UNIVERSITI TEKNOLOGI MARA

NEW SUBCLASSES OF UNIVALENT FUNCTIONS AND THEIR PROPERTIES

YUZAIMI BINTI YUNUS

Thesis submitted in fulfillment
of the requirements for the degree of
Doctor of Philosophy

Faculty of Computer and Mathematical Sciences

January 2023

ABSTRACT

In this thesis, the boundaries of knowledge on geometric function theory is extended with the discovery of interesting new subclasses of univalent functions plus their properties. The main theme of this thesis is establishing these new classes to conform with existing geometric domains and ascertaining their properties using proven techniques. There are four different type of classes of functions introduced and their main properties studied. First, two classes of starlike functions associated with geometric conditions, namely S^*_{GC} and S^*_{LC}, the generalised cardioid and the limaçon are established. The properties such as coefficient bounds, Fekete-Szegö estimates and radius problems are studied. Also included are bounds on the Hankel and Toeplitz determinants. Next, three new classes are formed using a new convolution operator. This new operator unifies various operators and the classes defined using this operator relates to the quantifier $zf'(z)/f(z)$, namely the classes are $S^{n,s,b}_{\alpha,\lambda}(\beta)$, a class of starlike functions of order β, $\mathcal{K}(n,s,b,\alpha,\lambda,\gamma,\beta)$, a unified class of starlike and convex functions of order β, and $SS_n(\delta,\gamma)$, the class of strongly starlike functions. For the first two classes, estimates on coefficients including the Fekete-Szegö functional are obtained, followed by growth and distortion results as well the extreme points determined. For the class of strongly starlike, Alexander-like relation and the preservation of the Bernardi integral operator are proven to also hold true. The class of harmonic functions are also the interest of this study. Two classes of Goodman-Rønning type harmonic functions are investigated. The classes $G_H(\lambda,\alpha,\gamma,\rho)$ and $\mathcal{HU}(\kappa,\beta,\lambda)$ generalises and unifies previous classes founded by earlier researchers. For these classes, properties such as coefficient bounds, growth and distortion bounds, extreme points and convex combination are considered. The last type of the classes formed relates to the class of bi-univalent functions. Bi-univalent functions have more stringent conditions. For these classes, $S_\sigma(\xi,\lambda)$ and $\mathcal{N}_\sigma(\zeta,\lambda)$, the upper bounds for the first few initial coefficients as well as the Fekete-Szegö estimate are obtained.

ACKNOWLEDGEMENT

Alhamdulillah, praise be to Allah, the Most Gracious, the Most Merciful for giving me strength and guidance to complete this thesis. I would never have been able to finish my thesis without the mercy of Allah. This thesis is dedicated to all those who have supported my study over the years.

I would like to express my first thanks to my supervisor, Associate Professor Dr Ajab Bai Akbarally for her continuous help, understanding, comments and cooperation in completing this thesis. I am also greatly indebted to my co-supervisor, Professor Dr Suzeini Abdul Halim for providing guidance, comments, her kind assistance and support during the years of study. Without the encouragement from both of them, I would not have the inner strength to complete this work.

The financial support of this study was funded by the Kementerian Pendidikan Malaysia together with Universiti Teknologi MARA, which is highly appreciated and gratefully acknowledged.

Last but not least, words cannot express how grateful I am to all members of my family especially my loving mum, my siblings and my friends, for all the prayers they have made for me. Also, my deepest appreciation goes to my husband, Ambo bin Madi who has been supporting and blessing me from the day of registration of my study until the day I submit the manuscript. Not forgetting my wonderful children, Ali, Adibah, Aisyah and Afifah, thank you for all your patience and sacrifices.

TABLE OF CONTENTS

CHAPTER ONE
PRELIMINARIES

1.1 Introduction

Complex analysis is one of the captivating areas in pure and applied mathematics. It started when mathematicians were perplexed by equations that could not be solved caused by the square roots of negative numbers. It had taken more than 250 years to come to terms with complex numbers, but the growth of a beautiful complex analysis was extremely fast. Most of the essential results were obtained during the 19th century by Cauchy, Riemann and others (Gray, 2015). Possibly the most amazing thing about complex analysis is the broad range of applications in physics and engineering. In any of these applications, complex analysis proves to be a useful tool. Complex analysis plays an important role in areas including fluid dynamics, the study of temperature, electrostatic and in evaluation of many real integrals of functions of real variables. Since a complex function needs four dimensions (two dimensions for input and two dimensions for output), it is difficult to imagine it as a graph. Geometric function theory is the study of the behaviour of functions defined by some geometric property, particularly the geometric properties of analytic functions.

Analytic functions are the central component of the study of complex analysis due to their interesting properties.

Definition 1.1.1. (Carathéodory, 1954) A complex function that is differentiable at every point of a region D is called an analytic function in D.

Analytic functions can be represented as a power series in its region of convergence. A complex-valued function f of a complex variable has a Taylor series expansion $f(z) = \sum_{n=0}^{\infty} A_n(z - z_0)^n$ where $A_n = f^{(n)}(z_0)/n!$ and is convergent in some open disk centered at z_0. One most important property is that every analytic function can be calculated in a neighborhood of any given point of its domain of definition by means of power series (Carathéodory, 1954).

Definition 1.1.2. (Jenkins, 1958) Let the function f be regular (analytic) or