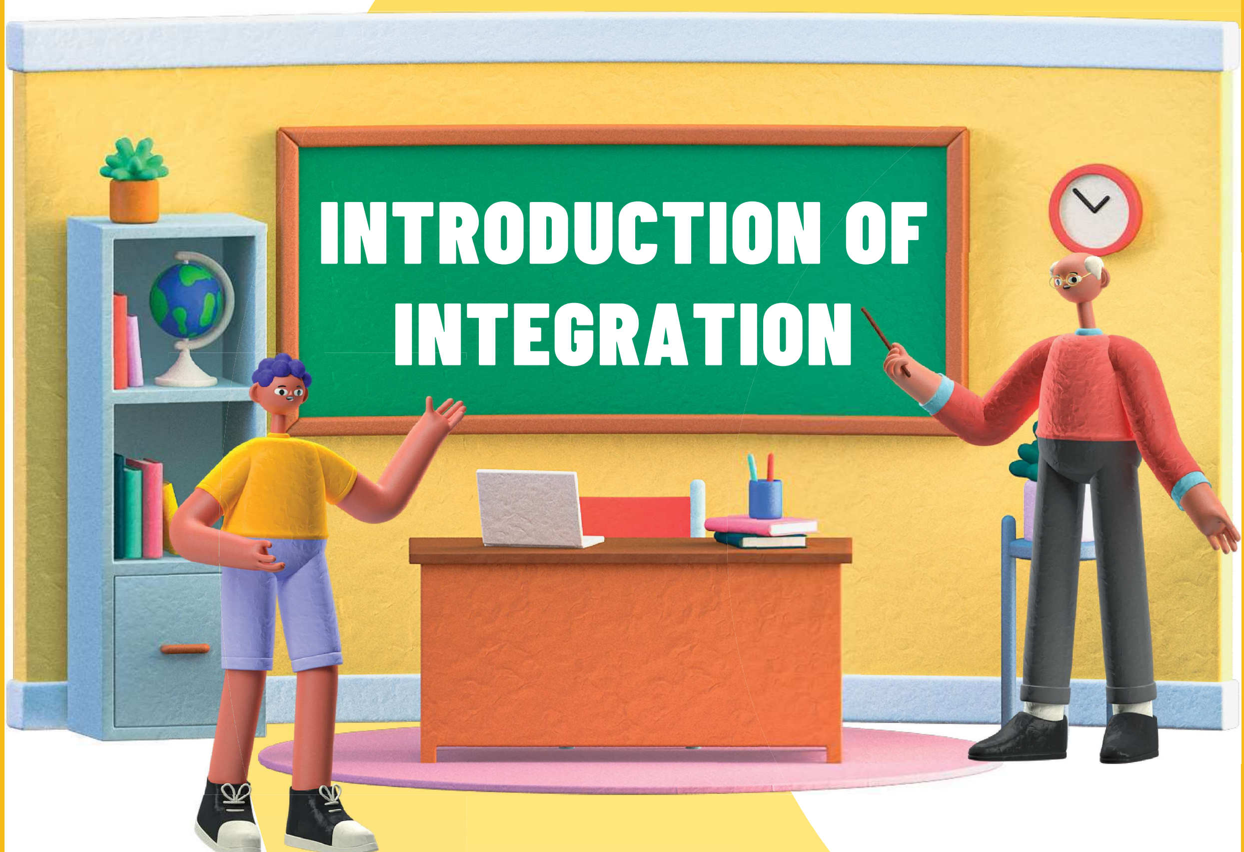

AISHAH BINTI MAHAT

INTRODUCTION OF INTEGRATION

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DISCLAIMER

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PREFACE

This e-book, Introduction of Integration aimed to help students in Calculus subject. Targeted users for this module is students who take Calculus course. This e-book is divided into four subtopics which includes Integration by Part, Integration of Trigonometric Functions, Integration of Substitutions, and Integration of Rational Functions. All the four topics above will be completed in accordance to lesson planned. Mathematical tips and formulas will be placed in accordance to the subtopics whilst each calculus questions will be displayed based on the syllabus carried out during the lesson. At the end of each topic, targeted students should meet up with the lecturer to discuss over the solution of Calculus problem. With the existence of this e-book, hopefully it will be beneficial and give positive impact towards teaching and learning for students and lecturers as a whole.

