GADING

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daftar isi:

INFINITESIMAL RIGIDITY OF FRAMEWORKS

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1.0 INTRODUCTION

This paper is a continuation of my first article in Gading Jilid 1, Bil 3. So the terms used in this paper should be the same as the first one unless stated. When one considers rigidity of frameworks he/she cannot avoid talking about infinitesimal rigidity. This is because they are inter relating to each other. Infinitesimal rigidity arises from one's ability to avoid infinitesimal motion, in other words, there are no external or internal forces which cannot be resolved reacted on the frameworks. This matter is very useful, especially to engineers and also interesting in the sense that it can be applied to models so that we can see its practicality. This paper will focus on the importance of infinitesimal rigidity in determining the rigidity of framework as a whole.

2.0 DEFINITIONS

Basically, finite motion will cause rigid motion in which a finite motion is defined as a continuous path p(t), $t \in [0,1]$ for each point of a framework G(p) preserving the required length for each t. Translation and rotation are examples of rigid motion. So we can defined a *rigid motion in* $\mathbb{R}^n$ as a mapping $T : \mathbb{R}^n \rightarrow \mathbb{R}^n$ such that

$$| Tx - Ty | = | x - y | \quad \text{for all} \quad x, y \in \mathbb{R}^n. \qquad \text{(Eqn. 1)}$$

that is, all distances are preserved.

A motion is said to be *infinitesimal* if there is an assignment of velocities u_i to each vertex p_i of a framework G(p) such that the length of $| p_i - p_j |$ is "initially preserved" for all i, j $\in$ E, the edge set of the framework. So if G = (V,E) is a framework in $\mathbb{R}^n$,f is its edge function $p \in \mathbb{R}^{nv}$, and $x = (x_1,\ldots,x_v)$ is infinitely differentiable function on [0,1] with x(0) = p,

$x(t) \in f^{-1}(f(p))$ for all $t \in [0,1]$, then

$$| x_i(t) - x_j(t) |^2 = | p_i - p_j |^2 \qquad \text{(Eqn. 2)}$$

for all $t \in [0,1]$, $\{ i,j \} \in E$.

Differentiating and evaluating at $t = 0$, that is $x_i'(0) = u_i$,

$$(x_i(t) - x_j(t)) \cdot (x_i'(t) - x_j'(t))$$

$$= (x_i(0) - x_j(0)) \cdot (x'_i(0) - x'_j(0))$$

$$= (p_i - p_j) \cdot (u_i - u_j)$$

$$= 0 \qquad \text{(Eqn. 3)}$$

2.1 Definition: A framework is said to be *infinitesimally rigid in* $\mathbb{R}^n$ if all the infinitesimal motions of a framework arise from rigid motions in $\mathbb{R}^n$. Otherwise it is said to have a internal motion - a set of velocity vectors which prevents rigid motions in $\mathbb{R}^n$.

Thus, $(p_i - p_j) \cdot (u_i - u_j) \neq 0$ for at least one pair of p_i, p_j and the framework is usually said to be *infinitesimally flexible* in $\mathbb{R}^n$

Sometimes an infinitesimal rigidity is also known as static rigidity [Whiteley] since, an infinitesimal motion exists due to unresolvable forces in the framework. Due to this, infinitesimal rigidity can be determined by row or column rank of the rigidity matrix for the framework (refer [Daud], Gading Jilid 1, Bil 3). To determine the infinitesimal rigidity from the statics point of view, we can apply an external force at the vertices of a framework and observe the tension and compression exist in the edge. Thus we can define $w_{ij}(p_i - p_j)$, $1 \leq i, j \leq v$ to be the force exerted by the edge on the vertex p_i where w_{ij} is a scalar, so that if $w_{ij} < 0$, $w_{ij}(p_i - p_j)$ is called the *tension* of the edge and if $w_{ij} > 0$, $W_{ij}(p_i - p_j)$ is called the *compression* in the edge. w_{ij} gives the magnitude of the force per unit length.

