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NUMERICAL SOLUTION OF HYPERBOLIC GOURSAT PARTIAL DIFFERENTIAL EQUATIONS WITH HYBRID CENTRAL DIFFERENCE - TAYLOR SERIES EXPANSIONS METHOD

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ABSTRACT

This paper investigates a new method for solving the Goursat partial differential equation (PDE) using a combination of the central finite difference method (FDM) and Taylor series expansion. The study evaluates the effectiveness and accuracy of this new approach, analyzing linear Goursat problems and conducting multiple numerical experiments. The simulation study demonstrates that the suggested approach surpasses the existing method in terms of performance and accuracy. Applying this proposed scheme will minimize the cost, especially for engineers that might apply this model in solving their real-life problems.

Keywords*: Central Finite Difference Method, Goursat Problem, Hyperbolic Partial Differential Equation, Numerical Differentiation, Taylor Series Expansions*

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1. Introduction

It has been recognized that a significant proportion of differential equations, which are used to represent real-world issues, cannot be solved using established analytical techniques. In such circumstances, one must settle for numerical approximations of the models that can be achieved by numerous numerical techniques of diverse characteristics (Fadugba *et al.*, 2021). The basic form of the Goursat partial differential equation problem is as follows (Wazwaz, 1993):

$$
\begin{aligned}
&u_{xy} = f(x, y, u, u_x, u_y) \\
&u(x,0) = g(x), \quad u(0,y) = m(y) \\
&g(0) = m(0) = u(0,0) \\
&0 \le x \le a, \quad 0 \le y \le b\ .
\end{aligned} \tag{1}
$$

where u_{xy} is the mixed derivative in space *x* and *y* and $f\left(x, y, u, u_x, u_y\right)$ is a function of the two independent variables *x* and *y*, the dependent variable *u* and the derivative terms u_x and u_y .

It appears in a variety of scientific and technological fields. There are researchers such as (Son & Thao, 2019; Tian *et al*., 2020; Mokdad, 2021) studied the applications of the Goursat problem in the trajectory of an economic dynamics, global optimal scheduling, geoscience, bio-medical engineering and Nordström-like black hole.

Several techniques have been suggested such as Newton-Cotes Integration (Deraman & Nasir, 2015), reduction differential transform (Naseem, 2022), the fuzzy transform (Saharizan & Zamri, 2019; Kim Son *et al*., 2021), iterative regularization (Meziani et al., 2021), Signature Kernel (Salvi *et al*., 2021) and method of transmutation operators (Sitnik & Karimov, 2023; Karimov & Yulbarsov, 2023). The established FDM is provided by and averages the functional values as (Nasir & Ismail, 2013):

$$
\frac{u_{i+1,j+1} + u_{i,j} - u_{i+1,j} - u_{i,j+1}}{h^2} = \frac{1}{4}\left(f_{i+1,j+1} + f_{i,j} + f_{i+1,j} + f_{i,j+1} \right), \tag{2}
$$

where *h* represents grid size.

Initially, second-order linear and nonlinear hyperbolic Goursat PDEs were solved using the FDM (Pandey, 2014a; 2014b; Nasir & Ismail, 2012; 2013). A deeper understanding and improvements of these schemes will improve the mathematical modelling of problems where the Goursat problem arises. However, the numerical schemes for solving Goursat problems deal with difficulties to preserve the linearity of approximate numerical solutions. Therefore, there is a motivation to develop a better method with higher accuracy and preserve linearity for the Goursat scheme. Moreover, Taylor series expansion can be an alternative method to resolve the problem (Tailor & Bhathawala, 2011; Jacquemin *et al*., 2020). Hence, we introduce a new approach named central difference - Taylor series expansion method (CD-TSE). The proposed scheme combines the central FDM with higher-order Taylor series expansion to solve homogenous and inhomogeneous linear Goursat PDE problems.

Section 2 (methodology) will formulate the CD-TSE, and Section 3 (results and discussions) will apply it to three different classes of linear Goursat problems. Furthermore, the numerical result of CD-TSE will be given at the last section and conclusion.

2. Methodology

The central difference is one of the basic formulae of FDM for the approximation of the first-order derivatives of a function. The second-order partial derivatives by using FDM is constructed based on the Taylor series expansion and written in the following form (Twizell, 1984):

$$
u_{xy} = \frac{1}{4h^2}\begin{bmatrix} u(x+h, y+h) - u(x+h, y-h) \\ -u(x-h, y+h) + u(x-h, y-h) + O\left(h\right) \end{bmatrix} \tag{3}
$$

known as central difference formulas for $\frac{\partial^2 u}{\partial x \partial y}$ at $\left(x, y\right)$ with the truncation error, $O\left(h\right)$ and step size, h .