NOTES

CHAPTER 1

METHODS OF INTEGRATION

- INTEGRATION BY PARTS

$\int u\,dv = uv - \int v\,du$

LIATE

L -In/log	I-Inverse	A-Algebra	T-trigonomet	E-Exponent

- TRIGONOMETRIC FUNCTIONS

SINE ODD	COSINE ODD	SIN/COS EVEN
Split to $\sin^{m-1} x$ and $\sin x$ $\sin^2 x = 1 - \cos^2 x$ $u = \cos x$	Split to $\cos^{m-1} x$ and $\cos x$ $\cos^2 x = 1 - \sin^2 x$ $u = \sin x$	$\sin^2 x = \frac{1}{2}(1 - \cos 2x)$@ $\cos^2 x = \frac{1}{2}(1 + \cos 2x)$

TAN ODD	SEC EVEN
Split to $\sec x \tan x$ $\tan^2 x = \sec^2 x - 1$ $u = \sec x$	Split to $\sec^{m-2} x$ $\sec^2 x = \tan^2 x + 1$ $u = \tan x$

- TRIGONOMETRIC SUBSTITUTION

EXPRESSION	SUBSTITUTION	RESULT
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$	$a\cos\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$	$a\sec\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$	$a\tan\theta$

CHAPTER 1

METHODS OF INTEGRATION

- RATIONAL FUNCTIONS

Completing the square (when $g(x)$ cannot be factorized)

$$ax^2+bx+c=a\left[\left(x+\frac{b}{2a}\right)^2-\left(\frac{b}{2a}\right)^2+\frac{c}{a}\right]$$

$$\int\frac{1}{u^2+a^2}du=\frac{1}{a}\tan^{-1}\frac{u}{a}+C$$

$$\int\frac{1}{\sqrt{a^2-u^2}}du=\sin^{-1}\frac{u}{a}+C$$

Partial functions (when $g(x)$ can be factorized)

Distinct linear factors : $\dfrac{f(x)}{(Ax+B)(Cx+D)}=\dfrac{a}{Ax+B}+\dfrac{b}{Cx+D}$

Repeated linear factors : $\dfrac{f(x)}{(Ax+B)^n}=\dfrac{a}{Ax+B}+\dfrac{b}{(Ax+B)^2}+\ldots+\dfrac{n}{(Ax+B)^n}$

Irreducible factors : $\dfrac{f(x)}{(Ax^2+B)(Cx^2+D)(Ex^2+Fx+G)}=\dfrac{ax+b}{Ax^2+B}+\dfrac{cx+d}{Cx^2+D}+\dfrac{ex+f}{Ex^2+Fx+G}$

G(x) cannot be factorized, f(x) is not a constant : $\dfrac{f(x)}{g(x)}=\dfrac{A[g'(x)]}{g(x)}+B$

INTEGRATION BY PART

INTEGRATION BY PART

1.

$$\int 2x\sin(2-x)\,dx$$

$$u = 2x \qquad dv = \sin(2-x)\,dx$$

$$du = 2\,dx \qquad v = \frac{-\cos(2-x)}{-1} + c$$

$$= \cos(2-x)$$

$$\int udv = uv - \int v\,du$$

$$= 2x\big(\cos(2-x)\big) - \int \cos(2-x)\,.\,2\,dx$$

$$= 2x\cos(2-x) - 2\int \cos(2-x)\;dx$$

$$= 2x\cos(2-x) - 2\left[\frac{\sin(2-x)}{-1}\right] + c$$

$$= 2x\cos(2-x) + 2\sin(2-x) + c$$

2.

$$\int \sin^{-1} 4x\,dx$$

$$u = \sin^{-1} 4x \qquad dv = dx$$

$$du = \frac{4}{\sqrt{1-16x^2}} \qquad v = x + c$$

$$\int udv = uv - \int v\,du$$

$$= \sin^{-1} 4x\,.\,x - \int x\,.\,\frac{4}{\sqrt{1-16x^2}}\,dx$$

let u $= 1-16x^2$

$$\frac{du}{dx} = -32x$$

$$= x\sin^{-1} 4x - \int \frac{4}{\sqrt{u}}\,.\,\frac{du}{-32x}$$

$$= x\sin^{-1} 4x + \int \frac{1}{8\sqrt{u}}\,du$$

$$= x\sin^{-1} 4x + \frac{1}{8}\int u^{-\frac{1}{2}}\,du$$

$$= x\sin^{-1} 4x + \frac{1}{8}\left[\frac{u^{-\frac{1}{2}}}{\frac{1}{2}}\right] + c$$

$$= x\sin^{-1} 4x + \frac{1}{4}\sqrt{1-16x^2} + c$$

INTEGRATION BY PART

3.

