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# COMPARATIVE STUDY OF NONLINEAR ROOT FINDING USING IMPROVISED SECANT METHODS

**Nurul Nabilah Rosli[1*], Nor Azni Shahari[2], Farah Atikah Mohamad Azraei[3], Siti Najwa Izaham[4]**

[1]*College of Computing, Informatics and Media, Universiti Teknologi MARA Shah Alam Selangor*

[2,3,4]*College of Computing, Informatics and Media, Universiti Teknologi MARA, Cawangan Negeri Sembilan, Kampus Seremban, Malaysia*

[1*]2022765939@student uitm.edu.my, [2]norazni@uitm.edu.my, [3]atikahazraei99@gmail.com, [4]snajwa669@gmail.com

## ABSTRACT

*The nonlinear transcendental or algebraic equation problem is one of the important research areas in numerical analysis, and the iterative methods are playing an important role to find approximate solutions. The Secant method is one of the best iterative methods since it only requires a single evaluation of function. However, the Secant method has low convergence order, thus many improvised Secant methods have been developed by other researchers. Even though improvise secant method has been developed vastly, comparative study of these methods is relatively scarce, and the novelty of this paper is to assess critical numerical performances of the methods. Therefore, in this study, two algorithms based on the Secant method which are the exponential method, and three-point Secant method were used to compare with the Secant method to evaluate the roots for nonlinear equations. The three methods were tested using different initial values in various transcendental functions such as polynomial, exponential, logarithm, trigonometric and some combinations of linear, exponential, polynomial, and trigonometric functions to determine the best method among three methods and to determine the behavior of these method. All the computation results were developed using Graphical User Interface (GUI) in MATLAB environment to get the results and as the visual indicator representations. The obtained results show that the three-point Secant method has the least number of iterations than the Secant method and exponential method in six numerical results. Conclusively, the three-point Secant method is the best iterative method since the method converged to the roots faster than other two.*

**Keywords** *: Nonlinear Equations, Numerical Analysis, Root Finding, Secant Method.*

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## 1. Introduction

Numerical methods have been widely used to find simple or multiple zeros from transcendental equations. The process of determining roots for nonlinear transcendental functions is engaged in many various fields such as mathematics, fuzzy systems, and mechanical and chemical engineering. However, the analytical solution for finding the exact roots of such nonlinear equations is very challenging, hence the iterative methods based on the numerical analysis is used (Thota & Srivastav, 2014).

There are many well-known existing methods in approximating roots for algebraic or transcendental equations such as the Bisection method, Newton-Raphson method, Secant method, False position (Regular Falsi) method and Muller's method (Azure *et al.*, 2019). These methods can be classified into two types of methods. Firstly, there are closed domain methods such as Bisection method and False position. Secondly, the open domain methods rely on a formula that requires one or two initial guesses that does not necessarily form the actual basis (Intep, 2018). In some cases, the open domain methods may diverge from the root as the iteration progresses. Some known open methods are the Newton Raphson method, Secant method and Muller's method. These methods use information about the nonlinear function itself to complete the estimates of the root, and that is why they are said to be more efficient than closed domain methods.

Since many numerical methods have developed, most comparative studies have been performed specially to find out the best among the well-known methods to solve the problems of finding the root. Example of such research are by Azure *et al.* (2019), Ebelechukwu *et al.* (2018), and Moheuddin *et al.* (2019), investigating the effectiveness of Newton Raphson method, Bisection method and Secant method based on the rate of performances. From the results obtained, it was observed that Newton's method is a robust method since it converges faster to the solution of the function compared to the other methods. These findings contradict the findings of some authors who put Secant method predates Newton Raphson method in efficiency requirements. Ehiwario & Aghamie (2014), Tasiu *et al.* (2020), and Yasir AbdulHassan (2016) had carried out a same comparative study and the results showed that the Secant method converges the least iterations to the root thus, it provenly can be said that Secant method is the most effective method of solving the nonlinear equation. These results correspond to the fact that the Secant method converges close to Newton Raphson method without the need of derivative of function (Gemechu, 2016). In general, the Secant method was used in this study instead of Newton Raphson method because Newton's formula requires more time to evaluate the derivative function which makes it difficult to compute the roots. The Secant method also does not require use of the derivative of the function, something that is not available in several applications.

