GADING

majalah akademik itm,
cawangan pahang.

BIL: 3 JILID: 1 JAN-JUN 1989

daftar isi :

RIGIDITY OF FRAMEWORKS

By: Daud Mohamad

1.0 Introduction

The term "frameworks" is normally referred to a collection of rods and connectors/hinges. However, some people tend to used terms such as linkages, linkworks and mechanisms. basically, in mathematics, a framework consists of two sets, a finite set of vertices and a finite set of edges. Many things can be considered as frameworks, from little things such as a cube or a triangle to larger constructions such as skycrapers and transmission line towers etc.

One important characteristic of a framework which the author would like to discuss is rigidity. Consider one simple framework, that is a triangle. It is said to be rigid in R^2 since we cannot change the relative position of its vertices. For a square, it is definitely not rigid, or we call it flexible since it can be transformed into a rhombus (refer Fig. 1) with the edge lengths remaining constant.

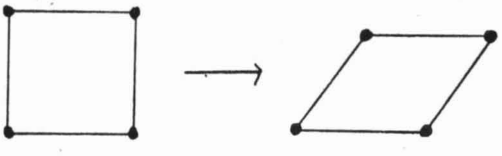


Fig. 1

The square can be made rigid in R^2 by adding an extra edge at one of the diagonals (Fig. 2).

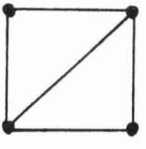


Fig.2

However, this is only true in R^2, but not in R^3 since it is possible to rotate one of its vertices along the diagonal edge (Fig. 3).

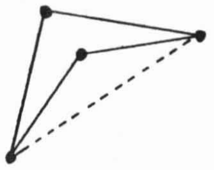


Fig.3

Looking back in R^2, by triangulating the square ie adding an edge along the diagonal, the framework can become rigid. In a way, we can say that triangulation does play a role in determining the rigidity of a framework.

2.0 Definitions

2.1 Definition : A framework G(p) in R^n is defined as a graph G(V,E), V is the finite set of vertices and E is the finite set of edges, with $P = (p_1, \ldots\ldots p_v) \in R^n \text{ x} \ldots\ldots \text{x } R^v = R^{nv}$. Here we identify p_i, $i \in V$, with the vertices and line segment $[p_i, p_j]$ in R^n for $\{i, j\} \in E$ with the edges of the graph G(p) in R^n.
We limit the definition of edges to only "straight" edges, that is, no bent edges are allowable.

2.2 Definition . The edge function of G(p) is the function $f : R^{nv} \rightarrow R^e$ such that
$f(p) = f(p_1, \ldots\ldots p_v) = (\ldots\ldots, |p_i - p_j|, \ldots\ldots)$
where v and e are the number of vertices and edges of G (p) respectively, $\{i, j\} \in E$, $p_k \in R^n$ and $|.|$ is an Euclidean norm in R^n. This is essentially the squares of length of the edges of G(p).

2.3 Definition : A framework is said to have a regular point $p \in V$, the set of vertices if rank df(p) = max { rank df(p), $p \in V$}.

A polyhedron is a special type of framework. It is a close portion of space bounded on all side by plane surfaces called faces. Basically it is built from edges and vertices such that;

a) each face is a cycle of K distinct vertices, $K \geqslant 3$
b) each edge occurs in exactly two faces
c) there is a cycle of m distinct surfaces, $m \geqslant 3$ such that each vertex p_i, occurs on a common edge between adjacent surfaces of the cycle and $p_i \neq p_j$ occurs on no others edge.

Furthermore, a convex polyhedron is a realization of an abstract polyhedron by distinct vertices $v_1, \ldots\ldots\ldots v_n$ such that

a) the edge are affinely spanned in R^3 ie $v_1, \ldots\ldots v_n$ are not coplanar in R^3.

b) for each face of it, there is a single plane containing all vertices of the faces (and affinely spanned by those vertices) which places all other vertices in a single half space.