$$\int e^x \sin 3x\, dx$$

$u = \sin 3x \qquad dv = e^x\, dx$

$du = 3\cos 3x\, dx \qquad v = e^x + c$

$$\int udv = uv - \int v\, du$$

$$= \sin 3x \,.\, e^x - \int e^x \,.\, 3\cos 3x\, dx$$

$$= \sin 3x\, e^x - \int 3e^x \cos 3x\, dx$$

$u = \cos 3x \qquad dv = 3e^x\, dx$

$du = -3\sin 3x\, dx \qquad v = 3e^x + c$

$$\int udv = uv - \int v\, du$$

$$= \sin 3x \,.\, e^x - \left[\cos 3x \,.3e^x - \int 3e^x \,.\, -3\sin 3x\, dx\right]$$

$$= \sin 3xe^x - \left[3e^x \cos 3x + \int 9e^x \sin 3x\, dx\right]$$

$$\int e^x \sin 3x\, dx = \sin 3xe^x - 3e^x\cos 3x - \int 9e^x \sin 3x\, dx$$

$$10 \int e^x \sin 3x\, dx = \sin 3xe^x - 3e^x \cos 3x$$

$$\int e^x \sin 3x\, dx = \frac{\sin 3xe^x - 3e^x\cos 3x}{10} + c$$

4.

$$\int \ln(3x-1)\, dx$$

$u = \ln(3x-1) \qquad dv = dx$

$du = \dfrac{3}{3x-1} \qquad v = x + c$

$$\int udv = uv - \int v\, du$$

$$= \ln(3x-1)\,.\,x - \int x\,.\frac{3}{3x-1}dx$$

$$= x\ln(3x-1) - \int \frac{3x}{3x-1}dx$$

By long division:

$$\begin{array}{r} 1 \\ 3x-1\,\overline{)\quad 3x} \\ (-)\quad 3x-1 \\ \hline 1 \end{array}$$

$$= 1 + \frac{1}{3x-1}$$

$$= x\ln(3x-1) - \left[1 + \frac{1}{3x-1}dx\right]$$

$$= x\ln(3x-1) - \left[x + \frac{1}{3}\ln(3x-1)\right] + c$$

$$= x\ln(3x-1) - x - \frac{1}{3}\ln(3x-1) + c$$

INTEGRATION BY TRIGONOMETRIC FUNCTIONS

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INTEGRATION BY TRIGONOMETRIC FUNCTIONS

1.

$$\sin^5 x\,dx = \int \sin^4 x \,.\, \sin x\,dx$$

$$= \int (\sin^2 x)(\sin^2 x) \,.\, \sin x\,dx$$

let $u = \cos x$

$$= \int (1-\cos^2 x)(1-\cos^2 x) \,.\, \sin x \,.\, \frac{du}{-\sin x}$$

$$\frac{du}{dx} = -\sin x$$

$$= -\int (1-u^2)(1-u^2)\,du$$

$$= -\int 1 - 2u^2 + u^4\,du$$

$$= -u + \frac{2u^3}{3} - \frac{u^5}{5} + c$$

$$= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + c$$

2.

$$\int \sin^2 2x \cos^3 2x\,dx$$

$$\int \sin^2 2x \,.\, \cos^2 2x \,\cos 2x\,dx$$

let $u = \sin 2x$

$$\int \sin^2 2x\,(1-\sin^2 2x) \,.\, \cos 2x\,dx$$

$$\frac{du}{dx} = 2\cos 2x\,dx$$

$$\int u^2(1-u^2) \,.\, \cos 2x \,.\, \frac{du}{2\cos 2x}$$

$$\frac{1}{2}\int u^2(1-u^2)\,du$$

$$\frac{1}{2}\int u^2 - u^4\,du$$

$$\frac{1}{2}\left[\frac{u^3}{3} - \frac{u^5}{5}\right] + c$$

$$\frac{1}{6}\sin^3 2x - \frac{1}{10}\sin^5 2x + c$$

INTEGRATION BY TRIGONOMETRIC FUNCTIONS

3.

