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Second Hankel Determinants for a Subclass of Close-to-Convex Function Related to Certain Generalized Koebe Function

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ABSTRACT

For ages, researchers have conducted numerous studies exploring every aspect of problems related to univalent functions. Most of the research has been concentrated on investigating the diverse properties of univalent functions. Notably, finding the upper bound of Hankel determinants has become an intriguing problem among researchers in this field. The aim of this paper is to solve on the second Hankel determinant problem for the class $C_{k_\beta(z)}$ of close-to-convex functions related with the certain generalized starlike functions. We first give the definition of the class $C_{k_\beta(z)}$ and use certain preliminary lemmas to achieve on the main goal of this research. The finding of this research generalizes certain results related to the second Hankel determinant of other classes of close-to-convex functions.

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1. Introduction

Let A denote the class of analytic functions normalized by $f(0) = f'(0) - 1 = 0$ in the unit disc U $= \{z \in \mathbb{C} : |z| < 1\}$. Functions $f \in$ A has the Taylor series expression of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1}$$

where a_n is the coefficient of *f*. Now, let Σ be the subclass of A consists of functions *f* which are univalent in Υ. Also, let ΣΤ , Κ and Χ be the subclasses of Σ containing functions *f* which represent the class of starlike functions, convex functions and close-to-convex functions respectively. The class ΣΤ , Κ and Χ are defined as follows.

Definition 1.1 [1] Let *f* be given by (1). Then $f \in ST$ if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0, \quad z \in U.$$

Definition 1.2 [1] Let *f* be given by (1). Then $f \in K$ if and only if

$$\operatorname{Re}\left\{\frac{(zf'(z))'}{f'(z)}\right\} > 0, \quad z \in U.$$

Definition 1.3 [1] Let *f* be given by (1). Then $f \in C$ if there exist the function $g(z) \in ST$ and a real number $\beta \in (-\pi/2, \pi/2)$ such that

$$\operatorname{Re}\left\{e^{i\beta}\frac{zf'(z)}{g(z)}\right\} > 0, \quad z \in U. \tag{2}$$

The most important example of a function in the Σ is the Koebe function given by

$$k(z) = \frac{z}{(1-z)^2} = \frac{1}{4}\left[\left(\frac{1+z}{1-z}\right)^2 - 1\right] = \sum_{n=1}^{\infty} nz^n.$$

This function plays a crucial role, as it is pivotal in numerous findings concerning univalent functions.

Noonan and Thomas [2] defined the q^{th} Hankel determinant of $f \in S$ for positive integers *n* and *q* by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q-2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q-2} & \cdots & a_{n+2(q-1)} \end{vmatrix}.$$

Easily, we can see that for the case $n = 2$ and $q = 2$ we have

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = a_2a_4 - a_3^2$$

known as the second Hankel determinant.

The study on the second Hankel determinant, that is finding the upper bound of the functional $\left|a_2a_4 - a_3^2\right|$ of $f \in S$ began since 1960. Many studies have been conducted to solve the problem of finding the upper bound of $\left|a_2a_4 - a_3^2\right|$ for various subclasses of Σ (see, for examples [3] [4] [5], [6] [7], [8], [9], [10], [11], [12]). Particularly, for $f \in C$ satisfying (2) with certain functions $g(z)$ and value β, Janteng et al. [13] obtained $\left|a_2a_4 - a_3^2\right| \le \frac{4}{9}$, Soh and Mohammad [14] obtained $\left|a_2a_4 - a_3^2\right| \le \frac{13}{36}$, Mehrok et al. [15] obtained $\left|a_2a_4 - a_3^2\right| \le \frac{73}{72}$ and Ullah et al. [16] obtained $\left|a_2a_4 - a_3^2\right| \le \frac{1}{36}$.

Inspired by the previous study, our objective in this paper is to establish the upper bound of the functional $\left|a_2a_4 - a_3{}^2\right|$ for the subclass of $f \in$ Cb with respect to generalized Koebe function, $C_{k_\alpha(z)}$ defined as follows.

Definition 1.4 Let $f \in$ S= be given by (1). Then $f \in C_{k_\beta(z)}$ if and only if for $0 \le \beta \le 2,$ there exist $k_\beta(z) = \dfrac{z}{(1-z)^\beta} \in$ ST such that

$$\operatorname{Re}\left\{\frac{zf'(z)}{k_\beta(z)}\right\} > 0, \quad z \in \text{U}.$$

To prove on the main theorem of this paper, we give some preliminaries results related to the class of function whose real parts are positive in Υ
.