The iterative process in the Secant method proves the approximate root development towards the real value, which effectively improves the calculation speed. Nevertheless, the Secant method converges slower than Newton method therefore, to solve these problems, many researchers have improvised the Secant method to find a new algorithm with better rate of convergence. Recently, the modification of secant method has been proposed (Thota, 2019; Thukral, 2018, 2020; Tiruneh *et al.*, 2013, 2019). However, none of the above makes comparisons in terms of behavior and shows how these methods work with different types of function. Furthermore, choosing an initial approximation for iterative methods is dependent on information's of the function and finding the right initial approximation is crucial to choose the best iterative methods. Thus, this study will determine nonzero real roots for solving some transcendental equations using the improvised Secant methods which are the exponential (Thota, 2019) and three-point Secant method (Tiruneh *et al.*, 2019). Both methods will be compared with the existing Secant method by testing and analyses in some selected test equations that have single and multiple roots. The effect of initial approximation values in determining the multiple roots of an equation was also tested.

## 2. Methodology

### 2.1 Mathematical Formulation of selected Secant and Improvise Secant Method

The Secant formula is given in equation (1) below:

$$x_{n+1} = x_n - \frac{f(x_n)\left(x_n - x_{n-1}\right)}{f(x_n) - f(x_{n-1})}, \quad n = 1,2,3,.... \tag{1}$$

Clearly $x_{n+1}$ in Secant method refers to two prior sequence elements. So as a start, two points such as $x_0$ and $x_1$ must be provided.

The new iterative formula using exponential series is shown below, for any two initials approximations $x_0$ and $x_1$ of the root. The exponential method (Thota, 2019) based on improvise secant method formula is given in equation (2):

$$x_{n+1} = x_n \exp\left(\frac{x_{n-1}f(x_n) - x_n f(x_n)}{x_n f(x_n) - x_n f(x_{n-1})}\right). \quad n = 1,2,3,.... \tag{2}$$

The three-point Secant method (Tiruneh *et al.*, 2019) formula is:

$$x_{n+1} = x_{n-2} - \frac{y_{n-2}\left(y_n - y_{n-1}\right)}{\left(\frac{y_n - y_{n-2}}{x_n - x_{n-2}}\right)\left(y_n - y_{n-1}\right) - y_n\left[\left(\frac{y_n - y_{n-2}}{x_n - x_{n-2}}\right) - \left(\frac{y_{n-1} - y_{n-2}}{x_{n-1} - x_{n-2}}\right)\right]}. \tag{3}$$

where $y_n = f(x_n),\ y_{n-1} = f(x_{n-1}),\ y_{n-2} = f(x_{n-2}).$

Equation (3) will be used for the iteration process to evaluate the next point of iteration from the previous most consecutive points of the iteration for *n*=2, 3, 4,…..

### 2.2 Select Suitable Function

There are six test equations that were selected to substitutes into the Secant method, exponential method and three-point Secant method to determine the root. This process needs to stop when the $x_n$ starts to converge to specific tolerance. For this study, all the results used $10^{-9}$ of tolerance. Table 1 shows the expansion of test functions used in this study with the respective roots. All the computation roots of test functions for the three methods are presented by Graphical User Interface (GUI) in MATLAB.

Table 1. Test Functions and the Respective Roots.

| No. | Test functions | Roots | References |
|---|---|---|---|
| 1 | $x^3 + 4x^2 - 10$ | 1.36523001 | (Thukral, 2020) |
| 2 | $e^{-x} - x - \sin(x)$ | 0.354463104 | (Suhadolnik, 2012) |
| 3 | $x - 3\ln(x)$ | 1.85718386 and 4.53640366 | (Tiruneh *et al.*, 2019) |
| 4 | $\sin^2(x) - x^2 + 1$ | 1.40449165 and -1.40449165 | |
| 5 | $e^{-x} + \cos(x)$ | 1.74613953, 4.70332376, 7.85436969, 10.9955575, … | (Özyapici *et al.*, 2016) |
| 6 | $e^{(x^2+7x-30)} - 1$ | 3.00000000 and -10.00000000 | (Hui *et al.*, 2009) |

For this study, all the three formulated methods are compared in various ways which are number of iterations, relative error, and elapsed time. Furthermore, the initial approximations are selected randomly depending on the root to determine the behaviors of each function.

### 2.3 Developed the Graphical User Interface (GUI)

The comparisons of the Secant method, exponential method and three-point Secant methods were collected from GUI that have been developed. To determine the results, the user must follow the steps given. Firstly, the user needs to run the MATLAB and given like Figure 1. The user must provide the function to be evaluated, followed by the initial points and also the tolerance desired. Then, the user may select whether the approach will be done using the Secant method, exponential method or three-point Secant method and the answers will be automatically displayed at results section. The interface calculates the maximum number of iterations for the processing, alongside with the approximate root. On top of that, the interface also shows the graph for the function entered. In command window, the estimated root for every iteration was shown for each selected method. However, the result may not be displayed if there are some errors identified. The most common problems may happen to the user is the filled function incorrect or cannot be read by MATLAB. Plus, the iterations also may display as Nan which defined as not a number due to initial points filled not suitable for the function entered. Then, the comparisons were characterized by the number of roots for respective functions.

