Conversion of Coordinate System (Circular Area)

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Introduction

The Cartesian coordinate system provides a straightforward way to describe the location of points in space. Some surfaces, however, can be difficult to model with equations based on the Cartesian system for example, cases in two-dimensional Polar Coordinates often offers as a useful alternative system for describing the location of a point in the plane, particularly in cases involving circles. Alternatively, for three-dimensional cases, extensions of polar coordinates are used, namely by the Cylindrical Coordinates and Spherical Coordinates.

Cylindrical Coordinates

When we expanded the traditional Cartesian coordinate system from two dimensions to three, we simply added a new axis to model the third dimension. Starting with polar coordinates, we can follow this same process to create a new three-dimensional coordinate system, called the cylindrical coordinate system. In this way, cylindrical coordinates provide a natural extension of polar coordinates to three dimensions.

Definition: The Cylindrical Coordinate System In the cylindrical coordinate system, a point in space (Figure 1) is represented by the ordered triple (r, θ, z) are the polar coordinates of the point's projection in the *xy*-plane

- z is the usual z-coordinate in the Cartesian coordinate system

Figure 1: The right triangle lies in the xy-plane. The length of the hypotenuse is r and θ is the measure of the angle formed by the positive *x*-axis and the hypotenuse. The z-coordinate describes the location of the point above or below the *xy*-plane. In the *xy*-plane, the right triangle shown in Figure 1 provides the key to transformation between cylindrical and Cartesian, or rectangular, coordinates.

Conversion between Polar / Cylindrical and Cartesian Coordinates: The rectangular coordinates (x, y, z) and the cylindrical coordinates (r, θ, z) were given by Isaac Newton in year 1670 (Vladimir Rovenski, 1999) of a point are related as follows:

These equations are used to convert from polar/cylindrical coordinates to rectangular coordinates.

- $x = rcos\theta$
- $y = rsin\theta$
- $z = z$

These equations are used to convert from rectangular coordinates to cylindrical coordinates

1. $r^2 = x^2 + y^2$
2. $\tan\theta = \frac{y}{x}$
3. $z = z$

As when we discussed conversion from rectangular coordinates to polar coordinates in two dimensions, it should be noted that the equation $tan\theta = \frac{y}{x}$ has an infinite number of solutions. However, if we restrict θ to values between 0 and 2π, then we can find a unique solution based on the quadrant of the *xy*-plane in which original point (x, y, z) is located. Note that if $x = 0$, then the value of θ is either $\frac{\pi}{2}$, $\frac{3\pi}{2}$, or 0 depending on the value of *y*.

Notice that these equations are derived from properties of right triangles. To make this easy to see, consider point *P* in the *xy*-plane with rectangular coordinates $(x, y, 0)$ and with cylindrical coordinates $(r, \theta, 0)$, as shown in Figure 1

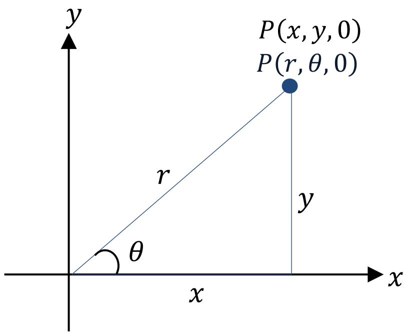


Figure 1: The Pythagorean theorem provides equation $r^2 = x^2 + y^2$.
Right-triangle relationships tell us that $x = rcos\theta$, $y = rsin\theta$, and $\tan\theta = \frac{y}{x}$.

Let's consider the differences between rectangular and cylindrical coordinates by looking at the surfaces generated when each of the coordinates is held constant. If c is a constant, then in rectangular coordinates, surfaces of the form $x=c$,$y=c$, or $z=c$ are all planes. Planes of these forms are parallel to the *yz*-plane, the *xz*-plane, and the *xy*-plane, respectively. When we convert to cylindrical coordinates, the *z*-coordinate does not change.