Essentially, a convex polyhedron is the convex hull of a finite set of non coplanar points in R^3. When discussing the rigidity of a polyhedron (or convex polyhedron), one can say that its rigidity is equivalent to rigidity of the associated framework with each face replaced by the complete graph of its vertices.

2.4 Definition : By differentiating the edge function $f(p)$ of a fremework we would obtain a system of equation of $df(p)$. This system may be represented as a matrix having nv columns, and e rows where v and e are the number of vertices and edges of the framework respectively. The matrix can be written in the form of

$$
\begin{array}{cc}
 & \begin{array}{ccccccc} \text{Vertex} & & \text{Vertex} & & \text{Vertex} & & \text{Vertex} \\ 1 & \cdots\cdots & i & \cdots\cdots & j & \cdots\cdots & v \end{array} \\
\begin{array}{c} \\ \text{edge }(i,j) \\ \\ \\ \\ \end{array} & \begin{bmatrix} \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ 0 & \cdots\cdots & 2(p_i - p_j) & \cdots\cdots & 2(p_j - p_i) & \cdots\cdots & 0 \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \\ \cdots\cdots & & \cdots\cdots & & \cdots\cdots & & \cdots\cdots \end{bmatrix}
\end{array}
$$

and this is known as a **rigidity matrix.**

3.0 Main Results

3.1 Theorem (Rigidity Predictor) : Let G(p) be a framework in R^n where p $= (p_1,......p_v) \in R^{nv}$ is a regular point and $p_1,......p_v$ do not lie on a hyperplane in R^n. Then G(p) is rigid if and only if

$$\text{rank } df(p) = \frac{nv - n(n+1)}{2} \qquad \text{(Eqn 3.1)}$$

and G(p) is flexible in R^n if and only if

$$\text{rank } df(p) < nv - \frac{n(n+1)}{2} \qquad \text{(Eqn 3.2)}$$

Note: There is no harm in generalizing (Eqn 3.1) to

$$\text{rank } df(p) \geqslant \frac{nv - n(n+1)}{2}$$

3.2 Corollary 1 : Let G(p) be a framework in R^2 where $p = (p_1,......p_v) \in R^{2v}$ is a regular point and $p_1,......p_v$ do not lie on a line in R^2. Then G(p) is rigid in R^2 if and only if

$$\text{rank } df(p) \geqslant 2v - 3$$

and flexible if and only if

$$\text{rank } df(p) < 2v - 3$$

3.3 Corollary 2 : Let G(p) be a framework in R^3 where $(p_1,.....p_v) \in R^{3v}$ is a regular point and $p_1,.....p_v$ do not lie on a plane in R^3. Then G(p) is rigid in R^3 if and only if

$$\text{rank } df(p) \geqslant 3v - 6$$

and flexible in R^3 if and only if

$$\text{rank } df(p) < 3v - 6$$

The rigidity predictor is a very useful but only gives a general behavior of a framework and does not necessarily determine the rigidity of it at all points. For example, in R^2, frameworks in Fig. 4 basically have the same abstract framework, that is, a framework with six vertices and eight edges. However, only (a)) is rigid since its bottom edges are collinear. The framework of (b) is flexible since it can have finite motion.

One of the possibilities of the deformed framework for (b) will look like Fig. 5.

Fig. 4

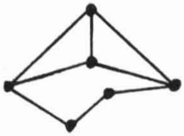
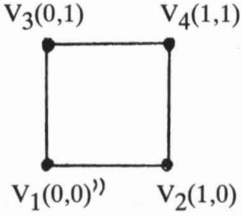


Fig. 5

One may check the rigidity of a triangle by using the rigidity predictor by taking (0,0), (2,0) and (1,1) as the locations of its vertices. And then by using rigidity matrix, one can easily show that rank df(p) = 2(3) - 3 = 3. So it is rigid in R^2.