$$\int \sin^2 2x \cos^2 2x \, dx$$

$$\int \frac{1}{2}(1-\cos 4x) \cdot \frac{1}{2}(1+\cos 4x)\, dx$$

$$\frac{1}{4}\int 1-\cos^2 4x \, dx$$

$$\frac{1}{4}\int 1-\frac{1}{2}[1+\cos 8x]\, dx$$

$$\frac{1}{4}\int 1-\frac{1}{2}-\frac{1}{2}\cos 8x \, dx$$

$$\frac{1}{4}\int \frac{1}{2}-\frac{1}{2}\cos 8x \, dx$$

$$\frac{1}{4}\left[\frac{1}{2}x-\frac{1}{2}\frac{\sin 8x}{8}\right]+c$$

$$\frac{1}{8}x-\frac{1}{64}\sin 8x+c$$

4.

$$\int \tan^5 x \sec x \, dx$$

$$\int \tan^4 x \tan x \sec x \, dx$$

$$\int (\tan^2 x)(\tan^2 x) \tan x \sec x \, dx \qquad u=\sec x$$

$$\int (\sec^2 x-1)\ (\sec^2 x-1)\ \tan x \sec x \, dx \qquad du=\sec x \tan x \, dx$$

$$\int (u^2-1)\ (u^2-1)\ \tan x \sec x \frac{du}{\sec x \tan x}$$

$$\int u^4-u^2-u^2+1 \, du$$

$$\int u^4-2u^2+1 \, du$$

$$=\frac{u^5}{5}-\frac{2u^3}{3}+u+c$$

$$=\frac{\sec^5 x}{5}-\frac{2}{3}\sec^3 x+\sec x+c$$

INTEGRATION BY TRIGONOMETRIC SUBSTITUTIONS

INTEGRATION BY TRIGONOMETRIC SUBSTITUTIONS

1. |

$$\int \frac{x}{\sqrt{x^2+16}}\,dx$$

$x = 4\tan\theta$

$\frac{dx}{d\theta} = 4\sec^2\theta$

$$\int \frac{4\tan\theta}{\sqrt{(4\tan\theta)^2+16}} . 4\sec^2\theta\,d\theta$$

$$\int \frac{4\tan\theta}{\sqrt{16\tan^2\theta+16}} . 4\sec^2\theta\,d\theta$$

$$\int \frac{4\ \tan\theta}{\sqrt{16\,(\tan^2\theta+1)}} . 4\sec^2\theta\,d\theta$$

$$\int \frac{4\ \tan\theta}{4\sqrt{\sec^2\theta}} . 4\sec^2\theta\,d\theta$$

$$\int \frac{4\ \tan\theta}{\sec\theta} . \sec^2\theta\,d\theta$$

$$\int 4\ \tan\theta\sec\theta\,d\theta$$

$$4\int \frac{\sin\theta}{\cos\theta} . \frac{1}{\cos\theta}\,d\theta$$

$$4\int \frac{\sin\theta}{\cos^2\theta}\,d\theta$$

$$4\int \frac{\sin\theta}{u^2} . \frac{du}{-\sin\theta}$$

$$-4\int u^{-2}\,du$$

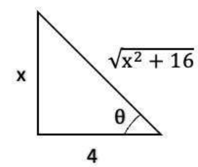
$$= -4\frac{u^{-1}}{-1} + c$$

$$= \frac{4}{u} + c$$

$$= \frac{4}{\cos\theta} + c$$

$$= \frac{4}{\frac{4}{\sqrt{x^2+16}}} + c$$

$$= \sqrt{x^2+16+c}$$



$x = 4\tan\theta$

$4\tan\theta = x$

$\tan\theta = \frac{x}{4}$

$\cos\theta = \frac{4}{\sqrt{x^2+16}}$

$\text{let } u = \cos\theta$

$du = -\sin\theta$

2.

