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STABILITY ANALYSIS OF A PROPOSED SCHEME OF ORDER FIVE FOR FIRST ORDER ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT

This paper presents the stability analysis of a proposed scheme of order five (FCM) for first order Ordinary Differential Equations (ODEs). The proposed FCM is derived by means of an interpolating function of polynomial and exponential forms. The properties of FCM were discussed extensively. The linear stability of FCM in the context of the Third Order One-Step Method (TCM) and Second Order One-Step Method (SCM) for the solution of initial value problems of first order differential equations is presented. The stability region of FCM, TCM and SCM is investigated using the Dahlquist's test equation. The numerical results obtained via FCM are compared with TCM and SCM. Moreover, by varying the step length, the accuracy and convergence of the methods in terms of the final absolute relative error are measured. The results show that FCM converges faster and more stable than its counterparts.

Keywords: *Fifth order scheme, final absolute relative error, initial value problem, second order method, stability, third order method.*

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1. Introduction

Most of the problems of mathematical physics are formulated in the form of differential equations. Such physical models represent future estimation for any real-world situation based on the data available in the past and present as detailed in (Bird, 2017; Butcher, 2016; Jain, 2003; Lambert, 1991; Lambert, 1973). It is a known fact that a huge number of differential equations that model real life problem cannot be solved via well-known analytical methods. In such situations, one has to compromise at numerical approximate solutions of the models achievable by various numerical techniques of different nature (Qureshi & Fadugba, 2018).

In this paper, the stability analysis of FCM in the context of TCM (Fadugba & Idowu, 2019) and SCM (Fadugba & Falodun, 2017) is presented and investigated. The rest of the paper is outlined as follows; Section Two presents the problem formulation and derivation of FCM. Section Three presents a brief review of two existing methods: TCM and SCM. The stability analysis of the methods is examined in Section Four. Section Five presents implementation of FCM, discussion of results and concluding remarks.

2. Problem Formulation and Derivation of the One-Step Scheme of Order Five

2.1 Problem Formulation

Consider an initial value problem of first order ordinary differential equation of the form:

$$y' = f(x,y), y(a) = y_0 x \in [a,b], y \in (-\infty,\infty) \quad (1)$$

The existence and uniqueness of solution of (1) have been guaranteed via the Lipschitz condition on the interval $D = [a, \text{b}]$. The exact solution of (1) at $x = x_n$ is given by $y(x_n)$.

2.2 Derivation of a Fifth Order One-Step Scheme

Consider an interpolating function of the form:

$$F(x) = \sum_{j=0}^{5} \eta_j x^j + \eta_6 e^{5c} \quad (2)$$

here $\eta_0, \eta_1, \eta_2, \eta_3, \eta_4, \eta_5, \eta_6$ are undetermined constants and c is a constant. The integration interval of $[a,b]$ is defined as $a = x_0 \le x \le x_n = b$. The step length is defined as:

$$h = \frac{b-a}{N} \quad (3)$$

where N is the number of integration steps. The mesh point is defined as:

$$x_{n+1} = x_0 + (n+1)h, n = 0,1,2,...N-1 \quad (4)$$

or

$$x_n = x_0 + nh, n = 1,2,...N \quad (5)$$

Expanding (2) at the points x_n and x_{n+1} yields:

$$F(x_n) = \sum_{j=0}^{5} \eta_j x_n^{\ j} + \eta_6 e^{5c} \quad (6)$$

and

$$F(x_{n+1}) = \sum_{j=0}^{5} \eta_j x_{n+1}^{\ j} + \eta_6 e^{5c} \quad (7)$$

respectively. Differentiating (6) five times and setting:

$$F'(x_n) = f_n, F''(x_n) = f_n^{(1)}, F'''(x_n) = f_n^{(2)}, F^{(iv)}(x_n) = f_n^{(3)}, F^{(v)}(x_n) = f_n^{(4)} \quad (8)$$

yields:

$$f_n = \sum_{j=1}^{5} j\eta_j x_n^{\ j-1} = F'(x_n) \quad (9)$$

$$f_n^{(1)} = \sum_{j=2}^{5} j(j-1)\eta_j x_n^{\ j-2} = F''(x_n) \quad (10)$$

$$f_n^{(2)} = \sum_{j=3}^{5} j(j-1)(j-2)\eta_j x_n^{\ j-3} = F'''(x_n) \quad (11)$$

$$f_n^{(3)} = \sum_{j=4}^{5} j(j-1)(j-2)(j-3)\eta_j x_n^{\ j-4} = F^{(iv)}(x_n) \tag{12}$$

$$f_n^{(4)} = \sum_{j=5}^{5} j(j-1)(j-2)(j-3)(j-4)\eta_j x_n^{\ j-5} = F^{(v)}(x_n) \tag{13}$$