The next step is to adopt difference formulas (3) into the left-hand side of the standard form of Goursat problem (1) is called discretizing process. Hence, the derivation of Goursat schemes can be written as shown below:

$$\frac{1}{4h^2}\begin{bmatrix} u(x+h,y+h)-u(x+h,y-h) \\ -u(x-h,y+h)+u(x-h,y-h)+O(h) \end{bmatrix} = f\left(x,y,u,u_x,u_y\right) \tag{4}$$

where $f(x,y,u,u_x,u_y)$ can be computed by referring to the selected Goursat problems and approximated by using the higher-order Taylor series expansion.

Moreover, the general form of the Taylor series expansions for two variables functions $f(x,y)$, and can be written as (Pantaleón & Ghosh, 2015):

$$f\left(x_0+\Delta x, y_0+\Delta y\right) \cong \sum_{n=0}^{k}\frac{1}{n!}\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)^n f\left(x_0,y_0\right) \tag{5}$$

where, n is the order starting from 0 till kth term, Δx and Δy are the step size and $f\left(x_0,y_0\right)$ is an origin.

Thus, this paper will use up to fiftieth terms (k is 15) of the Taylor series and (5) can be written as:

$$\begin{aligned} f\left(x_0+\Delta x, y_0+\Delta y\right) &\cong f\left(x_0,y_0\right)+\left(\Delta x\frac{\partial}{\partial x}+\Delta y\frac{\partial}{\partial y}\right)f\left(x_0,y_0\right) \\ &+\frac{1}{2!}\left(\Delta x^2\frac{\partial}{\partial x\partial x}+2\Delta x\Delta y\frac{\partial}{\partial x\partial y}+\Delta y^2\frac{\partial}{\partial y\partial y}\right)f\left(x_0,y_0\right) \\ &+\frac{1}{3!}\begin{pmatrix} \Delta x^3\frac{\partial}{\partial x\partial x\partial x}+\Delta y^3\frac{\partial}{\partial y\partial y\partial y} \\ +3\Delta x^2\Delta y\frac{\partial}{\partial x\partial x\partial y}+3\Delta x\Delta y^2\frac{\partial}{\partial x\partial y\partial y} \end{pmatrix} f\left(x_0,y_0\right)+\ldots \end{aligned} \tag{6}$$

By letting $x=x_0+\Delta x$, $y=y_0+\Delta y$ and step size $\Delta x=\Delta y=h$. Hence, the right-hand side of the standard form of Goursat problem (1) can be approximated as shown below:

$$\begin{aligned} f\left(x_0+h, y_0+h\right) &= f(x,y,u,u_x,u_y) \cong f(x-h,y-h) \\ &+h\left[f_x(x-h,y-h)\right]+h\left[f_y(x-h,y-h)\right] \\ &+\frac{1}{2!}\begin{bmatrix} h^2 f_{xx}(x-h,y-h)+h^2 f_{yy}(x-h,y-h) \\ +2h^2 f_{xy}(x-h,y-h) \end{bmatrix} \\ &+\frac{1}{3!}\begin{bmatrix} h^3 f_{xxx}(x-h,y-h)+h^3 f_{yyy}(x-h,y-h) \\ +3h^3 f_{xxy}(x-h,y-h)+3k^3 f_{xyy}(x-h,y-h) \end{bmatrix}+\ldots \end{aligned} \tag{7}$$

Therefore, the next stage is to adopt central difference formulas (3) and approximate via Taylor series expansion into the left and right-hand sides of (1), respectively.

Hence, the derivation of linear Goursat schemes using CD-TSE can be written as given:

$$\begin{aligned} u(x+h,y+h) &= u(x+h,y-h)+u(x-h,y+h)-u(x-h,y-h) \\ &+4h^2\left\{\begin{aligned} & f(x-h,y-h)+h\left[f_x(x-h,y-h)\right] \\ & +h\left[f_y(x-h,y-h)\right] \\ & +\frac{1}{2!}\begin{bmatrix} h^2 f_{xx}(x-h,y-h)+h^2 f_{yy}(x-h,y-h) \\ +2h^2 f_{xy}(x-h,y-h) \end{bmatrix} \\ & +\frac{1}{3!}\begin{bmatrix} h^3 f_{xxx}(x-h,y-h)+h^3 f_{yyy}(x-h,y-h) \\ +3h^3 f_{xxy}(x-h,y-h)+3k^3 f_{xyy}(x-h,y-h) \end{bmatrix}+\ldots \end{aligned}\right\}+O(h). \end{aligned} \tag{8}$$

