method for solving nonlinear boundary value problems," *Journal of Applied Mathematics and Computation,* vol. 183, pp. 1351-1358, 2006.
[13] S. Abbasbandy and E. Shivanian, "Application of Variational Iteration Method for nth-order Integro-Differential Equations," *Journal of Physical Sciences*, vol. 64a, pp. 439-444, 2009.
[14] A. M. Wazwaz, "The variational iteration method for analytic treatment for linear and nonlinear ODEs," *Journal of Applied Mathematics and Computation*, vol. 212, pp. 120–134, 2009.

[15] D. I. Lanlege, F. M. Edibo and M. S. Omeiza, “Solution Of Fredholm Integro-Differential Equation By Variational Iteration Method,” *FUDMA Journal Of Sciences*, vol. 7, no. 2, pp. 1-8, 2023.

[16] M. A. Shihab, W. M. Taha, R. A. Hameed, A. Jameel and I. M. Sulaiman, “Implementation of variational iteration method for various types of linear and nonlinear partial differential equations,” *International Journal of Electrical and Computer Engineering*, vol. 13, no. 2, pp. 2131-2141, 2023.

[17] K. A. Bello, O. A. Taiwo, M. A. Adeyanju, A. Abdulkareem, A. Abubakar and G. O. Ibrahim, “Adomian Decomposition Method and Variational Iteration Method for Solving Fourth Order Differential Equation with Initial Conditions,” *Journal of Science Technology and Education*, vol. 9, no. 4, pp. 240-249, 2021.

[18] A. M. Wazwaz, “A comparison between the variational iteration method and Adomian decomposition method,” *Journal of Computational and Applied Mathematics*, vol. 207, pp. 129-136, 2007.

[19] E. Hetmaniok, D. Słota, R. Wituła and A. Zielonka, “Comparison of the Adomian decomposition method and the variational iteration method in solving the moving boundary problem,” *Computers and Mathematics with Applications*, vol. 61, pp. 1931–1934, 2011.

[20] A. Abdirashidov, B. Ortikov, N. Kadirov and A. Abdurashidov, “Application of Adomian decomposition method, Taylor series method and a variational iterations method to solving a second order ordinary differential equations,” *International Scientific Journal Theoretical & Applied Science*, vol. 73, no. 5, pp. 1–5, 2019.