Equations (9)-(13) form a system of linear equations of the form:

$$AX = b \tag{14}$$

where,

$$A = \begin{bmatrix} 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 \\ 0 & 2 & 6x_n & 12x_n^2 & 20x_n^3 \\ 0 & 0 & 6 & 24x_n & 60x_n^2 \\ 0 & 0 & 0 & 24 & 120x_n \\ 0 & 0 & 0 & 0 & 120 \end{bmatrix} \tag{15}$$

$$X = \begin{bmatrix} f_n \\ f_n^{(1)} \\ f_n^{(2)} \\ f_n^{(3)} \\ f_n^{(4)} \end{bmatrix} \tag{16}$$

and

$$b = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \beta_3 \\ \beta_4 \\ \beta_5 \end{bmatrix} \tag{17}$$

Solving the system of linear equations in (14), one gets:

$$\eta_1 = \frac{1}{24}\left(24f_n - 24x_x f_n^{(1)} + 12x_n^2 f_n^{(2)}\right) - \frac{1}{24}\left(4x_n^3 f_n^{(3)} - x_n^4 f_n^{(4)}\right) \tag{18}$$

$$\eta_2 = \frac{1}{12}\left(6f_n^{(1)} - 6x_n f_n^{(2)}\right) + \frac{1}{12}\left(3x_n^2 f_n^{(3)} - x_n^3 f_n^{(4)}\right) \tag{19}$$

$$\eta_3 = \frac{1}{12}\left(2f_n^{(2)} - 2x_n f_n^{(3)} + x_n^2 f_n^{(4)}\right) \tag{20}$$

$$\eta_4 = \frac{1}{24}\left(f_n^{(3)} - x_n f_n^{(4)}\right) \tag{21}$$

$$\eta_5 = \frac{1}{120} f_n^{(4)} \tag{22}$$

Subtracting (6) from (7) yields:

$$F(x_{n+1}) - F(x_n) = \eta_1 \left(x_{n+1} - x_n\right) + \eta_2 \left(x_{n+1}{}^2 - x_n{}^2\right) + \eta_3 \left(x_{n+1}{}^3 - x_n{}^3\right) + \eta_4 \left(x_{n+1}{}^4 - x_n{}^4\right) + \eta_5 \left(x_{n+1}{}^5 - x_n{}^5\right) \quad (23)$$

Using (4) and (5), with $x_0 = 0$, yields:

$$x_n = nh \quad (24)$$

$$x_{n+1} = (n+1)h \quad (25)$$

$$x_{n+1} - x_n = h \quad (26)$$

$$x_{n+1}^2 - x_n^2 = (2n+1)h^2 \quad (27)$$

$$x_{n+1}^3 - x_n^3 = (3n^2 + 3n + 1)h^3 \quad (28)$$

$$x_{n+1}^4 - x_n^4 = (4n^3 + 6n^2 + 4n + 1)h^4 \quad (29)$$

$$x_{n+1}^5 - x_n^5 = (5n^4 + 10n^3 + 10n^2 + 5n + 1)h^5 \quad (30)$$

Using (24) into (18), (19), (20), (21) and (22) yield:

$$\eta_1 = \frac{1}{24}\left(24 f_n - 24nhf_n^{(1)} + 12(nh)^2 f_n^{(2)}\right) - \frac{1}{24}\left(4(nh)^3 f_n^{(3)} - (nh)^4 f_n^{(4)}\right) x_n = nh \quad (31)$$

$$\eta_2 = \frac{1}{12}\left(6 f_n^{(1)} - 6nhf_n^{(2)}\right) + \frac{1}{12}\left(3(nh)^2 f_n^{(3)} - (nh)^3 f_n^{(4)}\right) \quad (32)$$