2.2 Definition : A *stress* of a framework G(p) in $\mathbb{R}^3$ is a collection of scalars w_{ij} one for each edge $[p_i, p_j]$, $1 \leq i, j \leq v$ of G(p) such that

$$\sum_{j \in a(i)} w_{ij} (p_i - p_j) = 0 \quad 1 \leq i \leq v \qquad \text{(Eqn. 4)}$$

where a(i) = { $j \in V$, {i,j} $\in E$ } is the set of vertices of G(p) adjacent to vertex i.

A framework is said to *stress free* if for all edges, $w_{ij} = 0$. This is also known as the trivial stress.

It is clear that, referring back to the rigidity matrix, the coefficient of each entry is in fact the value of scalar w_{ij}. It also follows that a stress of a framework G(p) in $\mathbb{R}^3$ is a linear dependency among the rows of df(p).

The existence of a non trivial stress in a framework in $\mathbb{R}^3$ means the possibility of internal forces in any physical construction of the framework, thus leading to possible unforeseen collapse - as any scalar multiple of a stress is also a stress.

There are two types of forces to consider that is, equilibrium force and resolvable force.

2.3 Definition : a) An equilibrium force of $p = (p_1, \ldots, p_v)$ of a framework G(p) in $\mathbb{R}^3$ is a vector $F = (F_1, \ldots, F_v) \in \mathbb{R}^3$ such that the sum of forces F_i and the sum of moments or torques about any axis through the origin of the forces F_i applied at p_i are zero, that is

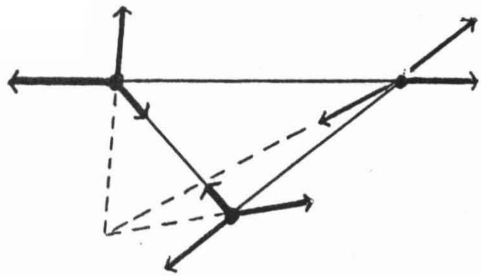
$$\sum_{i=1}^{v} F_i = 0 \quad \text{and} \quad \sum_{i=1}^{v} p_i \times F_i = 0 \qquad \text{(Eqn. 5)}$$

b) A *resolvable force* or a resolution of a framework G(p) in $\mathbb{R}^3$ is a vector $F = (F_1, \ldots, F_v) \in \mathbb{R}^{3v}$ with a scalar w_{ij}, one for each edge $[p_i, p_j]$ of G(p), $1 \leqslant i, j \leqslant v$ such that at each p_i, the sum of all forces at every vertex is zero, that is

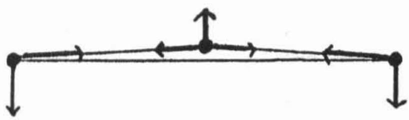
$$F_i + \sum_{j \in a(i)} w_{ij} (p_i - p_j) = 0 \qquad \text{(Eqn. 6)}$$

Here, we can conclude that the resolvable force is actually a set of tension and compression of the edges at the equilibrium state. A framework is said to be *statistically rigid* if and only if every equilibrium force applied to the framework has a resolution by the edges of the framework. Therefore, if a framework experiences the equilibrium force, it is actually the same as experiencing zero force, that is no effect at all.

The following are examples for resolvable and non resolvable forces in a framework.



a) All forces can be resolved on the triangle. So it is statistically rigid.



b) For a degenerate triangle (where the three vertices are collinear) the forces cannot be resolved. So a stress still exists in the edges.

3.0 THEOREMS

3.1 Theorem : A framework G(p) in $\mathbb{R}^{3v}$ is stress free if and only in rank df(p) = e, where e is the number of edges of the framework.

Proof : A stress of a framework is an element of the kernel of $df(p)^t : \mathbb{R}^e \longrightarrow \mathbb{R}^{3v}$, the transpose of df(p). Suppose G(p) is stress free. Then the dimension of ker $(df(p)^t) = 0$. Now, by the basic rule of linear algebra

$$\dim(\ker df(p)^t) + \dim(\operatorname{Im} df(p)^t) = \dim(\mathbb{R}^e)$$

Thus $\dim(\operatorname{Im} df(p)^t) = \dim \mathbb{R}^e = e$.

The converse is obvious.