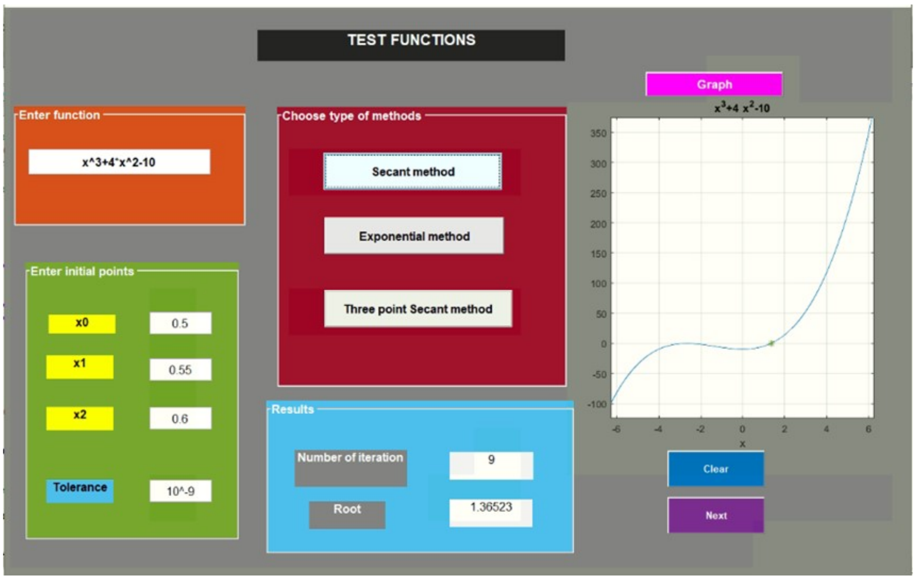


Figure 1. Simulation GUI MATLAB for Test Equations.

## 3. Results and Discussions

The three methods which are Secant method, exponential method developed by Thota, 2019 and three-point Secant method (Tiruneh *et al.*, 2019) were compared by six test equations (Table 1). All the computation roots and number of iterations for the three methods are determined using MATLAB, Graphical User Interface (GUI) that has been developed. Furthermore, all the examples of test functions were also illustrated in graph to show the root(s) for every function. The Secant method and exponential method only need two initial approximations, $x_0$ and $x_1$ while three initial approximations $x_0, x_1$ and $x_2$ for the three-point Secant method. The roots for all the test functions are correct to accuracy 9 significant digits.

### 3.1 Testing A Function with One Root

For the first comparison, the function is a polynomial, $x^3+4x^2-10$ and the initial approximations used are closer to the root which are $x_0=1.20,\ x_1=1.30$ and $x_2=1.40$. Based on the Table 2, the root for $x^3+4x^2-10$ is 1.36523001. To confirm the root, the graph for $x^3+4x^2-10$ was illustrated as shown in Figure 2a. All the three methods converge to the exact root, 1.36523001 which results in both Secant method and exponential method converge at 4th iteration, but the three-point Secant method only need 3 iterations.

Table 2. Comparison of Different Methods with Different Initial Values for Two Numerical Test Functions.

| Function | Root | Initial points | Number of iterations required | | |
|---|---|---|---|---|---|
| | | | Secant method | Exponential method | Three points Secant method |
| $x^3+4x^2-10$ | 1.36523001 | 1.20, 1.30, 1.40 | 4 | 4 | 3 |
| $e^{-x}-x-\sin(x)$ | 0.354463104 | -12.0, -12.5, -13.0 | 22 | 24 | 12 |

For the second comparison, the combination function of exponential, linear and trigonometric was selected which is $e^{-x}-x-\sin(x)$, with initial approximations $x_0=-12.0,\ x_1=-12.5$ and $x_2=-13.0$ which are far from the exact root, 0.354463104 (Table 2). The Three-point Secant method converged to the root at 12th iteration, but both the Secant method and exponential method are still far away from the exact root. Furthermore, the Secant method converges to the root, 0.354463104 at 22nd iteration while the exponential method converges to the same root at 24th iteration. In comparison, Thukral, 2020 proposed a new three-point Secant-type methods and two of the methods which is three point secant method and new three point secant type method have higher order convergence than the Secant method, 1.84 and 1.80, which indicates the both three point secant type method converge faster than the Secant method. This result is equivalent with findings on Table 2, where number of iterations for three-point Secant method is less than Secant method and exponential method thus, it can be said that the three-point Secant converge faster than the other two methods.