$$\int \sqrt{9-x^2}\,dx$$

$x = a\sin\theta$

$x = 3\sin\theta$

$\frac{dx}{d\theta} = 3\cos\theta$

$$\int \sqrt{9-(3\sin\theta)^2} . 3\cos\theta\ d\theta$$

$$\int \sqrt{9-9\sin^2\theta} . 3\cos\theta\ d\theta$$

$$\int \sqrt{9(1-\sin^2\theta)} . 3\cos\theta\ d\theta$$

$$\int 3\sqrt{\cos^2\theta} . 3\cos\theta\ d\theta$$

$$9\int \cos^2\theta\ d\theta$$

$$9\int \frac{1}{2}(1+\cos 2\theta)\,d\theta$$

$$\frac{9}{2}\int (1+\cos 2\theta)\ d\theta$$

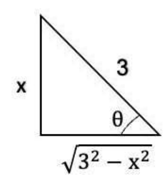
$$= \frac{9}{2}\left[\theta + \frac{\sin 2\theta}{2}\right] + c$$

$$= \frac{9}{2}\theta + \frac{9}{4}\sin 2\theta + c$$

$$= \frac{9}{2}\theta + \frac{9}{4}\ 2\sin\theta\cos\theta + c$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} + \frac{9}{2}\left(\frac{x}{3}\right)\frac{\sqrt{9-x^2}}{3} + c$$

$$= \frac{9}{2}\sin^{-1}\frac{x}{3} + \frac{x}{2}\sqrt{9-x^2} + c$$



$3\sin\theta = x$

$\sin\theta = \frac{x}{3}$

$\cos\theta$

$= \frac{\sqrt{9-x^2}}{3}$

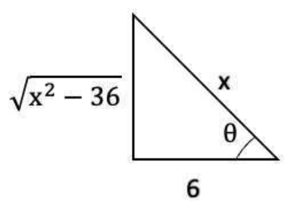
INTEGRATION BY TRIGONOMETRIC SUBSTITUTIONS

3.

$$\int \frac{\sqrt{x^2 - 36}}{x}\, dx$$

$x = a \sec\theta$
$x = 6 \sec\theta$
$dx = 6 \sec\theta \tan\theta\, d\theta$

$$\int \frac{\sqrt{(6\sec\theta)^2 - 36}}{6\sec\theta} . 6\sec\theta\tan\theta\, d\theta$$
$$\int \frac{\sqrt{36\sec^2\theta - 36}}{6\sec\theta} . 6\sec\theta\tan\theta\, d\theta$$
$$\int \sqrt{36(\sec^2\theta - 1)}\, .\tan\theta\, d\theta$$
$$\int 6\sqrt{\tan^2\theta}\, .\tan\theta\, d\theta$$
$$\int 6\tan^2\theta\, d\theta$$



$$6\int \tan^2\theta\, d\theta$$
$$6\int \sec^2\theta - 1\, d\theta$$
$$= 6(\tan\theta - \theta)$$
$$= 6\tan\theta - 6\theta + c$$
$$= 6\left[\frac{\sqrt{x^2 - 36}}{6}\right] - 6\sec^{-1}\frac{x}{6} + c$$

$6\sec\theta = x$

$\sec\theta = \frac{x}{6}$

$\frac{1}{\cos\theta} = \frac{x}{6}$

$x\cos\theta = 6$

$\cos\theta = \frac{6}{x}$

INTEGRATION BY RATIONAL FUNCTIONS

INTEGRATION OF RATIONAL FUNCTIONS

1.

$$\int \frac{16x-8}{(x^2-4)(x+5)}\,dx = \int \frac{16x-8}{(x-2)(x+2)(x+5)}$$

$$\frac{16x-8}{(x-2)(x+2)(x+5)} = \frac{A}{x-2} + \frac{B}{x+2} + \frac{C}{x+5}[(x-2)(x+2)(x+5)]$$

$$16x-8 = A(x+2)(x+5) + B(x-2)(x+5) + C(x-2)(x+2)$$

$$\begin{aligned} \text{when } x-2 &= 0 \\ x &= 2 \\ 16(2)-8 &= A(4)(7) \\ 24 &= 28\,A \\ 28\,A &= 24 \\ A &= \frac{24}{28} \\ &= \frac{6}{7} \end{aligned}$$

$$\begin{aligned} \text{when } x+2 &= 0 \\ x &= -2 \\ 16(-2)-8 &= 0 + B(-4)(3) \\ -40 &= -12\,B \\ -12\,B &= -40 \\ B &= \frac{-40}{-12} \\ &= \frac{10}{3} \end{aligned}$$

$$\begin{aligned} \text{when } x-5 &= 0 \\ x &= 5 \\ 16(5)-8 &= 0 + 0 + C(-7)(-3) \\ -88 &= 21\,C \\ 21\,C &= -88 \\ C &= \frac{-88}{21} \end{aligned}$$

