Journal of Mathematics and Computing Science, 2022,
Volume 8, No 2, 51-58

The Connectivity and Wiener Index of Order Graph in Symmetric Group

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Abstract: Let G be a finite group and x is an element of G. Then, the order graph of a finite group denoted by Γ_{OG}, is a digraph and for any two distinct vertices x and y, there is an edge from x to y if and only if x divide y. The Wiener index is defined as the summation of distances between all pairs of vertices in a graph. It is one of the topological indices which can be used for analyzing intrinsic properties of molecule structure in chemistry. In this paper, the connectivity and Wiener index of Γ_{OG} are evaluated from the order graph of symmetric groups of degree up to 10.

Keywords: diameter; order graph; Wiener index; symmetric group

1 Introduction

Graph theory is the study of graphs, which are mathematical structures, used to model pair wise relations between objects. Group is a set of objects with a rule of combination. Given any two elements of the group, the rule yields another group element depending upon the elements chosen. The information in a group can be represented by a graph, which is a collection of points, called vertices and lines between them, called edges. There are many ways to establish a link between graph and a finite group, which results into many of the group properties. The association between a graph and a group is usually determined by the adjacency of the vertices. To be more specific, to make the edges, we pick some elements from the group. The combinatorial properties of graphs have been employed to investigate the theoretic algebraic properties of groups and vice versa. The relationship between a graph and a group (finite) was first introduced by Arthur Cayley in 1878 [1, 2] in which a graph represents a finite group. Cayley graphs geometrically display the actions of a finite group. There are some other well-known graphs associated with finite groups such as the power graph [3], commuting graph [4], non-commuting graph [5] and generalized conjugacy class graph [6].

The symmetric group on a finite set S is the group whose elements are all bijective functions from S to S and whose group operation is that of function of composition [7]. The symmetric group of degree n, $Sym(n)$ is the symmetric group on the set $S = \{1, 2, \ldots, n\}$. The order graph of a finite group denoted by Γ_{OG}, is the directed graph whose vertices are the elements of the group order classes, and for two distinct vertices x and y, there is an edge between them if and only if x divides y. The main objective of the paper is to focus on the connectivity and Wiener index of order graph Γ_{OG} obtained in symmetric group, $Sym(n)$, namely for degree $n \le 10$.

The true nature of number theory emerges from the study of the integers [8]. Suppose that a divides b. There is an integer k such that $ak = b$. It can be denoted by $a|b$. For example, $7|63$ because $7 \cdot 9 = 63$ but 63 does not divide 7. This seems simple enough, and let us play this definition by adapting the behaviour of the order graph Γ_{OG}. Assume that $a = 7$ and $b = 63$ are vertices of Γ_{OG}. Then, there is

the directed graph shown from a to b which presents that a divides b, but the contrary depicts no connection.
The paper is organized as follows. In Section 2, we recall some terminologies and notations for groups and graphs. Section 3 and Section 4 respectively deals with the main results on the order graph with its properties and summary of the study.

2 Preliminaries

This section will introduce the definitions and notations used throughout this research.

The graph terminologies in graph theory will be an ingredient of this work. Let Γ be a simple graph with vertex set $V(\Gamma)$ and edge set $E(\Gamma)$. The order of the graph Γ is the number of its vertices, while the size of the graph Γ is the number of its edges. A directed graph, also called a digraph, is a graph in which the edges have a direction. This is usually indicated with an arrow on the edge to connect an ordered pair of vertices. For $u,v \in V(\Gamma)$, let $d(u,v)$ be the distance between the vertices u and v in Γ. The diameter of graph Γ is the largest distance between the pair of vertices or it can be indicated by $Diam(\Gamma)$. A graph Γ is said to be connected, if there is a path between any two distinct vertices, otherwise it is a disconnected graph [9].

The Wiener index, $W(u)$ is the sum of distances between u and all other vertices of Γ [10]. Numerous results involving the Wiener index have been published, see, for example, the surveys [11 – 14]. Of particular interest to us is a problem of [15] which asked to find the directed graph of order graph and introduced the most important properties of such graph when the associated groups are the prime order classes groups and the dihedral group of order 2^n, $n \geq 3$.

The group definitions are also included in this study. Let x be an element in a finite group G. The order of $x \in G$ denoted by $o(x)$. For all $x \in G$, the set of all $y \in G$ which have the same order as x is the order class of x. The main interest in the obtained classes is because of each order class has a unique order. Two elements $x, y \in G$ are conjugated if there exists an element $g \in G$ such that $gxg^{-1} = y$ in which y is called a conjugate of x and x is called a conjugate of y [16].