Therefore, in cylindrical coordinates, surfaces of the form $z=c$ are planes parallel to the *xy*-plane. Now, let's think about surfaces of the form $r=c$. The points on these surfaces are at a fixed distance from the *z*-axis. In other words, these surfaces are vertical circular cylinders. Last, what about $\theta=c$? The points on a surface of the form $\theta=c$ are at a fixed angle from the *x*-axis, which gives us a half-plane that starts at the *z*-axis (Figures 3 and 4).

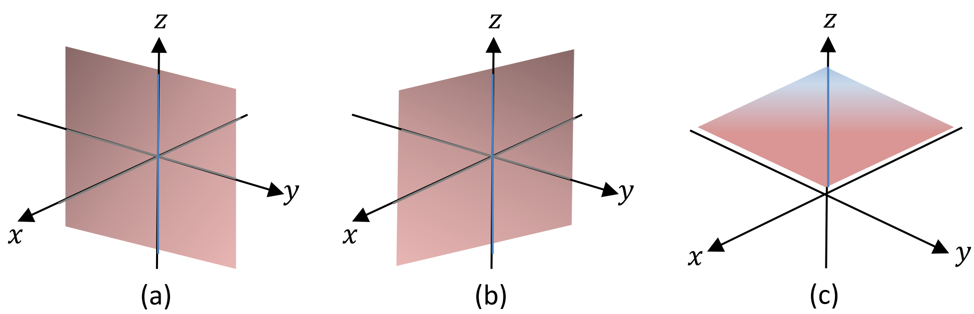


Figure 3: In rectangular coordinates, (a) surfaces of the form $x=c$ are planes parallel to the *yz*-plane, (b) surfaces of the form $y=c$ are planes parallel to the *xz*-plane, and (c) surfaces of the form $z=c$ are planes parallel to the *xy*-plane.

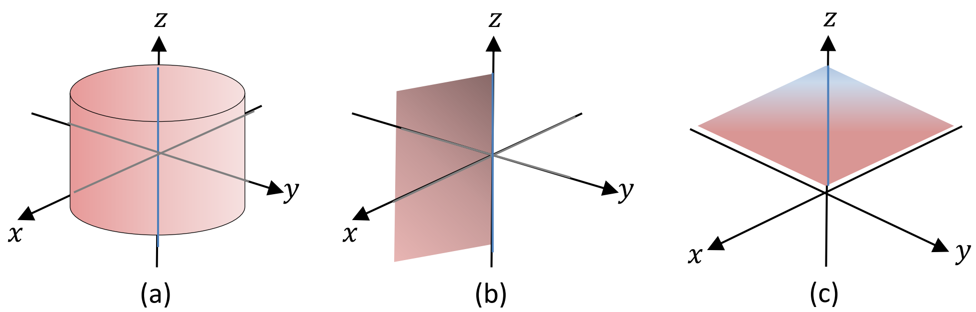


Figure 4: In cylindrical coordinates, (a) surfaces of the form *r=c* are vertical cylinders of radius r, (b) surfaces of the form θ=c are half-planes at angle θ from the x-axis, and (c) surfaces of the form z=c are planes parallel to the xy-plane.

Spherical Coordinates

In the Cartesian coordinate system, the location of a point in space is described using an ordered triple in which each coordinate represents a distance. In the cylindrical coordinate system, location of a point in space is described using two distances (*r* and *z*) and an angle measure (θ). In the spherical coordinate system, we again use an ordered triple to describe the location of a point in space. In this case, the triple describes one distance and two angles. Spherical coordinates make it simple to describe a sphere, just as cylindrical coordinates make it easy to describe a cylinder. Grid lines for spherical coordinates are based on angle measures, like those for polar coordinates.

Definition: Spherical Coordinate System In the spherical coordinate system, a point *P* in space (Figure 5) is represented by the ordered triple (ρ, θ, φ) where

- ρ (the Greek letter rho) is the distance between P and the origin $(\rho \neq 0)$;
- θ is the same angle used to describe the location in cylindrical coordinates;
- φ (the Greek letter phi) is the angle formed by the positive *z*-axis and line segment $\overrightarrow{OP}$, where *O* is the origin and $0 \leq \varphi \leq \pi$.