2. Preliminaries

Let Π denote the class of analytic functions $p(z)$ whose real part are positive in Υ normalized by $p(0) = 1$ and satisfying $\operatorname{Re} p(z) > 0$. All functions $p \in$ P has the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n. \tag{3}$$

Lemma 2.1. [17]
Let $p \in$ P be analytic functions in Υ given by (3). Then the sharp inequality $|c_n| \le 2$ hold for all $n \ge 1$. Equality occurs for the function $p(z) = \dfrac{1+z}{1-z}.$

Lemma 2.2. [18]
Let $p \in$ P be analytic functions in Υ given by (3). Then,

$$2c_2 = c_1^2 + x\left(4 - c_1^2\right)$$

for some x, $|x| \le 1$ and

$$4c_3 = c_1^3 + 2\left(4 - c_1^2\right)c_1 x - c_1\left(4 - c_1^2\right)x^2 + 2\left(4 - c_1^2\right)\left(1 - |x|^2\right)z$$

for some z, $|z| \le 1$.

3. Results and Discussion

To achieve on the main result for this paper, we first find the coefficients a_2, a_3 and a_4 by relating the $f \in C_{k_\beta(z)}$ with $p \in$ P.. Then, we apply Lemma 2.1 and Lemma 2.2 to establish on the upper bound of $\left|a_2a_4 - a_3{}^2\right|$ for the class $C_{k_\beta(z)}$.

Theorem 1
If $f(z)$ given by (1) is in the class $C_{k_\beta(z)}$, then

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{144}\left(-32\beta + 6\beta^2 + \beta^3 - \beta^4\right) - \frac{49\beta^2}{36\left(7\beta^2 - 9\beta - 18\right)} - \frac{\beta\left(2\beta^2 - \beta - 30\right)\sqrt{-12\beta^2 + 6\beta + 180}}{486} + \frac{4}{9}.$$

Proof.

From definition 1.4, we see that $f \in C_{k_\beta(z)}$ if and only if there exist $p \in P$ such that

$$f'(z) = \left[\frac{1}{(1-z)^\beta}\right]p(z) \tag{4}$$

where, $f'(z)$ were obtained from the series given by (1), $p(z)$ is the series given by (3) and

$$\frac{1}{(1-z)^\beta} = 1 + \beta z + \left(\frac{1}{2}\beta^2 + \frac{1}{2}\beta\right)z^2 + \left(\frac{1}{6}\beta^3 + \frac{1}{2}\beta^2 + \frac{1}{3}\beta\right)z^3 + \ldots.$$

Now, by comparing the coefficients in (4), we have

$$a_2 = \frac{1}{2}(\beta + c_1),$$

$$a_3 = \frac{1}{3}\left(\frac{1}{2}\beta^2 + \frac{1}{2}\beta + \beta c_1 + c_2\right),$$

and

$$a_4 = \frac{1}{4}\left[\frac{1}{6}\beta^3 + \frac{1}{2}\beta^2 + \frac{1}{3}\beta + \left(\frac{1}{2}\beta^2 + \frac{1}{2}\beta\right)c_1 + \beta c_2 + c_3\right].$$

Then, the application of the triangle inequality gives

$$\left|a_2a_4 - a_3^2\right| \le \frac{1}{144}\left|-\beta^4 + \beta^3 + 2\beta^2\right| + \frac{1}{72}|A| + \frac{1}{144}|B| \tag{5}$$

where,

$$A = \left(-2\beta^3 + \beta^2 + 3\beta\right)c_1 + \left(\beta^2 - 8\beta\right)c_2 + 9\beta c_3$$

and

$$B = \left(-7\beta^2 + 9\beta\right)c_1^2 - 14\beta c_1c_2 + 18c_1c_3 - 16c_2^2.$$

Next, to establish the upper bound of $\left|a_2a_4 - a_3^2\right|$, we first, seek the upper bound of $|A|$ and $|B|$ using the following approach.

Let $c_1 = c$, $\eta = |x| \le 1$ and applying Lemma 2.1 and Lemma 2.2 along with the triangle inequality we obtain

$$\begin{aligned}|A| &= \left|\left(-2\beta^3 + \beta^2 + 3\beta\right)c_1 + \left(\beta^2 - 8\beta\right)c_2 + 9\beta c_3\right|,\\ &\le \frac{9}{4}\beta c^3 + \frac{1}{2}\left(\beta^2 - 8\beta\right)c^2 + \left(-2\beta^3 + \beta^2 + 3\beta\right)c + \frac{1}{2}\left(\beta^2 - 8\beta + 9\beta c\right)\left(4 - c^2\right)\eta + \frac{9}{4}\beta c\left(4 - c^2\right)\eta^2\\ &\quad - \frac{9}{2}\beta\left(4 - c^2\right)\eta^2 + \frac{9}{2}\beta\left(4 - c^2\right),\end{aligned}$$

$$\leq\left(\frac{9}{4}\beta c-\frac{9}{2}\beta\right)\left(4-c^2\right)\eta^2+\frac{1}{2}\left(\beta^2-8\beta+9\beta c\right)\left(4-c^2\right)\eta+\frac{9}{4}\beta c^3+\frac{1}{2}\left(\beta^2-8\beta\right)c^2$$

$$+\left(-2\beta^3+\beta^2+3\beta\right)c+\frac{9}{2}\beta\left(4-c^2\right)=F(\beta,c,\eta).$$