$$\eta_3 = \frac{1}{12}\left(2 f_n^{(2)} - 2nhf_n^{(3)} + (nh)^2 f_n^{(4)}\right) x_{n+1} - x_n = h \quad (33)$$

$$\eta_4 = \frac{1}{24}\left(f_n^{(3)} - nhf_n^{(4)}\right) \quad (34)$$

$$\eta_5 = \frac{1}{120} f_n^{(4)} \quad (35)$$

Suppose that:

$$y_{n+1} - y_n \equiv F(x_{n+1}) - F(x_n) \quad (36)$$

By using (23), (31), (32), (33), (34) and (35), (36) yields:

$$
\begin{aligned}
y_{n+1} - y_n = {} & \frac{h}{24}\left(24f_n - 24nhf_n^{(1)} + 12(nh)^2 f_n^{(2)}\right) - \frac{h}{24}\left(4(nh)^3 f_n^{(3)} - (nh)^4 f_n^{(4)}\right) \\
& + \frac{h^2}{12}\left(6f_n^{(1)} - 6nhf_n^{(2)}\right)(2n+1) + \frac{h^2}{12}\left(3(nh)^2 f_n^{(3)} - (nh)^3 f_n^{(4)}\right)(2n+1) \\
& + \frac{h^3}{12}\left(2f_n^{(2)} - 2nhf_n^{(3)}\right)(3n^2+3n+1) \quad + \frac{h^3}{12}\left((nh)^2 f_n^{(4)}\right)(3n^2+3n+1) \\
& + \frac{h^4}{24} f_n^{(3)}(4n^3+6n^2+4n+1) - \frac{h^4}{24} nhf_n^{(4)}(4n^3+6n^2+4n+1) \\
& + \frac{h^5}{120} f_n^{(4)}(5n^4+10n^3+10n^2+5n+1)
\end{aligned} \tag{37}
$$

Setting:

$$K_1 = \frac{1}{2}\left(24f_n - 24nhf_n^{(1)} + 12(nh)^2 f_n^{(2)}\right) - \frac{1}{2}\left(4(nh)^3 f_n^{(3)} - (nh)^4 f_n^{(4)}\right) \tag{38}$$

$$K_2 = h\left(6f_n^{(1)} - 6nhf_n^{(2)}\right)(2n+1) + h\left(3(nh)^2 f_n^{(3)} - (nh)^3 f_n^{(4)}\right)(2n+1) \tag{39}$$

$$K_3 = h^2\left(2f_n^{(2)} - 2nhf_n^{(3)}\right)(3n^2+3n+1) \quad + h^2\left((nh)^2 f_n^{(4)}\right)(3n^2+3n+1) \tag{40}$$

$$K_4 = \frac{h^3}{2} f_n^{(3)}(4n^3+6n^2+4n+1) - \frac{h^3}{2} nhf_n^{(4)}(4n^3+6n^2+4n+1) \tag{41}$$

$$K_5 = \frac{h^4}{10} f_n^{(4)}(5n^4+10n^3+10n^2+5n+1) \tag{42}$$

Therefore,

$$y_{n+1} = y_n + \frac{h}{2(3!)}\left(K_1 + K_2 + K_3 + K_4 + K_5\right) \tag{43}$$

Equation (43) is known as FCM for the solution of initial value problems of ordinary differential equations. The local truncation error, order of accuracy, consistency, zero stability and convergence of (43) were summarized in the following remarks.

Remarks:

a. Local Truncation Error and Order of Accuracy

According to Fadugba & Idowu (2019), the analysis of the local truncation error determines the order of convergence of any numerical integration method designed for the numerical solutions of initial value problems in ordinary differential equations. In order to check the order of the method, the algorithm of the numerical method is subtracted from the Taylor's series expansion for $y(x_n)$ in powers of *h* and by means of the localizing assumptions.