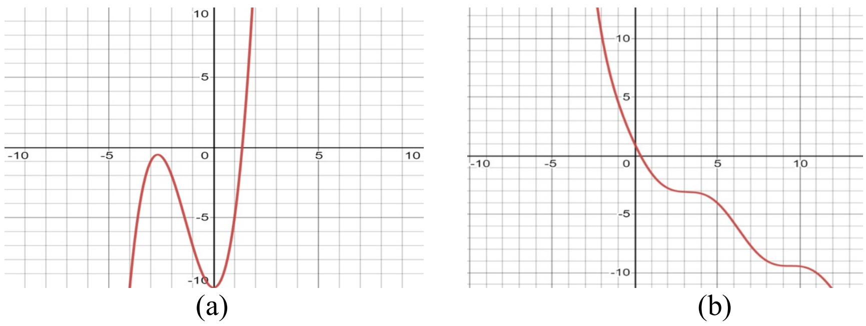


Figure 2. Graph for (a) $x^3+4x^2-10$ and (b) $e^{-x}-x-\sin(x)$ .

### 3.2 Testing Function with Two Positive Roots

Figure 3 shows the graph for the function, $x-3\ln(x)$. The exact roots for this logarithm function are 1.85718386 and 4.53640366 as shown in Figure 3. This function was tested with three different positive initial points to find the root of the function by using the Secant method, exponential method, and three-point Secant method since the exact roots are also positive. The results were illustrated in Table 3.

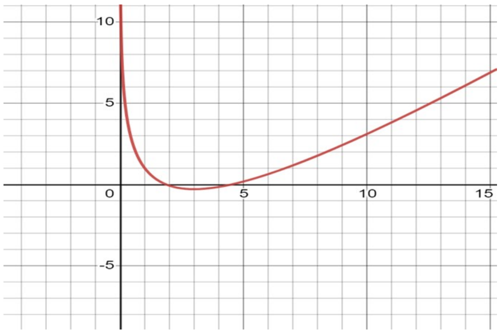


Figure 3. Graph for $x-3\ln(x)$.

Table 3. Comparison of Different Methods with Different Initial Values for $x-3\ln(x)$.

| **Initial approximations** | | | **Number of iterations required** | | |
|---|---|---|---|---|---|
| $x_0$ | $x_1$ | $x_2$ | **Secant** | **Exponential** | **Three-point Secant** |
| 1.30 | 1.50 | 1.75 | 7 | 7 | 5 |
| 2.00 | 2.05 | 2.10 | 6 | 6 | 5 |
| 10.0 | 10.5 | 11.0 | 8 | 9 | 7 |

Based on Table 3, it can be observed that for first starting points, $x_0=1.30,\ x_1=1.50$ and $x_2=1.75,$ the three-point Secant method need least number of iterations which are 5 to converge to the exact root of 1.85718386 while both Secant method and exponential method need 7 number of iterations. In addition, when the initial approximations used much closer to the root, 1.85718386 which are $x_0=2.00,\ x_1=2.05$ and $x_2=2.10,$ Secant method and exponential method are influenced in number of iterations where both methods converge at similar iterations but decreased from the first initial results which is 6 while the number of iterations for three-point Secant method remains unchanged. This is in line with Tiruneh *et al.*, (2019) that make a comparison for the same function and the results presented that when using initial points that closed to the exact value, the three-point Secant method will converge to the root at the smallest number of iterations compared to the Newton method and Secant method. However, when the initial approximations selected far from the first root, which are $x_0=10.0,\ x_1=10.5$ and $x_2=11.0,$ all the three methods converge to the second exact root, 4.53640366. The number of iterations for the three-point Secant method have increased to 7 iterations followed by the Secant method and exponential method converges at 8$^{th}$ and 9$^{th}$ iteration, respectively. These significant results show that if the initial point is far from the exact value, the number of iterations will increase when applied to this logarithm function and otherwise.

Figure 4 illustrated a comparison of relative error to find root of the function for $x-3\ln(x)$ using $x_0=10.0$, $x_1=10.5$ and $x_2=11.0$. By analyzing graph from Figure 4, the three-point Secant method decreases drastically from approximately error, 1.123 to 0.104 and subsequently stops at six iterations when error approaches to zero. For the Secant method and exponential method, both Secant method and exponential method still took longer time to descend to zero approximately 7 and 8 iterations respectively to converge at the root, 4.53640366. These results surely can be concluded as the three-point Secant method better than exponential method and Secant method since three-point Secant method need least iterations to get exact root even the starting relative error quite larger than other methods. It also can be proved by comparisons of error from Wang *et al.*, (2010) where the new proposed method which is also namely as three-point Secant method had the least error compared to Secant method and Zhang, Li and Liu method (ZLLM). Therefore, from Figure 4 below, it can be concluded that the three-point Secant method is the best method in comparison for error with the Secant and exponential method.