By rewriting in index form and shifting $i\to i+1$, $j\to j+1$ will yields:

$$u_{i+2,j+2}=u_{i+2,j}+u_{i,j+2}-u_{i,j}$$
$$+4h^2\left\{\begin{array}{l} f_{i,j}+h\left[\frac{\partial}{\partial_i}f_{i,j}+\frac{\partial}{\partial_j}f_{i,j}\right]+\frac{h^2}{2!}\left[\frac{\partial^2}{\partial_i\partial_i}f_{i,j}+\frac{\partial^2}{\partial_j\partial_j}f_{i,j}+2\frac{\partial^2}{\partial_i\partial_j}f_{i,j}\right] \\ +\frac{h^3}{3!}\left[\frac{\partial^3}{\partial_i\partial_i\partial_i}f_{i,j}+\frac{\partial^3}{\partial_j\partial_j\partial_j}f_{i,j}+3\frac{\partial^3}{\partial_i\partial_i\partial_j}f_{i,j}+3\frac{\partial^3}{\partial_i\partial_j\partial_j}f_{i,j}\right]+.... \end{array}\right\}+O(h). \quad (9)$$

In the next section, the approximation (9) will be tested into homogenous and inhomogeneous linear Goursat PDE problems.

3. Results and Discussions

Example 1:

Take into consideration the Goursat linear homogeneous problem.

$$\begin{array}{l} u_{xy}=u, \\ u(x,0)=e^x, \\ u(0,y)=e^y, \\ 0\le x\le 4,\ 0\le y\le 4. \end{array} \quad (10)$$

The exact to the problem stated in (10) is $u(x,y)=e^{x+y}$ (Naseem, 2022).

By applying formula (9) to problem (10), differentiating the derivative terms and adopting the initial condition. Thus, the new scheme using CD-TSE formula can be rewritten as:

$$u_{i+2,j+2}=u_{i+2,j}+u_{i,j+2}-u_{i,j}$$
$$+4h^2\left[\begin{array}{l} 1+ih+jh+\frac{(ih)^2}{2}+\frac{(jh)^2}{2}+(ij)h^2 \\ +\frac{(ih)^3}{6}+\frac{(jh)^3}{6}+\frac{(ih)^2(jh)}{2}+\frac{(ih)(jh)^2}{2}+... \end{array}\right]+O(h). \quad (11)$$

Comparative study between scheme (2) versus CD-TSE scheme (13) have been done for problem (10).

Table 1. Approximate numerical solution at $h=1$ for problem (10).

Scheme	**u(x, y)**			
	u(1,1)	**u(2,2)**	**u(3,3)**	**u(4,4)**
Exact	7.3891	54.5982	4.0343e+02	2.9811e+03
Standard (2)	8.0609	68.7114	5.9443e+02	5.1802e+03
CD-TSE (11)	8.0609	43.3343	3.3928e+02	2.1882e+03

Table 2. Average relative errors for problem (10).

Step size (h)	**Standard (2)**	**CD-TSE (11)**
1	3.2861e-01	1.8976e-01
0.8	1.7329e-01	1.1441e-01
0.4	3.2760e-02	2.8814e-02
0.2	7.3749e-03	7.3720e-03

The approximate solution and average relative errors at selected step sizes are illustrated in Table 1 and Table 2. The average relative error in Table 2 becomes smaller as the grid size decreases for both schemes and CD-TSE is powerful than standard scheme. The numerical experiments signify that, the accuracy level of the CD-TSE scheme (11) is higher than standard scheme (2) for linear homogeneous Goursat problem (10). Furthermore, the proposed scheme is also preserving the linearity.

Example 2:

Consider the following linear inhomogeneous Goursat problem

$$\begin{aligned} & u_{xy} = u - y, \\ & u(x,0) = e^x, \\ & u(0,y) = y + e^y, \\ & 0 \le x \le 4,\ 0 \le y \le 4. \end{aligned} \tag{12}$$

The exact is $u(x,y) = y + e^{x+y}$ (Ahmad & Mustaq, 2015).

By applying formula (9) to problem (12), differentiating the derivative terms and adopting the initial condition. Thus, the new scheme using CD-TSE formula can be rewritten as:

$$\begin{aligned} & u_{i+2,j+2} = u_{i+2,j} + u_{i,j+2} - u_{i,j} \\ & +4h^2 \begin{bmatrix} 1 + ih + 2jh + \frac{(ih)^2}{2} + \frac{(jh)^2}{2} + (ij)h^2 \\ + \frac{(ih)^3}{6} + \frac{(jh)^3}{6} + \frac{(ih)^2(jh)}{2} + \frac{(ih)(jh)^2}{2} + \ldots \end{bmatrix} + O(h). \end{aligned} \tag{13}$$

Comparative study between scheme (2) and scheme (13) have been done for problem (12). The approximate solution and average relative errors at selected step sizes are illustrated in Table 3 and Table 4.

Table 3. Approximate numerical solution at $h = 1$ for problem (12).