Hence,

$$\int \frac{16x-8}{(x-2)(x+2)(x+5)}\,dx = \int \frac{\frac{6}{7}}{x-2} + \frac{\frac{10}{3}}{x+2} - \frac{\frac{88}{21}}{x+5}\,dx$$

$$= \frac{6}{7}\ln|x-2| + \frac{10}{3}\ln|x+2| - \frac{88}{21}\ln|x+5| + c$$

INTEGRATION OF RATIONAL FUNCTIONS

2.

$$\int \frac{5x+7}{x(x-1)^2}\,dx$$

$$\frac{5x+7}{x(x-1)^2} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{(x-1)^2}[x(x-1)^2]$$
$$5x+7 = A(x-1)^2 + B(x)(x-1) + C(x)$$

when $x = 0$
$$5(0)+7 = A(-1)^2$$
$$7 = A$$

when $x = 1$
$$5(1)+7 = 7(1-1)^2 + B(1)(1-1) + C(1)$$
$$12 = C$$

when $x = 2$
$$5(2)+7 = 7(1)^2 + B(2)(1) + 12(2)$$
$$17 = 7 + 2B + 24$$
$$2B = -14$$
$$B = -7$$

$$\int \frac{5x+7}{x(x-1)^2}\,dx = \int \frac{7}{x} - \frac{7}{x-1} + \frac{12}{(x-1)^2}\,dx$$

let $u = x - 1$

$$\frac{du}{dx} = 1$$

$$\int \frac{12}{u^2}\,du$$
$$= \int 12u^{-2}\,du$$
$$= \frac{12u^{-1}}{-1} + c$$
$$= \frac{12}{-u} + c$$
$$= \frac{-12}{x-1} + c$$
$$= 7\ln x - 7\ln|x-1| - \frac{12}{x-1} + c$$

INTEGRATION OF RATIONAL FUNCTIONS

3.

$$\int \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)}\,dx$$

$$\frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)} = \frac{A}{x-1} + \frac{B}{(x-1)^2} + \frac{Cx + D}{x^2+1}\ [(x-1)^2(x^2+1)]$$