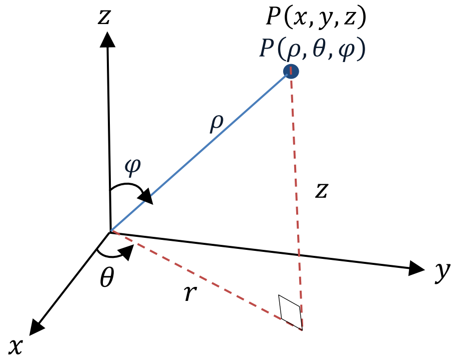


Figure 5: The relationship among spherical, rectangular, and cylindrical coordinates. By convention, the origin is represented as $(0,0,0)$ in spherical coordinates.

Conversions among Rectangular, Cylindrical / Polar and Spherical Coordinates: Rectangular coordinates (x, y, z), cylindrical or polar coordinates (r, θ, z) or $(r, \theta, 0)$ and spherical coordinates (ρ, θ, φ) of a point are related as follows:

Convert from spherical coordinates to rectangular coordinates

These equations are used to convert from spherical coordinates to rectangular coordinates.

- $x = \rho sin\varphi cos\theta$
- $y = \rho sin\varphi sin\theta$
- $z = \rho cos\varphi$

Convert from rectangular coordinates to spherical coordinates

These equations are used to convert from rectangular coordinates to spherical coordinates.

- $\rho^2 = x^2 + y^2 + z^2$
- $\tan\theta = \frac{y}{x}$
- $\varphi = cos^{-1}\left(\frac{z}{\sqrt{x^2+y^2+z^2}}\right)$.

Convert from spherical coordinates to cylindrical coordinates

These equations are used to convert from spherical coordinates to cylindrical coordinates.

- $r = \rho sin\varphi$
- $\theta = \theta$
- $z = \rho cos\varphi$

Convert from cylindrical coordinates to spherical coordinates

These equations are used to convert from cylindrical coordinates to spherical coordinates.

- $\rho = \sqrt{r^2 + z^2}$
- $\theta = \theta$
- $\varphi = cos^{-1}\left(\frac{z}{\sqrt{r^2+z^2}}\right)$

The formulas to convert from spherical coordinates to rectangular coordinates may seem complex, but they are straightforward applications of trigonometry. Looking at Figure, it is easy to see that $r = \rho sin\varphi$. Then, looking at the triangle in the *xy*-plane with r as its hypotenuse, we have $x = rcos\theta = \rho sin\varphi cos\theta$. The derivation of the formula for *y* is similar. Figure 6 also shows that $\rho^2 = r^2 + z^2 = x^2 + y^2 + z^2$ and $z = \rho cos\varphi$. Solving this last equation for φ and then substituting $\rho = \sqrt{r^2 + z^2}$ (from the first equation) yields $\varphi = cos^{-1}\left(\frac{z}{\sqrt{r^2+z^2}}\right)$. Also, note that, as before, we must be careful when using the formula $\tan\theta = \frac{y}{x}$ to choose the correct value of θ.

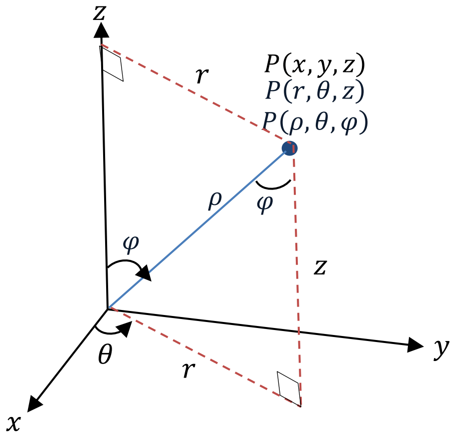


Figure 6: The equations that convert from one system to another are derived from right-triangle relationships.