In addition, the vertices of the order graph are represented by the order of G-conjugacy class elements of the symmetric group which is symbolized by $OC_G(x)$. For $x, y \in OC_G(x)$, the edges show the connection between x that divide y. Hence, the directed order graph will be obtained by finding its connectivity and the Wiener index.

3 Main Results

The computation involved in this study used MAGMA [17] in determining the order of G-conjugacy class elements in $Sym(n)$. The following results are obtained. Table 1 below presents each $Sym(n)$ for $n \leq 10$ and the largest order of its conjugacy classes. It is symbolized by the prime factorization K. These data have been used to identify the diameter of order graph, $Diam(\Gamma_{OG})$ in Theorem 3.1.

Table 1: The Prime Factorization of the Largest Order of $OC_G(x)$ in $Sym(n)$ for $n \le 10$

n	2	3	4	5	6	7	8	9	10
K	2	3	2^2	$2 \cdot 3$	$2 \cdot 3$	$2^2 \cdot 3$	$3 \cdot 5$	$2^2 \cdot 5$	$2 \cdot 3 \cdot 5$
i	1	1	2	2	2	3	2	3	3

Theorem 3.1 Let $n \le 10$, a_m is an integer, m is a natural number and $K = \prod_{m=1}^{i} a_m$ be the prime factorization of the largest order of conjugacy class elements in $G = Sym(n)$. Then, $Diam(\Gamma_{OG}) = i$.

Proof. For $n \le 10$, assume that $G = Sym(n)$. The order graph Γ_{OG} is constructed by considering the vertex set of the order of conjugacy class elements, $OC_G(x)$ in $Sym(n)$. The distances between the pair of vertices are computed according to the definition of Γ_{OG}. Let $o(x)$ be the order of $x \in G$ and K be the largest $o(x)$ in the particular $Sym(n)$. Then, K is a product of the prime factorization of the largest order of conjugacy class elements in G such that $K = \prod_{m=1}^{i} a_m$, where a_m is an integer. Therefore, the total number of integers involved in K has proved that $Diam(\Gamma_{OG}) = i$.

Each proposition below has been proved in obtaining the Wiener index for $Sym(n)$ of degree $n \le 10$.

Proposition 3.2 If $G = Sym(2)$ and $x \in G$ be the conjugacy class elements, then the Wiener index of Γ_{OG} is $W(x) = 1$.

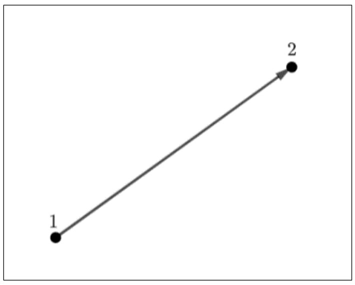

Figure 1: Γ_{OG} of $Sym(2)$

Proof. Let $G = Sym(2)$ and the vertex set of Γ_{OG} is $OC_G(x) = \{1,2\}$, as presented in Figure 1. Since $1|x$ for all $\{(1,x) | x \in OC_G(x) - \{1\}\} \subseteq E$. Since $x \in OC_G(x) - \{1\}$ does not divide $\{1\}$, there is no connection. Therefore, $W(x) = 1$.

Proposition 3.3 If $G = Sym(3)$ and $x \in G$ be the conjugacy class elements, then the Wiener index of Γ_{OG} is $W(x) = 2$.

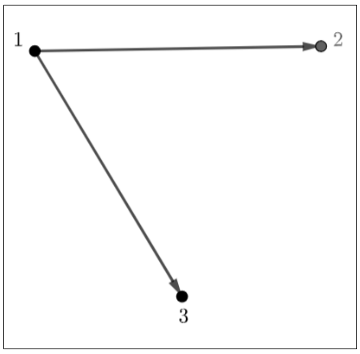

Figure 2: Γ_{OG} of $Sym(3)$

Proof. Let $G = Sym(3)$ and the vertex set of Γ_{OG} is $OC_G(x) = \{1,2,3\}$, as shown in Figure 2. Since $1|x$ for $x \in OC_G(x)$, then $\{(1,x) | x \in OC_G(x) - \{1\}\} \subseteq E$. There is no connection for $x \in OC_G(x) - \{1\}$. Therefore, $W(x) = 2$.

Proposition 3.4 If $G = Sym(4)$ and $x \in G$ be the conjugacy class elements, then the Wiener index of Γ_{OG} is $W(x) = 4$.