Scheme	**u(x, y)**			
	u(1,1)	**u(2,2)**	**u(3,3)**	**u(4,4)**
Exact	8.3891	56.5982	4.0643e+02	2.9850e+03
Standard (2)	9.0610	70.7114	5.9743e+02	5.1842e+03
CD-TSE (13)	9.0609	49.3343	3.5028e+02	2.2242e+03

Table 4. Average relative errors for problem (12).

Step size (h)	**Standard (2)**	**CD-TSE (13)**
1	3.2330e-01	1.6859e-01
0.8	1.7012e-01	9.0166e-02
0.4	3.2032e-02	1.9896e-02
0.2	7.1979e-03	7.1645e-03

The average relative error in Table 4 becomes smaller as the grid size decreases for both and CD-TSE is superior to standard scheme. The numerical experiments signify that, the accuracy level of the CD-TSE scheme (13) is higher than standard scheme (2) for linear inhomogeneous Goursat problem (12). Furthermore, the proposed scheme is also preserving the linearity.

Example 3:

Consider the following linear inhomogeneous Goursat problem.

$$\begin{aligned} & u_{xy} = u + 4xy - x^2y^2 \\ & u(x,0) = e^x \\ & u(0,y) = e^y \\ & 0 \le x \le 4,\ 0 \le y \le 4. \end{aligned} \tag{14}$$

The problem has been used by (Wazwaz, 2009). The problem's exact solution is $u(x,y) = x^2y^2 + e^{x+y}$ (Datta *et al.*, 2021).

By applying formula (9) to problem (14), differentiating the derivative terms and adopting the initial condition. Thus, the new scheme using CD-TSE formula can be rewritten as:

$$u_{i+2,j+2} = u_{i+2,j} + u_{i,j+2} - u_{i,j} + 4h^2 \begin{bmatrix} 1 + ih + jh + \frac{(ih)^2}{2} + \frac{(jh)^2}{2} + (ij)h^2 \\ + \frac{(ih)^3}{6} + \frac{(jh)^3}{6} + \frac{(ih)^2(jh)}{2} \\ + \frac{(ih)(jh)^2}{2} + \frac{(ih)^4}{24} + \frac{(jh)^4}{24} \\ + \frac{(ih)^3(jh)}{6} + \frac{(ih)(jh)^3}{6} + \frac{5(ih)^2(jh)^2}{4} \ldots \end{bmatrix} + O(h). \tag{15}$$

Below are the approximate numerical solutions and average relative errors at various selected grid points for the problem (14) results comparing the standard scheme (2) and CD-TSE scheme (15).

Table 5. Approximate numerical solution at $h = 1$ for problem (14).

Scheme	**u(x, y)**			
	u(1,1)	**u(2,2)**	**u(3,3)**	**u(4,4)**
Exact	8.3891	7.0598e+01	4.8443e+02	3.2370e+03
Standard (2)	6.3943	4.4119e+01	3.1079e+02	2.4908e+03
CD-TSE (15)	6.3943	5.9334e+01	3.7435e+02	2.4442e+03

Table 6. Average relative errors for problem (14).

Step size (h)	**Standard (2)**	**CD-TSE (15)**
1	2.6296e-01	2.1293e-01
0.8	2.4627e-01	1.4867e-01
0.4	1.8820e-01	4.0671e-02
0.2	1.5288e-01	1.0175e-02

The approximate solution and average relative errors at selected step sizes are illustrated in Table 5 and Table 6. The average relative error in Table 6 becomes smaller as the grid size decreases for both and hence CD-TSE is efficient than standard scheme. The numerical experiments signify that, the accuracy level of the CD-TSE scheme (15) is higher than standard scheme (2) for linear inhomogeneous Goursat problem (14). Furthermore, the proposed scheme is also preserving the linearity.

4. Conclusions

The aim of this paper was to create novel approaches for solving linear Goursat partial differential equations (PDEs) using the central finite difference method in combination with Taylor series expansion. Our goal was successfully accomplished, resulting in the CD-TSE scheme, which proved to be highly efficient and precise in solving both homogeneous and inhomogeneous linear Goursat PDE problems. The numerical analysis showed that the CD-TSE scheme outperformed the standard method (previous study), and its greatest advantage was its ability to preserve linearity effectively. There are many applications involving Goursat partial differential equation problems found in various fields of sciences and mathematical engineering. Applying this proposed scheme will minimize the cost, especially for engineers that might apply this model in solving their real-life problems. The cost will be calculated in terms of derivation time, running time, software development, energy, and production.

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Author Contribution

Author1 conducted the numerical analysis and interpreted the results. Author2 wrote the research methodology. Author3 prepared the literature review and oversaw the article writing.

Conflict of Interest

The authors have no conflicts of interest to declare.

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