As we did with cylindrical coordinates, let's consider the surfaces that are generated when each of the coordinates is held constant. Let c be a constant, and consider surfaces of the form $\rho=c$. Points on these surfaces are at a fixed distance from the origin and form a sphere. The coordinate θ in the spherical coordinate system is the same as in the cylindrical coordinate system, so surfaces of the form $\theta=c$ are half-planes, as before. Last, consider surfaces of the form $\varphi=0$. The points on these surfaces are at a fixed angle from the z-axis and form a half-cone (Figure 7).

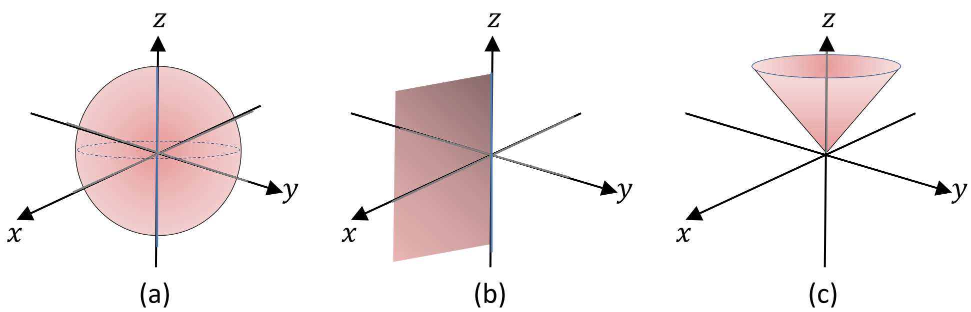
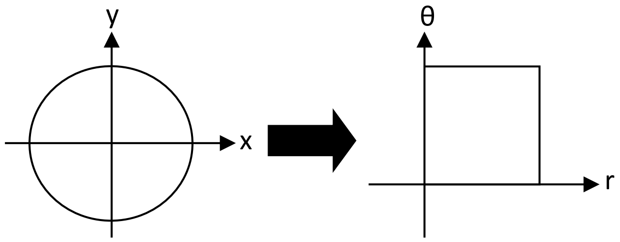
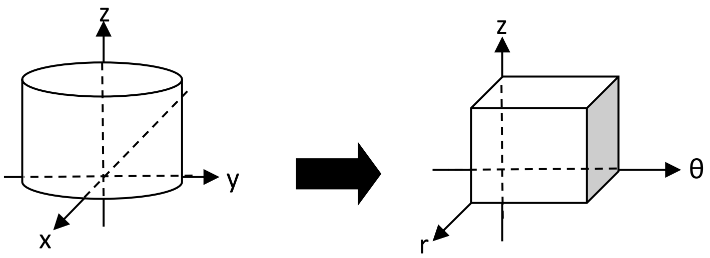
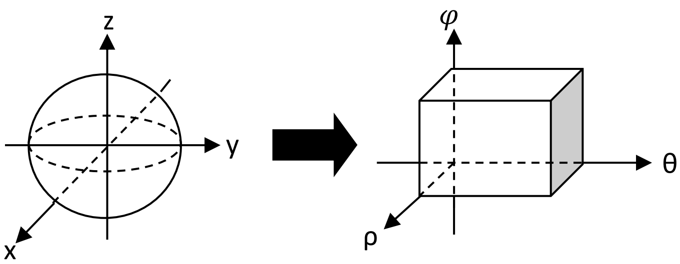


Figure 7: In spherical coordinates, surfaces of the form $\rho=c$ are spheres of radius ρ (a), surfaces of the form $\theta=c$ are half-planes at an angle θ from the x-axis (b), and surfaces of the form $\phi=c$ are half-cones at an angle ϕ from the z-axis (c).

Table 1 graphically describes the conversion of rectangular or Cartesian coordinate system to polar, cylindrical and spherical coordinate for the circular area.

Table 1. Conversion of Coordinate System.

Polar Coordinate (2-D) $x = rcos\theta$ $y = rsin\theta$	
Cylindrical Coordinate (3-D) $x = rcos\theta$ $y = rsin\theta$ $z = z$	
Spherical Coordinate (3-D) $x = \rho sin\varphi cos\theta$ $y = \rho sin\varphi sin\theta$ $z = \rho cos\varphi$	

Cylindrical coordinates are convenient for representing cylindrical surface and surface of revolution, which the z-axis is the axis of symmetry. Spherical coordinates are desirable when representing spheres, cones, or certain planes (Bradley & Smith, 1999). Figure 8 shows the relations between Cartesian, Cylindrical, and Spherical coordinates.