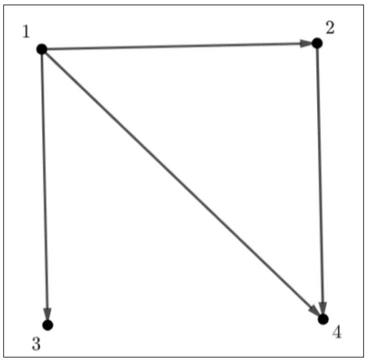

Figure 3: Γ_{OG} of $Sym(4)$

Proof. Let $G = Sym(4)$ and the vertex set of Γ_{OG} is $OC_G(x) = \{1,2,3,4\}$, as presented in Figure 3. Since $1|x$ for all $x \in OC_G(x)$, then $\{(1,x) | x \in OC_G(x) - \{1\}\} \subseteq E$. Suppose that $m \in \{2\}$ and $t \in OC_G(x) - \{1\}$. A connection exists in Γ_{OG} such that $m|t$ for $t \in OC_G(x) - \{1\}$, hence $W(x) = 4$.

Proposition 3.5 If $G = Sym(n)$ and $x \in G$ be the conjugacy class elements, for $n \in \{5,6\}$, then the Wiener index of Γ_{OG} is $W(x) = 8$.

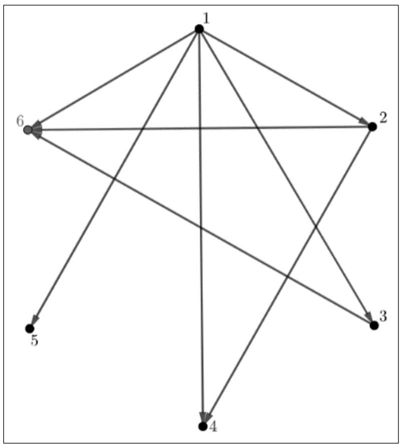

Figure 4: Γ_{OG} of $Sym(5)$ and $Sym(6)$

Proof. For $n \in \{5,6\}$, let $G = Sym(n)$ and the vertex set of Γ_{OG} is $OC_G(x) = \{1,2,3,4,5,6\}$, as illustrated in Figure 4. Since $1|x$ for all $x \in OC_G(x)$, then $\{(1,x) | x \in OC_G(x) - \{1\}\} \subseteq E$. Suppose that $m \in \{2,3\}$ and $t \in OC_G(x) - \{1\}$. The connections exist in Γ_{OG} such that $m|t$ for some t's, hence $W(x) = 8$.

Proposition 3.6 If $G = Sym(7)$ and $x \in G$ be the conjugacy class elements, then the Wiener index of Γ_{OG} shows that $W(x) = 17$.

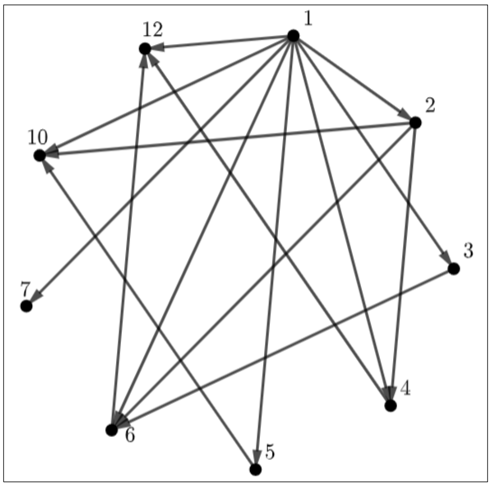

Figure 5: Γ_{OG} of $Sym(7)$

Proof. Let $G = Sym(7)$ and the vertex set of Γ_{OG} is $OC_G(x) = \{1,2,3,4,5,6,7,10,12\}$, as presented in Figure 5. Since $1|x$ for all $x \in OC_G(x)$, then $\{(1,x) | x \in OC_G(x) - \{1\}\} \subseteq E$. Suppose that $m \in \{2,3,4,5,6\}$ and $t \in OC_G(x) - \{1\}$. Connections exist in Γ_{OG} such that $m|t$ for some t's, hence $W(x) = 17$.

Proposition 3.7 If $G = Sym(8)$ and $x \in G$ be the conjugacy class elements, then the Wiener index of Γ_{OG} shows that $W(x) = 23$.

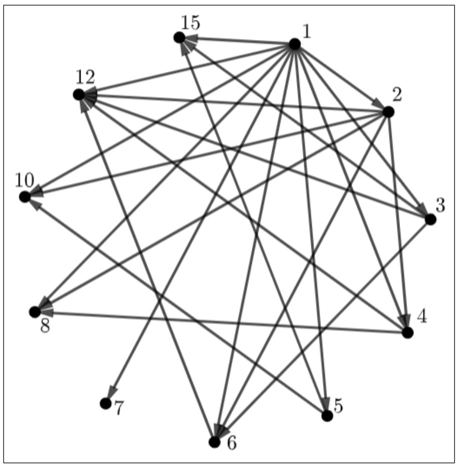

Figure 6: Γ_{OG} of $Sym(8)$

Proof. Let $G = Sym(8)$ and the vertex set of Γ_{OG} is $OC_G(x) = \{1,2,3,4,5,6,7,8,10,12,15\}$, as shown in Figure 6. Since $1|x$ for all $x \in OC_G(x)$, then $\{(1,x)|x \in OC_G(x) - \{1\}\} \subseteq E$. Suppose that $m \in \{2,3,4,5,6\}$ and $t \in OC_G(x) - \{1\}$. Connections exist in Γ_{OG} such that $m|t$ for some t's, hence $W(x) = 23$.