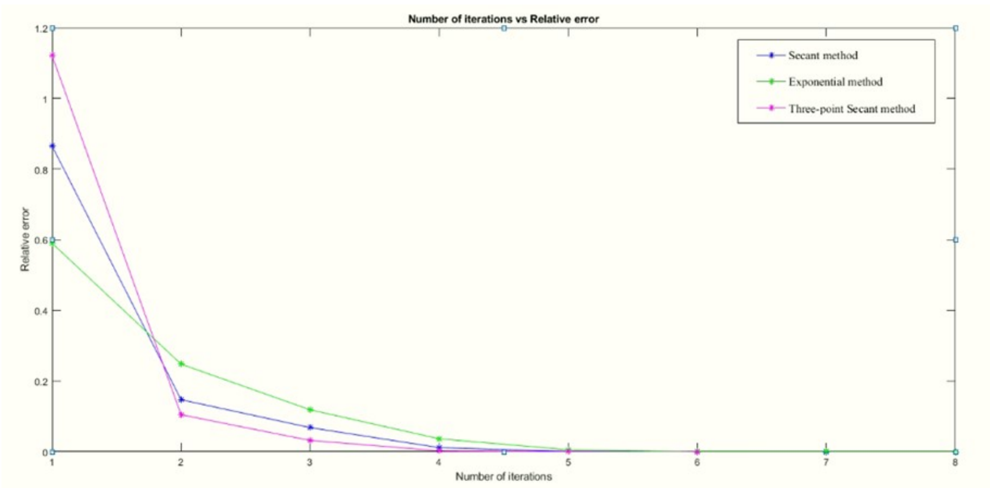


Figure 4. Graph Relative Error for $x-3\ln(x)$ .

### 3.3 Testing the Function with Positive and Negative Roots

In this simulation, there are two types of functions that will be discussed. Firstly, a trigonometric function, $\sin^2(x)-x^2+1$ and the exponential function, . $e^{(x^2+7x-30)}-1$.

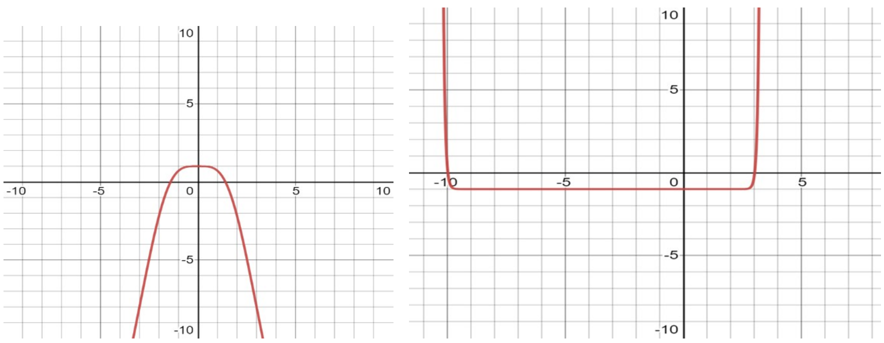

Figure 5. Graph for $\sin^2(x)-x^2+1$ and $e^{(x^2+7x-30)}-1$.

Based on Figure 5, the trigonometric function graph has two x-intercepts which equal to exact roots, 1.40449165 and -1.40449165. The results for this function are computed in Table 4. From Table 4, the first initial approximations tested are $x_0 = -1.60,\ x_1 = -1.70$ and $x_2 = -1.80$. The results showed that all the methods converge at the negative roots, -1.40449165. The number of iterations for the three-point Secant method is the fastest among other methods since it converges at 5th iteration, but the Secant method need 6 iterations and followed by the exponential method converges as the slowest method with 7 iterations. These results are similar with the findings from Tiruneh *et al.*, (2019). They have tested the same functions by using negative initial approximations. When $x_0 = -1.0,\ x_1 = -0.075$ and $x_2 = -0.95$ are used, both the three-point Secant method and Secant method converged to the root -1.40449165 and three-point Secant method need less iterations which is 8 than the Secant method (10 iterations).