$$x^2 - 2x - 1 = A(x-1)(x^2+1) + B(x^2+1) + (Cx+D)(x-1)^2$$

$$x^2 - 2x - 1 = (Ax - A)(x^2+1) + Bx^2 + B + (Cx+D)(x-1)(x-1)$$

$$= Ax^3 + Ax - Ax^2 - A + Bx^2 + B + (Cx+D)(x^2 - 2x + 1)$$

$$= Ax^3 + Ax - Ax^2 - A + Bx^2 + B + Cx^3 - 2Cx^2 + Cx + Dx^2$$

$$- 2Dx + D$$

$$Ax^3 + Cx = 0x^3$$

$$(A + C)x^3 = 0x^3$$

$$A + C = 0 \longrightarrow 1$$

$$-Ax^2 + Bx^2 - 2Cx^2 + Dx^2 = x^2$$

$$(-A + B - 2C + D)x^2 = 1x^2$$

$$-A + B - 2C + D = 1 \longrightarrow 2$$

$$-A + B + D = -1 \longrightarrow 4$$

$$Ax + Cx - 2Dx = -2x$$

$$(A + C - 2D)x = -2x$$

$$A + C - 2D = -2 \longrightarrow 3$$

substitute $\boxed{1}$ into $\boxed{3}$

$$A + C - 2D = -2$$

$$0 - 2D = -2$$

$$D = 1$$

substitute $\boxed{C = -A}$ into $\boxed{2}$

$$-A + B - 2(-A) + D = 1$$

$$-A + B + 2A + D = 1$$

$$A + B + D = 1 \longrightarrow 5$$

INTEGRATION OF RATIONAL FUNCTIONS

substitute $\boxed{D = 1}$ into $\boxed{5}$ and $\boxed{4}$

$$A + B + 1 = 1$$
$$A + B = 0 \longrightarrow 6$$

$$-A + B + 1 = -1$$
$$-A + B = -2 \longrightarrow 7$$

$\boxed{6} - \boxed{7}$

$$2A = 2$$
$$A = 1$$

substitute $\boxed{A = 1}$ into $\boxed{7}$

$$-A + B = -2$$
$$-1 + B = -2$$
$$B = -2 + 1$$
$$B = -1$$

substitute $\boxed{A = 1}$ into $\boxed{C = -A}$

$$C = -1$$

$$\int \frac{x^2 - 2x - 1}{(x-1)^2(x^2+1)}\, dx = \int \frac{1}{x-1} - \frac{1}{(x-1)^2} - \frac{x+1}{x^2+1}\, dx$$
$$= \ln|x-1| - \left(\frac{-1}{x-1}\right) - \int \frac{-x}{x^2+1} + \frac{1}{x^2+1}\, dx$$
$$= \ln|x-1| + \frac{1}{x-1} - \left[\left(-\frac{1}{2}\ln|x^2+1|\right) + 1\tan^{-1}x\right] + c$$
$$= \ln|x-1| + \frac{1}{x-1} + \frac{1}{2}\ln|x^2+1| - \tan^{-1}x + c$$

INTEGRATION OF RATIONAL FUNCTIONS

let $u = x - 1$

$$\frac{du}{dx} = 1$$

$$du = dx$$

$$\int \frac{1}{u^2}\, dx$$

$$\int \frac{1}{u^2}\, du$$

$$\int u^{-2}\, du$$

$$= \frac{u^{-1}}{-1} + c$$

$$= -\frac{1}{u} + c$$

$$= -\frac{1}{x-1} + c$$

let $u = x^2 + 1$

$$\frac{du}{dx} = 2x$$

$$dx = \frac{du}{2x}$$

$$\int \frac{-x}{u} \cdot \frac{du}{2x}$$

$$-\frac{1}{2}\int \frac{1}{u}\, du$$

$$= -\frac{1}{2}\ln u + c$$

$$= -\frac{1}{2}\ln|x^2 + 1| + c$$

$$\frac{1}{x^2+1} = \frac{1}{u^2+a^2}$$

$x^2 = u^2$ $\quad$ $1^2 = a^2$

$x = u$ $\quad$ $1 = a$

$$\int \frac{1}{x^2+1}\, dx = \int \frac{1}{u^2+a^2}\, du$$

$$= \frac{1}{a}\tan^{-1}\frac{u}{a} + c$$

$$= 1\tan^{-1} x + c$$

INTEGRATION OF RATIONAL FUNCTIONS

4.

$$\int \frac{2}{16x^2 + 25}\, dx$$

$$u^2 + a^2 = 16x^2 + 25$$

$$u^2 = 16x^2 \qquad a^2 = 25$$

$$(u)^2 = (4x)^2 \qquad a^2 = 5^2$$

$$u = 4x \qquad a = 5$$

$$\frac{du}{dx} = 4$$

$$dx = \frac{du}{4}$$

$$\int \frac{1}{16x^2 + 25} dx$$

$$2\int \frac{1}{u^2 + a^2} \frac{du}{4}$$

$$= \frac{2}{4}\left[\frac{1}{a}\tan^{-1}\frac{u}{a}\right] + c$$

$$= \frac{1}{2}\left[\frac{1}{5}\tan^{-1}\frac{4x}{5}\right] + c$$

$$= \frac{1}{10}\tan^{-1}\frac{4x}{5} + c$$

INTEGRATION OF RATIONAL FUNCTIONS

5.

$$\int \frac{7}{4x^2 - 4x + 3}\,dx$$

By Applying Completing The Square

$$4x^2 - 4x + 3$$
$$4\left(x^2 - x + \frac{3}{4}\right)$$
$$4\left[\left(x + \frac{-1}{2}\right)^2 - \left(\frac{-1}{2}\right)^2 + \frac{3}{4}\right]$$
$$4\left[\left(x - \frac{1}{2}\right)^2 + \frac{2}{4}\right]$$
$$4\left[\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}\right]$$

$$7\int \frac{1}{4x^2 - 4x + 3}\,dx$$

$$\frac{7}{4}\int \frac{1}{\left(x - \frac{1}{2}\right)^2 + \frac{1}{2}}\,dx \qquad u^2 = \left(x - \frac{1}{2}\right)^2 \qquad a^2 = \frac{1}{2}$$

$$\frac{7}{4}\int \frac{1}{u^2 + a^2}\,du \qquad u = \left(x - \frac{1}{2}\right) \qquad a = \sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$$

$$\frac{7}{4}\left[\frac{1}{a}\tan^{-1}\frac{u}{a}\right] + c$$

$$\frac{7}{4}\left[\frac{1}{\frac{1}{\sqrt{2}}}\tan^{-1}\frac{\left(x - \frac{1}{2}\right)}{\frac{1}{\sqrt{2}}}\right] + c$$

$$\frac{7}{4}\left[\sqrt{2}\tan^{-1}\sqrt{2}\left(x - \frac{1}{2}\right)\right] + c$$