For $\eta\leq 1$, we obtain

$$F(\beta,c,\eta)\leq F(\beta,c,1)=-\frac{9}{2}\beta c^3+\left(-2\beta^3+\beta^2+30\beta\right)c+2\beta^2-16\beta.$$

The application of derivatives test with respect to *c*, for $c\in[0,2]$ and $\beta\in[0,2]$, we obtain that, the maximum of $F(\beta,c,1)$ occur at $c=\frac{1}{9}\sqrt{180+6\beta-12\beta^2}$. Hence,

$$|A|\leq\frac{4}{27}\beta\left[\frac{27}{2}\beta-\left(2\beta^2-\beta-30\right)\sqrt{180+6\beta-12\beta^2}-108\right]. \tag{6}$$

Next, we seek the bound for $|B|$ by using the same approach as seeking the bound for $|A|$. Replacing $|x|=\upsilon$ we have,

$$|B|=\left|\left(-7\beta^2+9\beta\right)c_1^2-14c_1c_2+18c_1c_3-16c_2^2\right|$$

$$\leq c^2\left(-\frac{1}{2}c^2+7\beta c+7\beta^2-9\beta\right)+9c\left(4-c^2\right)+\left(7\beta c-c^2\right)\left(4-c^2\right)\upsilon$$

$$+\left(\frac{1}{2}c^2-9c+16\right)\left(4-c^2\right)\upsilon^2=G(\beta,c,\upsilon)$$

where for $\upsilon\leq 1$,

$$G(\beta,c,\upsilon)\leq G(\beta,c,1)=\left(7\beta^2-9\beta-18\right)c^2+28\beta c+64$$

and for $c\in[0,2]$ and $\beta\in[0,2]$, the maximum of $G(\beta,c,1)$ occur at $c=-\frac{14}{7\beta^2-9\beta-18}$. Thus, we have,

$$|B|\leq 64-\frac{196}{7\beta^2-9\beta-18} \tag{7}$$

Now, applying (6) and (7) into (5), we obtain

$$\left|a_2a_4-a_3^2\right|\leq\frac{1}{144}\left(-32\beta+6\beta^2+\beta^3-\beta^4\right)-\frac{49\beta^2}{36\left(7\beta^2-9\beta-18\right)}-\frac{\beta\left(2\beta^2-\beta-30\right)\sqrt{-12\beta^2+6\beta+180}}{486}+\frac{4}{9}$$

as required. This completes the proof of Theorem 1.

Corollary 3.1

By setting $\beta=0$, we get the same results $a_2=\frac{c_1}{2}$, $a_3=\frac{c_2}{3}$, $a_4=\frac{c_3}{4}$ and $\left|a_2a_4-a_3^2\right|\leq\frac{4}{9}$ which were earlier obtained in [13].

4. Conclusion

This study focuses on finding the upper bound for the functional $\left|a_2a_4-a_3^2\right|$ of the class $C_{k_\beta(z)}$. The objective of this study was achieved by applying Lemma 2.1 and Lemma 2.2. The result obtained can be reduced to the class studied by [13], as given in Corollary 3.1. Additionally, from Theorem 1 for $\beta=1$ and $\beta=2$, we can obtain new results of second Hankel determinant for the class $C_{k_1(z)}$ and $C_{k_2(z)}$ satisfying the conditions $\operatorname{Re}\left\{(1-z)f'(z)\right\}>0$ and $\operatorname{Re}\left\{(1-z)^2f'(z)\right\}>0$

respectively. Future studies are suggested to further generalized the class $C_{k_\beta(z)}$ such that the results obtained can be reduced to many subclasses studied by previous researchers. Furthermore, further studies could be conducted to solve other problems such as finding the third Hankel determinant and logarithmic coefficients for the class $C_{k_\beta(z)}$.

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Conflict of Interest

The authors declare no conflict of interest in the subject matter or materials discussed in this manuscript.

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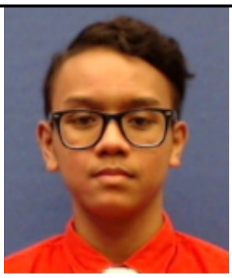	**Wan Muhammad Fakhrorrazzi Wan Ibrahim** is a graduate of Universiti Teknologi MARA in Bachelor of Science (Hons.) Computational Mathematics	Methodology, result analysis and editing.
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