Table 4. Comparison of Different Methods with Different Initial Values for $\sin^2(x) - x^2 + 1$.

| **Initial approximations** | | | **Number of iterations required** | | | **Convergence of exact roots** |
|---|---|---|---|---|---|---|
| $x_0$ | $x_1$ | $x_2$ | **Secant** | **Exponential** | **Three-point Secant** | |
| -1.60 | -1.70 | -1.80 | 6 | 7 | 5 | -1.40449165 |
| 1.60 | 1.70 | 1.80 | 6 | 7 | 5 | 1.40449165 |
| 13.1 | 13.2 | 13.3 | 11 | 13 | 9 | 1.40449165 |

However, if the positive initial values, $x_0 = 1.60,\ x_1 = 1.70$ and $x_2 = 1.80$ used, the three-point Secant method, Secant method, and exponential converged to the positive root, 1.40449165. Meanwhile, the number of iterations for all the three methods remains unchanged. Furthermore, if the positive initial points far from the root continually selected, such as $x_0 = 13.1,\ x_1 = 13.2$ and $x_2 = 13.3$, the three methods also converged to 1.40449165 but need a greater number of iterations than the second initial values. The three-point Secant method needs the smallest iterations which are 9, Secant method with 11 iterations, and exponential method converges at 13th iteration.

Table 5. Comparison of Different Methods with Different Initial Values for $e^{(x^2+7x-30)} - 1$.

| **Initial approximations** | | | **Number of iterations required** | | | **Convergence of exact roots** |
|---|---|---|---|---|---|---|
| $x_0$ | $x_1$ | $x_2$ | **Secant method** | **Exponential method** | **Three-point Secant method** | |
| -10.50 | -10.30 | -10.00 | 13 | 13 | 1 | -10.0 |
| 3.80 | 3.90 | 4.00 | 23 | 24 | 16 | 3.00 |
| 10.00 | 10.30 | 10.50 | 211 | 212 | 135 | 3.00 |
| -5 | -6 | -7 | fail | fail | 33 | -10.00 |

For the second testing for $e^{(x^2+7x-30)} - 1$ also has positive and negative roots, similar with the trigonometric function (Figure 5). However, the exact roots for this exponential function are 3.00 and -10.00. Table 5 shows that the three-point Secant method method takes only one number of iteration to converge at the negative root, -10.00 when using $x_0 = -10.50,\ x_1 = -10.30$ and $x_2 = -10.00$ as initial points. For the Secant method and exponential method, both methods need the same number of iterations, which are 13. Then, the comparisons continued by using the positive initial values that close to the exact roots, 3.00 which are $x_0 = 3.8,\ x_1 = 3.9$ and $x_2 = 4.0$. The three-point Secant method converge to the root with less iterations which are 16, meanwhile, the Secant method stay as the second-best method which takes 23 and the exponential method as the slowest method where it takes 24 number of iterations.

For the third comparisons, the positive initial points were selected that far from positive root which are $x_0 = 10.00,\ x_1 = 10.30$ and $x_2 = 10.50$, the number of iterations drastically increase for Secant method, exponential method and three-point Secant method results in 211, 212 and 135, correspondingly. These results indicated that if the positive initial approximations used, the three methods would converge to positive root since the distance between them closer than negative root and otherwise, when applied to $\sin^2(x) - x^2 + 1$ and $e^{(x^2+7x-30)} - 1$. Testing with initial approximation $x_0 = -5,\ x_1 = -6$ and $x_2 = -7$ show that secant method and exponent method fail to converge to the root, however three-point method can converge to the root at 33 iterations. Based on Tu *et al.*, (2013), iteration or convergence failures is the difference between a fully converged solution of a finite number of discrete points and a solution that has not entirely reached convergence. According to Aboamemeh et.al, (2021), secant method needs the initial approximation $x_0$ and $x_1$ are at the endpoints of the interval that contains the root of the function. In these testing all the $x_0$ and $x_1$ are not at the endpoint of the interval that contains the root and secant method only fail on case 4 ($x_0 = -5,\ x_1 = -6$ ), however three-point secant method can overcome the weaknesses of secant method.

The average of elapsed time is one of essential metrics to evaluate an algorithm. (Badr *et al.*, 2021). However, since the values of time elapsed always changeable every time the MATLAB was run, the average of elapsed time for 5 times were computed in confirmation of the best method among the Secant method, exponential method and three-point Secant method. The comparison was continued using elapsed time recorded in MATLAB is $e^{(x^2+7x-30)} - 1$ in Table 6.