TUTORIALS

1. $\int 6x \ln 2x \, dx$

Ans: $3x^2 \ln 2x - \frac{3}{2}x^2 + c$

2. Find $\int (1 + \cos 4x)^2 \, dx$

Ans: $\frac{3}{2}x + \frac{\cos 4x}{2} + \frac{\sin 8x}{16} + c$

3. Find $\int x^3\sqrt{25 - x^2} \, dx$ **by using the trigonometric substitution.**

Ans: $3125\left[\frac{\left(\sqrt{25 - x^2}\right)^3}{15} - \frac{\left(\sqrt{25 - x^2}\right)^4}{20}\right] + c$

4. Make a substitution of u= $\cos x$ **and then find** $\int \frac{\sin x}{\cos^2 x + \cos x - 2} \, dx$

Ans: $-\frac{1}{3}\ln|\cos x - 1| + \frac{1}{3}\ln|\cos x + 2| + c$

5. $\int \sin^{-1} 4x \, dx$ **by using integration by parts.**

Ans: $x \sin^{-1} 4x + \frac{1}{4}\sqrt{1 - 16x^2} + c$

6. Find $\int_0^1 \cos^3\theta \sin^4\theta \, d\theta$

Ans: 0.0417

7. By a suitable trigonometric substitution, evaluate $\int \frac{x}{\sqrt{x^2+9}} dx$

Ans: $\sqrt{x^2+9} + c$

8. If $\frac{3x+6}{x^3-3x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x-3}$ **, find values of A, B and C.**

Ans: A= $-\frac{5}{3}$, B= -2 , C= $\frac{5}{3}$

9. $\int_0^7 5x \sec^2 x \, dx$

Ans: 29.08822

10. Find $\int (\sin 3x - \cos 2x)(\sin x + \cos x) \, dx$

Ans: $\frac{1}{2}\left[\frac{\sin 2x}{2} - \frac{\sin 4x}{4} - \frac{\cos 2x}{2} - \frac{\cos 4x}{4} + \cos x + \frac{\cos 3x}{3} - \sin x - \frac{\sin 3x}{3}\right] + c$

11. Evaluate $\int \frac{4x}{\sqrt{x^2-4}}\,dx$ by using the trigonometric substitution, $x = 2\sec\theta$

Ans: $4\sqrt{x^2-4}+c$

12. $\int x^2 e^{-3}\ dx$ by using integration by parts.

Ans: $\frac{x^2e^{-3x}}{-3} - \frac{2xe^{-3x}}{9} - \frac{2e^{-3x}}{27} + c$

13. Find $\int \tan^3 x\ \sec^3 x\,dx$

Ans: $\frac{\sec^5 x}{5} - \frac{\sec^3 x}{3} + c$

14. By using integration by parts and trigonometric substitution, determine $\int x^2 \cos^{-1} x\ dx$

Ans: $\frac{x^3\cos^{-1}x}{3} - \frac{\sqrt{1-x^2}}{3} + \frac{\left(\sqrt{1-x^2}\right)^3}{9} + c$

15. Find $\int \frac{14}{9x^2+25}\,dx$

Ans: $\frac{14}{15}\tan^{-1}\frac{3x}{5} + c$

16. $\int x^2 \tan^{-1} x \; dx$ **by using integration by parts.**

Ans: $\frac{x^3 \tan^{-1} x}{3} - \frac{x^2}{6} + \frac{1}{6} \ln|1 + x^2| + c$

17. **Prove that** $\int_0^3 \tan\theta \; \sec^4\theta \; d\theta = 0.01026$

18. **Find** $\int \frac{x}{\sqrt{36 + x^2}} dx$ **by using trigonometric substitution,** $x = 6\tan\theta$

Ans: $\sqrt{36 + x^2} + c$

19. **Find** $\int \frac{12x^2 - 6x + 1}{x^2 + x^4} dx$

Ans: $-6\ln|x| - \frac{1}{x} + 3\ln|1 + x^2| + 11\tan^{-1} x + c$

20. **Determine** $\int \frac{5t + 8}{t^2 + 2t + 2} dt$

Ans: $\frac{5}{2} \ln|t^2 + 2t + 2| + 3\tan^{-1}(t + 1) + c$

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