Similarly for a square, we can test its rigidity by letting the vertices of the square be labelled with coordinates (0,0), (0,1), (1,0) and (1,1). We can construct a 4 x 8 rigidity matrix of a square as follow:

Fig. 6



$$
\begin{array}{c}
 \\ (V_1, V_2) \\ (V_1, V_3) \\ (V_3, V_4) \\ (V_2, V_4)
\end{array}
\begin{array}{c}
\begin{array}{cccc} \quad V_1 \quad & \quad V_2 \quad & \quad V_3 \quad & \quad V_4 \quad \end{array} \\
\begin{bmatrix}
-1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 1
\end{bmatrix}
\end{array}
$$

Obviously, the rank of the matrix is 4. Thus by using the rigidity predictor

$$\text{rank } df(p) = 4 < 2(4) - 3 = 5$$

and therefore the square is flexible. Again, one can easily deduce that the square with an extra edge along the diagonal is rigid by checking the rank of its rigidity matrix.

In the case of a polyhedron, the following theorem allows us to substitute rank df(p) in the rigidity predictor theorem and corollaries by e, the number of edges in a framework.

3.4 Theorem. Let $G(p)$, $p \in R^{3v}$, be the framework in R^3, given by a convex polyhedron P and suppose $f(p)$ is the edge function of $G(p)$. Then

$$\text{rank } df(p) = e$$

where e is the number of edges of G.

This theorem makes life easier in determining the rigidity of a framework only by counting the number of edges and vertices. Hence, one can ensure, by rigidity predictor that a framework $G(p)$ in R^n is rigid if

$$e = nv - \frac{n(n+1)}{2}$$

provided that $p = (p_1, \ldots\ldots p_v) \in R^{nv}$ is a regular point and does not lie on a hyperplance in R^n. To simplify the formula, we can write

$$e = 3V - 6 \text{ in } R^3$$

and $\quad e = 2V - 3 \text{ in } R^2$

Since we have stated the significance of tringulation of a framework, now, we put a theorem regarding to triangulation of a convex polyhedron.

3.5 Theorem : The framework $G(p)$ given by a convex polyhedron P is rigid in R^3 if and only if every face of P is a triangles.

Proof : By 3.4

$$\text{rank } df(p) = e = \max \{ \text{rank } df(p) : p \in R^{3v} \}$$

where f is the edge function of G. Thus all vertices $p = (p_1, \ldots\ldots p_v)$ is a regular point of f and clearly $p_1, \ldots p_v$ are not coplanar. By the rigidity predictor, $G(p)$ is rigid if and only if $e = \text{rank } df(p) = 3V - 6$. Since a convex polyhedron is essentially a spherical polyhedron (ie homeomorphic to a sphere), by using Euler's formula

$$e - f = v - 2$$

where e, f and v are the number of edges, faces and vertices of the polyhedron. Thus, for all regular points,

$$e = 3V - 6 = 3(v - 2) = 3(e - f)$$

Hence $2e = 3f$, that is, every face of the polyhedron is a triangle and the result follows.

4.0 Examples

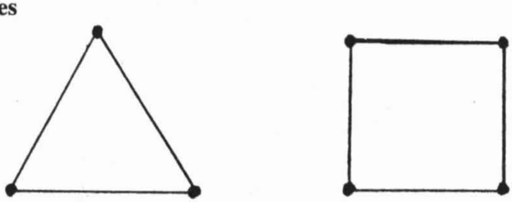


Fig. 7

A triangle is rigid in R^2 since $e = 3$, $v = 3$, so by using

$$e = 2v - 3$$
$$3 = 2(3) - 3 = 3$$

It is also rigid in R^3 since

$$3 = 3(3) - 6 = 3$$

A square with an extra edge along its diagonal is rigid in R^2. This is because, $e = 5$, $v = 4$, so

$$5 = 2(4) - 3 = 5$$

but it is not rigid in R^3 because

$$5 \neq 3(4) - 6 = 6$$

5.0 Concluding remarks

There is no doubt that many common rigid structures have triangles in their designs which may indicate the significance of triangulation. Obvious examples are Eifel Tower of Paris and the structure of the Tower of Liberty, New York. And we already have the result of 3.4 which related to triangulation. This result was basically based upon the result of the rigidity predictor.