Proposition 3.8 If $G = Sym(9)$ and $x \in G$ be the conjugacy class elements, then the Wiener index of Γ_{OG} shows that $W(x) = 33$.

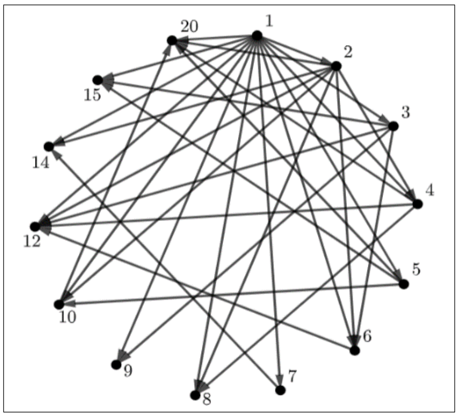

Figure 7: Γ_{OG} of $Sym(9)$

Proof. Let $G = Sym(9)$ and the vertex set of Γ_{OG} is $OC_G(x) = \{1,2,3,4,5,6,7,8,10,12,14,15,20\}$, as presented in Figure 7. Since $1|x$ for all $x \in OC_G(x)$, then $\{(1,x)|x \in OC_G(x) - \{1\}\} \subseteq E$. Suppose that $m \in \{2,3,4,5,6,7,10\}$ and $t \in OC_G(x) - \{1\}$. Connections exist in Γ_{OG} such that $m|t$ for some t's, hence $W(x) = 23$.

Proposition 3.9 If $G = Sym(10)$ and $x \in G$ be the conjugacy class elements, then the Wiener index of Γ_{OG} shows that $W(x) = 29$.

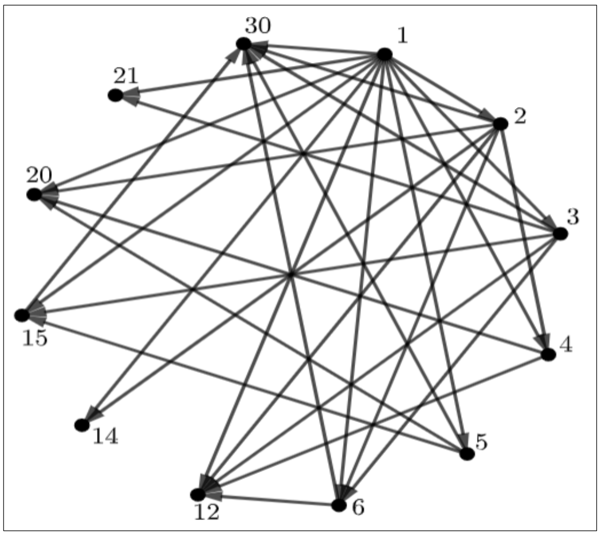

Figure 8: Γ_{OG} of $Sym(10)$

Proof. Let $G = Sym(10)$ $G = Sym(10)$ and the vertex set of Γ_{OG} is $OC_G(x) = \{1,2,3,4,5,6,12,14,15,20,21,30\}$, as illustrated in Figure 8. Since $1|x$ for all $x \in OC_G(x)$, then $\{(1,x) | x \in OC_G(x) - \{1\}\} \subseteq E$. Suppose that $m \in \{2,3,4,5,6,15\}$ and $t \in OC_G(x) - \{1\}$. The connections exist in Γ_{OG} such that $m|t$ for some t's, hence $W(x) = 29$.

4 Conclusion

Algebraic graph theory was initially developed as an intersection of algebra and graph theory. Many concepts of abstract algebra have facilitated the study of graphs from algebraic structures. On the other hand, graph theory has helped to characterize certain properties of algebraic structures. In this paper, we have dealt with the order graph of groups. The symmetric groups of degrees no more than 10 have been selected to construct the graph by taking the order of conjugacy class elements in the symmetric group as the vertex set. The order graph in the symmetric group has proved that it is all connected for the selection degree. Besides that, we have obtained the Wiener index for each case by calculating the sum of distances between all pairs of vertices. It is known that the Wiener index of a molecular graph correlates with physical and chemical properties of a molecule. Overall, the study of the order graph can be extended to a wide variety of graph properties.

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