Table 6. Comparison the Average of Elapsed Times of Different Methods with Different Initial Values for $e^{(x^2+7x-30)} - 1$.

| **Initial approximations** | | | **Average elapsed time (seconds)** | | |
|---|---|---|---|---|---|
| $x_0$ | $x_1$ | $x_2$ | **Secant** | **Exponential** | **Three-point Secant** |
| 3.10 | 3.20 | 3.30 | 0.00214 | 0.003104 | 0.003419 |
| 6.40 | 6.45 | 6.50 | 0.004808 | 0.006711 | 0.005765 |
| -11.5 | -11.3 | -11.0 | 0.002765 | 0.003465 | 0.00348 |
| -18.0 | -17.5 | -17.0 | 0.008764 | 0.0111 | 0.009445 |

The results indicated that when using both positive and negative initial points, mostly the time elapsed for the Secant method recorded as the fastest compared to the exponential and three-point Secant methods since the formula for the Secant method simpler compared to exponential method and three-point Secant method. These finding are verified to the comparison study results from Badr *et al.*, (2021) showed that the average Central Processing Unit (CPU) time for Secant method is the fastest when compared to Newton's method, Bisection method, False Position method and many more methods.I

### 3.4 Testing a function with infinite roots

If a function repeats itself at frequent intervals, it is referred to periodic function Therefore, function $e^{-x}+\cos(x)$ said to be periodic function. From Figure 6, the function only shows the intersection at positive x-axis and has more than two roots since it is a periodic function. From figure 6, the function curve contains so many minimum and maximum points, where the value of the function slope is either zero or very small. Thus, several initial approximations were being tested to some of the roots with number of iterations required for the three methods to converge as illustrated in Table 7.

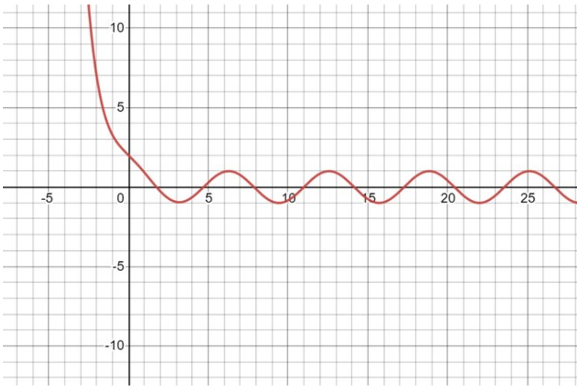


Figure 6. Graph for $e^{-x}+\cos(x)$ .

Table 7 shows the comparisons of different methods to converge at the exact roots while in the bracket displays comparisons of the three methods in number of iterations. Both comparisons are made for a combination function and using the same 11 initial values. Based on Figure 6 it can be seen that is a periodic function thus, the convergence of the three methods depends on how close the initial estimations with any roots.

Table 7. Comparisons of the Convergence Exact Roots (Number of Iterations) for Different Methods using Different Initial Values for $e^{-x}+\cos(x)$

| **Initial approximations** | | | **The convergence of different methods to respective roots (Number of iterations)** | | |
|---|---|---|---|---|---|
| $x_0$ | $x_1$ | $x_2$ | **Secant** | **Exponential** | **Three-point Secant** |
| -1.9 | -1.8 | -1.7 | 1.74613953(8) | **Fail** | 1.74613953(8) |
| 1.6 | 1.7 | 1.8 | 1.74613953(4) | 1.74613953(5) | 1.74613953(4) |
| 2.00 | 2.25 | 2.50 | 1.74613953(6) | 1.74613953(6) | 1.74613953(4) |
| 3.00 | 3.5 | 4.0 | **Diverge** | 158.650429(7) | 4.70332376(7) |
| 3.00 | 4.00 | 5.00 | 4.70332376(7) | 4.70332376(9) | 4.70332376(5) |
| 4.69 | 4.70 | 4.71 | 4.70332376(3) | 4.70332376(4) | 4.70332376(2) |
| 5.0 | 5.5 | 6.0 | 4.70332376(5) | 4.70332376(5) | 4.70332376(4) |
| 6 | 6.1 | 6.2 | 1.74613953(7) | 4.70332376(8) | 4.70332376(9) |
| 7.90 | 8.0 | 8.1 | 7.85436969(4) | 7.85436969(4) | 7.85436969(3) |
| 8.3 | 8.4 | 8.5 | 7.85436969(5) | 7.85436969(5) | 7.85436969(5) |
| 9.8 | 9.9 | 10 | 10.9955575(6) | 10.9955575(7) | 10.9955575(7) |
| 10.7 | 10.8 | 10.9 | 10.9955575(4) | 10.9955575(5) | 10.9955575(4) |