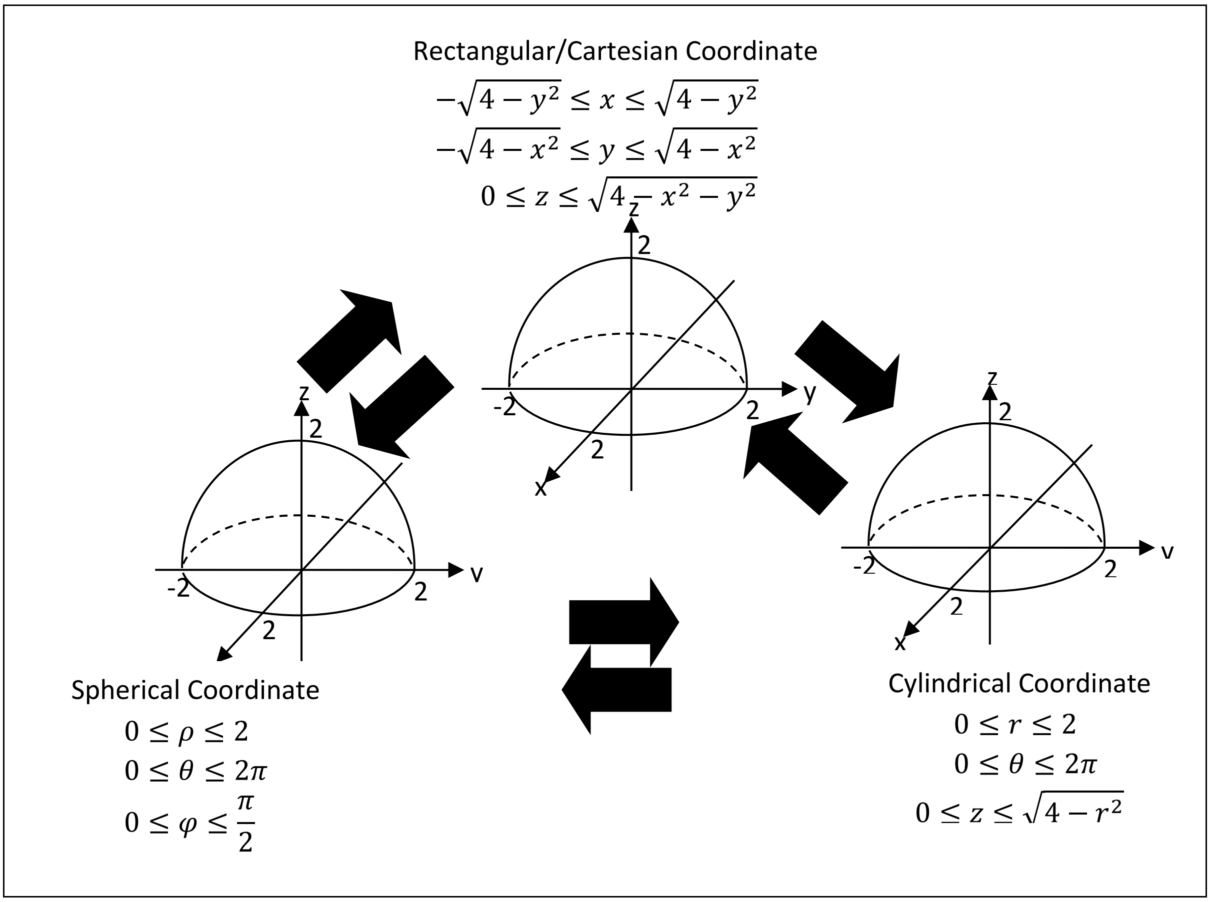


Figure 8: Relations between Cartesian, Cylindrical, and Spherical Coordinates.

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