3.2 Theorem: Suppose G(p) is a framework in $\mathbb{R}^3$, $p = (p_1, \ldots, p_v) \in \mathbb{R}^3$, where $p_1, \ldots, p_v$ are not coplanar. Then G(p) is infinitesimally rigid in $\mathbb{R}^3$ if and only if every equilibrium force for p is a resolvable force for G(p). Moreover, each equilibrium force is uniquely resolvable if and only if G(p) is infinitesimally rigid and stress free.

Proof: Let E and R be the collections of all equilibrium and resolvable force for a framework G(p) respectively. Then dim R = rank df(p) since R is the image of the transpose $df(p)^t : \mathbb{R}^e \to \mathbb{R}^{3v}$ of df(p) and dim E= 3v - 6 since E is the kernel of linear map $L : \mathbb{R}^{3v} \to \mathbb{R}^6$ defined by

$$L(F_1, \ldots, F_v) = (\Sigma F_i, \Sigma p_i \times F_i) \quad 1 \leq i \leq v$$

provided that $p_1, \ldots, p_v$ are not coplanar in $\mathbb{R}^3$

Thus E and R are subspaces of $\mathbb{R}^{3v}$. $R \subseteq E$ since $F = (F_1, \ldots, F_v) \in \mathbb{R}$, that is F is a linear combination of the rows of df(p) and each row of df(p) satisfies the condition defining E.

Now, $E \subseteq R$ if and only if rank df(p) = 3v - 6, but the tangent space T_p to the manifold is 6 dimensional (in $\mathbb{R}^3$) since the points $p_1, \ldots, p_v$ are not coplanar. Therefore G(p) is infinitesimally rigid in $\mathbb{R}^3$ if and only if dim (ker df(p)) = 6 that is rank df(p) = 3v - 6. The uniqueness result follows since G(p) is stress free if and only if kernel df(p) is trivial which is equivalent to the unique resolvability of the trivial equilibrium force. The following theorem gives us the importance of regular point (refer [Daud], pg 99) in determining infinitesimal rigidity.

3.3 Theorem : Suppose G(p) is a framewok in $\mathbb{R}^n$. $p = (p_1, \ldots, p_v) \in \mathbb{R}^{nv}$, and the affine span of $p_1, \ldots, p_v$ is $\mathbb{R}^n$. Then G(p) is infinitesimally rigid in $\mathbb{R}^n$ if and only if p is a regular point and G)p) is rigid in $\mathbb{R}^n$.

Therefore there are two things we have to consider for infinitesimal rigidity, that is

a) the forces reacted on the framework, either internal forces such as compression and tension or external forces such as wind etc.

b) The position of the vertices as in Theorem 3.3
One has to bear in mind that a framework maybe rigid but not necessarily infinitesimally rigid and furthermore if it is infinitesimally rigid then we can be sure that it is rigid.

3.4 Corollary: Consider a framework G(p) in $\mathbb{R}^{nv}$ Then at regular points of G(p), G(p) is rigid if and only if it is infinitesimally rigid and flexible if and only if it is infinitesimally flexible.

4.0 SOME EXAMPLES

It is very easy to see that a triangle of three vertices and three edges is infinitesimally rigid since any application of a velocity vector u_i will not change its chape. However a triangle with an extra vertex inn the middle of one of the edges in $\mathbb{R}^2$ is not infinitesimally rigid (Fig. 1)

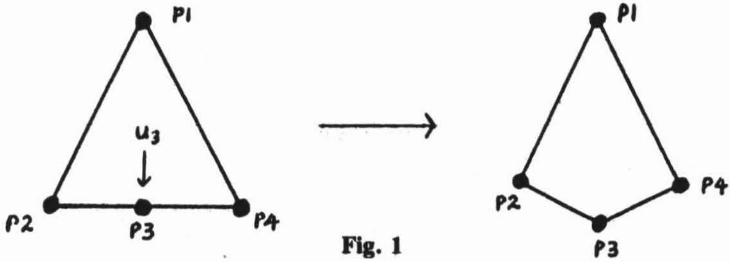


Fig. 1

This is because, we can apply a velocity vector u_3, say, to vertex p3, perpendicular to p2p4. This gives an infinitesimal motion to the framework and thus a finite motion as shown. Obviously, this triangle is also not infinitesimally rigid in $\mathbb{R}^3$. By Theorem 3.3, this is true because p2,p3 and p4 are collinear and thus, they are not at regular point.