Consider the Taylor's series expansion of the form:

$$\begin{aligned} y(x_n+h) &= y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2} f^{(1)}(x_n, y(x_n)) + \frac{h^3}{6} f^{(2)}(x_n, y(x_n)) \\ &+ \frac{h^4}{24} f^{(3)}(x_n, y(x_n)) + \frac{h^5}{120} f^{(4)}(x_n, y(x_n)) + \ldots \end{aligned} \tag{44}$$

Define the local truncation error for (43) as:

$$L_{TE} = y(x_n + h) - y_{n+1} \tag{45}$$

Substituting (43) and (44) into (45) yields:

$$\begin{aligned} L_{TE} &= y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2} f^{(1)}(x_n, y(x_n)) + \frac{h^3}{6} f^{(2)}(x_n, y(x_n)) \\ &+ \frac{h^4}{24} f^{(3)}(x_n, y(x_n)) + \frac{h^5}{120} f^{(4)}(x_n, y(x_n)) + \ldots - y_n \\ &- \frac{h}{12}\left(K_1 + K_2 + K_3 + K_4 + K_5\right) \end{aligned} \tag{45}$$

Simplifying further and using (38), (39), (40), (41) and (42), one gets:

$$L_{TE} = A - B \tag{47}$$

where,

$$\begin{aligned} A &= y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2} f^{(1)}(x_n, y(x_n)) + \frac{h^3}{6} f^{(2)}(x_n, y(x_n)) \\ &+ \frac{h^4}{24} f^{(3)}(x_n, y(x_n)) + \frac{h^4}{120} f^{(4)}(x_n, y(x_n)) + \ldots \end{aligned} \tag{48a}$$

and

$$B = y_n + \frac{h}{5!}\left[120 f_n + 60 h f_n^{(1)} + 20 h^2 f_n^{(2)} + 5 h^3 f_n^{(3)} + h^4 f_n^{(4)}\right] \tag{48b}$$

By means of the localizing assumptions, (48b) yields:

$$\begin{aligned} B &= y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2} f^{(1)}(x_n, y(x_n)) + \frac{h^3}{6} f^{(2)}(x_n, y(x_n)) \\ &+ \frac{h^4}{24} f^{(3)}(x_n, y(x_n)) + \frac{h^4}{120} f^{(4)}(x_n, y(x_n)) \end{aligned} \tag{49}$$

By using (48a) and (49), (47) becomes:

$$L_{TE} = \frac{h^6}{120} f^{(5)}(x_n, y(x_n)) + \ldots \tag{50}$$

Equation (50) is the local truncation error for FCM. It also shows that the scheme has accuracy of order five.

b. Consistency of the Scheme

The proposed scheme is consistent, since:

a) It has fifth order accuracy.

b) $$\lim_{h\to 0}\left(\frac{L_{TE}}{h}\right)=\lim_{h\to 0}\left(\frac{\frac{h^6}{120}f^{(5)}(x_n,y(x_n))+\ldots}{h}\right)=0 \tag{51}$$

c) $$\lim_{h\to 0}\phi(x_n,y_n,h)=\lim_{h\to 0}\left\{\frac{1}{5!}\left[120f_n+60hf_n^{(1)}+20h^2f_n^{(2)}+5h^3f_n^{(3)}+h^4f_n^{(4)}\right]\right\}=f_n \tag{52}$$

c. Zero Stability of the Scheme

A linear multistep method of step $k = 1$ is said to be zero stable if the roots of the first characteristic polynomial of the method given by $Q(a)=\sum_{j=0}^{1}\alpha_j a^j=\alpha_0+a\alpha_1$ satisfy the Dahlquist's root condition:

i) all roots *r* satisfy $a\le 1$

ii) multiple roots *r* satisfy $a<1$

From (43), $\alpha_1 = 1$ and $\alpha_0 = -1$ were deduced, then the characteristic polynomial is obtained as:

$$Q(a) = a-1 \tag{53}$$

Therefore,

$$Q(a) = 0 \Rightarrow a - 1 = 0 \Rightarrow a = 1 \tag{54}$$

Since the root of (54) satisfies the Dahlquist's root condition. Hence, it is concluded that the scheme is zero stable.

d. Convergence of the Scheme

The convergence of the scheme is discussed as follows. From (43), the increment function is obtained as:

$$\phi(x_n,y_n,h)=f(x_n,y_n)+Rf_n^{(1)}(x_n,y_n)+Sf_n^{(2)}(x_n,y_n)+Tf_n^{(3)}(x_n,y_n)+Uf_n^{(4)}(x_n,y_n) \tag{55}$$

where $R=\frac{h}{2!},S=\frac{h^2}{3!},T=\frac{h^3}{4!},U=\frac{h^4}{5!}$.