Nevertheless it is not that easy to apply the "'simple formula" of rigidity predictor, $e \geqslant 3v - 6$ in R^3 or $e \geqslant 2v - 3$ in R^2 at regular points. This has been of interest to many profesionals expecially engineers instead of taking the rank of df(p). Actually, one also has to make sure that the framework is convex (by Theorem 3.4) before using this formula.

For instance, the following framework (Fig. 8) satisfies the condition of the rigidity predictor $e = 3v - 6$ in R^3 and one is tempted to immediately deduce that this framework is rigid.

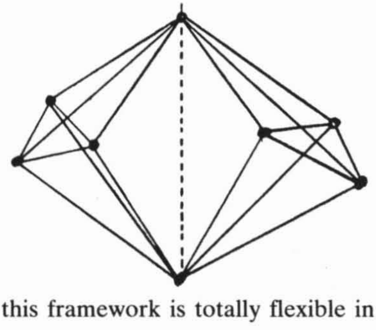


Fig. 8

However, in actual sense, this framework is totally flexible in R^3 since it can rotate about the dotted line.

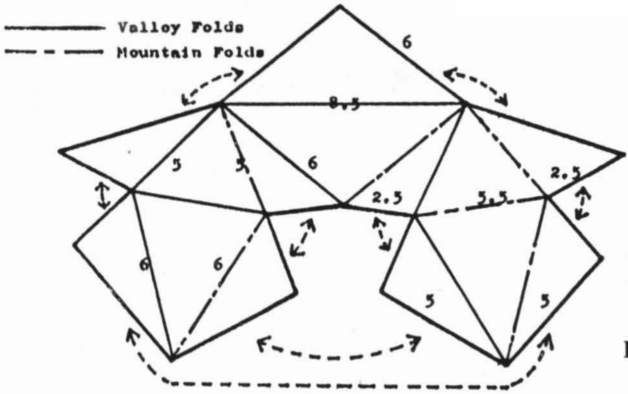


Fig. 9

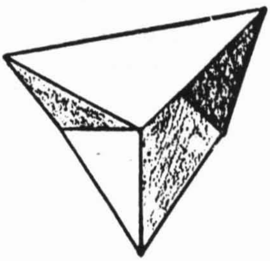


Fig. 10

Similarly, triangulation is not enough in determining the rigidity of a framework without having consider the convexity of the framework. There is a simple model which shows that not all triangulated polyhedron is rigid. It has 9 vertices and 14 faces. The construction of this model is based ona a measured triangulated flat surfaces (Fig. 9). The model mainly consists of two large triangles and two crinkle parts. The surfaces are symmetric. The model should look like Fig. 10 after folding the surfaces accordingly. The flexing may be observed by holding the top two triangles in one hand and moving the bottom vertex to the right and lift. So far, this is the simplest flexible triangulated polyhedron ever known in R^3

REFERENCES

1. L. ASIMOW AND B. ROTH, Rigidity of Graph I, American Math. Society Trans. 245(1978), 279 - 289.

2. R. CONNELLY, The Rigidity of Polyhedral Surfaces, Mathematics Mag. Vol. 52, No.5, Nov., 1979, 275 - 283.

3. H. GLUCK, Almost all simply connected surfaces are rigid, Lecture Notes in Math : 438, Geometric Topology, Springer Verlag, 225 - 239.

4. B. ROTH, Rigid and Flexible Frameworks, American Mathematical Monthhly, Vol. 88, No. 1, 6 - 21.

5. B. ROTH, Questions on the Rigidity of Structures, Structural Topology 4, 1980, 67 - 71.

6. W. WHITELEY, Infinitesimal Rigid Polyhedra I, Statics of framework, pre print (2980), Champlain Regional College, St. Lambert, Quebec, Canada.