By taking the degenerate case of a triangle again with assigned coordinates, this framework is infinitesimally flexible but not flexible in $\mathbb{R}^2$.

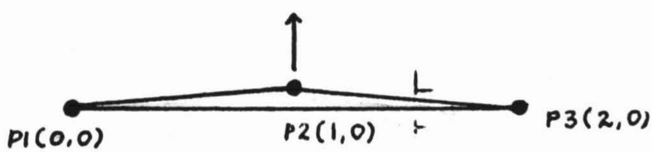


Fig. 2

If we denote u_i as the velocity at vertex p and $u_1 = u_3 = 0, u_2 = (0,c)$ where $c \neq 0$ Eqn. 3 is satistied but for any continuous motion x(t) of the framework as arigid body in $\mathbb{R}^2$, certainly at $t = 0$,

$$x'(0) \neq u(0) = (u_1(0), u_2(0), u_3(0))$$

Now, with the same example, let V = { 1,2,3}, E= {(1,2), (2,3), (1,3)}

a(1) = {2,3} a(2) = {1,3} a(3) = {1,2}

For $i = 1 \quad \sum_{j=2,3} w_{1j}(p_1 - p_j) = 0$

$i = 2 \quad \sum_{j=1,3} w_{2j}(p_2 - p_j) = 0$

$i = 3 \quad \sum_{j=1,2} w_{3j}(p_3 - p_j) = 0$

Thus, we have

$$w_{12}(p_1 - p_2) + w_{13}(p_1 - p_3) = 0 \qquad (1)$$

$$w_{21}(p_2 - p_1) + w_{23}(p_2 - p_3) = 0 \qquad (2)$$

$$w_{31}(p_3 - p_1) + w_{32}(p_3 - p_2) = 0 \qquad (3)$$

Clearly, adding (1) and (2) gives)3) since $w_{ij} = w_{ji}$. Taking w = 1, then w = 2 and w = 2. Hence the triangle has non trivial stress. By theorem 3.2 the forces are not resolvable and it follows that it is infinitesimally flexible.

5.0 REMARKS

In practical, a building cannot have a finite motion or else it will easily collapse. On the other hand, an infinitesimal motion is also unacceptable because by its existence, any framework built by elastic materials (bearing in mind that iron bars are also elastic!) will at least have finite motion, in which the result will be the same as the former one. This finite motion is closely related to the statics of the structure in resolving a corresponding external force. Thus, this suggests that engineers have to be more concerned with infinitesimal rigidity instead of rigidity only. So, even though a framework looks stable, one has to test its (infinitesimal) rigidity thoroughly especially in the case where are external forces factors such as strong winds, earth quake etc.

REFERENCES

1. *DAUD, Rigidity of Framework, Gading, Bil. 3, Jilid 1, Jan-Jun 1989, 98 - 106.*

2. *H. CRAPO AND WHITELEY, Statics of Frameworks and Motions of Panel Structures, a Projective Geometric Introduction, Structural Topology # 6, 1982, 43 - 82.*

3. *H. CRAPO, Structural Rigidity, Structural Topology # 1, 1979, 26 - 45.*

4. *B. ROTH, Questions on the Rigidity of Structures, Structural Topology # 4, 1980, 67 - 71.*

5. *B. ROTH, Rigid and Flexible Frameworks, American Mathematical Monthly, Vol. 88, No. 1, 6 - 21.*

6. *W. WHITELEY, Infinitesimal Rigid Polyhhedra I: Statics of frameworks, preprint (1980), Champlain Regional College, St. Lambert, Quebec, Canada.*

7. *W. WHITELEY, Intro to Structural Geometry I: Statics and Stresses, Draft (1978), Champlain Regional College, St. Lambert Quebec, Canada.*

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