Suppose that:

$$\phi(x_n,\bar{y}_n,h)=f(x_n,\bar{y}_n)+Rf_n^{(1)}(x_n,\bar{y}_n)+Sf_n^{(2)}(x_n,\bar{y}_n)+Tf_n^{(3)}(x_n,\bar{y}_n)+Uf_n^{(4)}(x_n,\bar{y}_n) \tag{56}$$

Subtracting (56) from (55), one gets:

$$\phi(x_n, y_n, h) - \phi(x_n, \bar{y}_n, h) = f(x_n, y_n) - f(x_n, \bar{y}_n) + R\left(f_n^{(1)}(x_n, y_n) - f_n^{(1)}(x_n, \bar{y}_n)\right) + S\left(f_n^{(2)}(x_n, y_n) - f_n^{(2)}(x_n, \bar{y}_n)\right) + T\left(f_n^{(3)}(x_n, y_n) - f_n^{(3)}(x_n, \bar{y}_n)\right) + U\left(f_n^{(4)}(x_n, y_n) - f_n^{(4)}(x_n, \bar{y}_n)\right) \quad (57)$$

Define $\hat{y}_n$ as a point in the interior of the interval whose end points are y_n and $\bar{y}_n$. Using the Mean Value Theorem (MVT), one obtains:

$$f(x_n, y_n) - f(x_n, \bar{y}_n) = \frac{\partial f(x_n, \hat{y}_n)}{\partial y_n}(y_n - \bar{y}_n) \quad (58)$$

$$f^{(1)}(x_n, y_n) - f^{(1)}(x_n, \bar{y}_n) = \frac{\partial f^{(1)}(x_n, \hat{y}_n)}{\partial y_n}(y_n - \bar{y}_n) \quad (59)$$

$$f^{(2)}(x_n, y_n) - f^{(2)}(x_n, \bar{y}_n) = \frac{\partial f^{(2)}(x_n, \hat{y}_n)}{\partial y_n}(y_n - \bar{y}_n) \quad (60)$$

$$f^{(3)}(x_n, y_n) - f^{(3)}(x_n, \bar{y}_n) = \frac{\partial f^{(3)}(x_n, \hat{y}_n)}{\partial y_n}(y_n - \bar{y}_n) \quad (61)$$

$$f^{(4)}(x_n, y_n) - f^{(4)}(x_n, \bar{y}_n) = \frac{\partial f^{(4)}(x_n, \hat{y}_n)}{\partial y_n}(y_n - \bar{y}_n) \quad (62)$$

Let

$$A = \frac{\partial f(x_n, \hat{y}_n)}{\partial y_n}, B = \frac{\partial f^{(1)}(x_n, \hat{y}_n)}{\partial y_n}, C = \frac{\partial f^{(2)}(x_n, \hat{y}_n)}{\partial y_n}, D = \frac{\partial f^{(3)}(x_n, \hat{y}_n)}{\partial y_n}, E = \frac{\partial f^{(4)}(x_n, \hat{y}_n)}{\partial y_n} \quad (63)$$

Using (63), (58)-(62) become:

$$f(x_n, y_n) - f(x_n, \bar{y}_n) = A(y_n - \bar{y}_n) \quad (64)$$

$$f^{(1)}(x_n, y_n) - f^{(1)}(x_n, \bar{y}_n) = B(y_n - \bar{y}_n) \quad (65)$$

$$f^{(2)}(x_n, y_n) - f^{(2)}(x_n, \bar{y}_n) = C(y_n - \bar{y}_n) \quad (66)$$

$$f^{(3)}(x_n, y_n) - f^{(3)}(x_n, \bar{y}_n) = D(y_n - \bar{y}_n) \quad (67)$$

$$f^{(4)}(x_n, y_n) - f^{(4)}(x_n, \bar{y}_n) = E(y_n - \bar{y}_n) \quad (68)$$

Substituting (64)-(68) into (57), yields:

$$\phi(x_n, y_n, h) - \phi(x_n, \bar{y}_n, h) = (A + RB + SC + TD + UE)(y_n - \bar{y}_n) \quad (69)$$