Based on Table 7, the Secant method, exponential method and three-point Secant method will converge to the first root, 1.74613953 since the first, second and third testing initial values are closer to that first root rather than the second root, 4.70332376. According to Aboamemah *et al.*, (2021), the initial point that have been chosen are at the endpoints of the interval that achieves Intermediate Value theorem(IVT) condition will converge to the nearest root, however this is not always true for secant method which is an open domain method. In addition, only exponential method fails to converge to the first root when using the negative initial approximation ( $x_0 = -1.9,\ x_1 = -1.8$ and $x_2 = -1.7$ ). This is due to the negative initial values is not suitable for the exponential formula to get the respective root of this function, $e^{-x} + \cos(x)$ Nevertheless, for the initial $x_0 = 3.0,\ x_1 = 3.5$ and $x_2 = 4.0$ , the Secant method diverge, and exponential method converge to root of 158.650429 which is the root that is far away from the initial approximation point while three-point Secant method converge to the second root which is 4.70332376. However, when the initial roots are $x_0 = 3.0,\ x_1 = 4.0$ and $x_2 = 5.0,$ all the three methods converge to root of 4.70332376. Divergent series are infinite series that are not convergent, for example, the infinite sequence of partial sums of the series has no finite limit. This is happened because of the approximate root, $x_{n+1}$ that replacing $x_n$ and $x_{n-1}$ can sometimes lie on the same side of the root which lead to divergence. In this case, a good starting initial value is important, and it depends on the function itself.

From overall observation of Table 7, the three-point Secant method gives the least iterations at most of the initial approximations but only difference 1 or 2 iterations than exponential method and Secant method. However, for $x_0 = 6.0,\ x_1 = 6.1$ and $x_2 = 6.2,$ the three-point Secant method converge to the root, 4.70332376 with more iterations which are 9 than exponential method converges at 8th iteration, but the Secant method converge to the first root, 1.74613953. Meanwhile, all the three methods converge to the third root, 7.85436969 with similar iterations which are 5 iterations when using $x_0 = 8.3,\ x_1 = 8.4$ and $x_2 = 8.5.$ Based on Table 6, the findings show that this combination function have many roots nonetheless, Özyapici *et al.*, 2016 only managed to discover the first root (1.74613953) since only two initial approximations used and both of the initial points are closer to the first root, which are $x_0 = 0.75$ and $x_2 = 2.5.$

To conclude, based on Figure 6 and Table 7, the periodic function converges to infinite roots hence the increase of the initial approximations affects the increase of the roots. Besides, if the initial approximations are negative, only exponential method will diverge while the Secant method and three-point Secant method converge to the closest root, which is 1.74613953. In addition, the results show that the closer initial approximations to the root, the smaller number of iterations needed for the three methods to converge at the respective roots. Lastly, the three-point Secant method is the best method compared to Secant method and exponential method since need the smallest number of iterations for most of the initial approximations selected.

## 4. Conclusions

The present study was designed to determine the behavior of the existing Secant method and its improvised methods such as exponential method and three-point Secant method in solving the transcendental equation. These three methods were used to determine the nonzero real roots for several test functions like polynomial, logarithm, trigonometric, exponential as well as combination functions. These test functions have been calculated using several different starting initial points correct to accuracy root for each type of function. All the computation results were implemented in GUI to verify and get the results.

The results validate that the three-point Secant method needs the least iterations to converge to the exact roots in comparison with the Secant method and exponential method. Moreover, the three methods will converge to the root that closest to the selected initial approximations. Not only that, the number of iterations for each method also will be increase if the initial approximations used, far from the exact root and otherwise. However, the Secant method and exponential method sometimes may fail, and diverge when using certain initial approximations. Meanwhile, the three-point Secant method has no problems converging at every root of the six functions, including the periodic function.

The findings indicated that the three-point Secant method and Secant method are less influenced by the selected initial approximations. These results contradicted the exponential method since the number of iterations for this method will be increased if the initial points are far from exact value. Not only that, the exponential method also will be diverge if the negative initial approximations are used since the exponential formula unable to compute the number of iterations needed. Thus, this study has provided a deeper insight into the existing knowledge of the Secant method by providing more effective and better methods in computing roots of nonlinear equations.

Although the objectives of this study have been accomplished based on the results and discussions, the generalizability of these results is subject to certain limitations. For instance, improvements could be made on the three-point Secant method by using four initial points and to explore whether it gives the better rate convergence or otherwise. In addition, further study could be done by exploring whether the three-point Secant method shall successfully converge for all types of functions or not. This is because most iterative methods do not converge when using combination functions. Lastly, another research needs to be carried out to validate the order of convergence for this three-point Secant method is by solving the other applications problems like biology, chemistry, and engineering.

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**Author Contribution**

All authors contributed in every part of the work. Specifically, author 1 wrote the program in MATLAB Software and run the results. Author 2 created the Graphical User Interface (GUI) and oversaw the article writing. Author 3 prepared the results and wrote the discussion of the results. Author 4 prepared the literature review and wrote the conclusion.

**Conflict of Interest**

The authors have no conflicts of interest to declare.

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