Taking the absolute value of (69), one obtains:

$$\left|\phi(x_n, y_n, h) - \phi(x_n, \bar{y}_n, h)\right| \le \left|(A + RB + SC + TD + UE)\right|\left|(y_n - \bar{y}_n)\right| \quad (70)$$

Setting $L = A + RB + SC + TD + UE$, therefore:

$$\left|\phi(x_n, y_n, h) - \phi(x_n, \bar{y}_n, h)\right| \le L\left|(y_n - \bar{y}_n)\right| \quad (71)$$

Hence, the scheme (43) is convergent and ϕ is Lipschitzian.

3. A Brief Review of TCM and SCM

The brief review of TCM (Fadugba & Idowu, 2019) and SCM (Fadugba & Falodun, 2017) are detailed as below.

3.1 A One-Step Method of Order Three (TCM)

A third order one-step method for the solution of (1) given by:

$$y_{n+1} = y_n + hf_n + \frac{h^2}{2}(f_n^{(1)} + f_n^{(2)}) + (1 - e^{-h} - h)f_n^{(2)} \tag{72}$$

was derived via the transcendental function of exponential type of the form:

$$F(x) = \sum_{i=0}^{2} \alpha_i x^i + \alpha_3 e^{-x} \tag{73}$$

where $\alpha_0, \alpha_1, \alpha_2, \alpha_3$ are undetermined constants.

3.2 A One-Step Method of Order Two (SCM)

A second order one-step method for the solution of (1) given by:

$$y_{n+1} = y_n + hf_n + (h + e^{-h} - 1)f_n^{(1)} \tag{74}$$

was derived via the interpolating function of the form:

$$F(x) = \sum_{i=0}^{1} \alpha_i x^i + \alpha_2 e^{-x} \tag{75}$$

where $\alpha_0, \alpha_1, \alpha_2$ are undetermined constants.

4. Stability Analysis of the Scheme

According to Fadugba & Qureshi (2019), a numerical method is said to be stable if it is capable of damping out the small fluctuations carried out in the input data. The notion of stability may be taken in different contexts: it may be associated with the specific numerical technique used, or the step size *h* used in numerical computations or with the particular problem being solved. To discuss the stability analysis of FCM in the context of TCM and SCM, consider the Dahlquist's test equation given by:

$$y' = \lambda y, y(0) = 1, \lambda < 0 \tag{76}$$

where λ is a complex constant.

The exact solution of (76) is obtained as:

$$y(x) = e^{\lambda x} \tag{77}$$

Expanding (77) at the points $x = x_n$ and $x = x_{n+1}$, yields respectively:

$$y(x_n) = e^{\lambda x_n} \tag{78}$$

and

$$y(x_{n+1}) = e^{\lambda x_{n+1}} = y(x_n)e^{\lambda h} \tag{79}$$

By means of (5), (43), (78) and localizing assumptions, the numerical approximation is obtained as:

$$
\begin{aligned}
y_{n+1} &= y(x_n) + hf(x_n, y(x_n)) + \frac{h^2}{2} f^{(1)}(x_n, y(x_n)) + \frac{h^3}{6} f^{(2)}(x_n, y(x_n)) \\
&+ \frac{h^4}{24} f^{(3)}(x_n, y(x_n)) + \frac{h^4}{120} f^{(4)}(x_n, y(x_n)) \\
&= \left[1 + \frac{h}{120}\left(120\lambda + 60\lambda^2 h + 20\lambda^3 h^2 + 5\lambda^4 h^3 + \lambda^5 h^4\right)\right] y_n
\end{aligned} \tag{80}
$$

Setting

$$\Psi = \left[1 + \frac{h}{120}\left(120\lambda + 60\lambda^2 h + 20\lambda^3 h^2 + 5\lambda^4 h^3 + \lambda^5 h^4\right)\right] \tag{81}$$

Equation (80) becomes:

$$y_{n+1} = \Psi y_n \tag{82}$$

It is clearly seen that (82) is the sixth term of $e^{\lambda h}$. Hence the stability function of (43) requires that:

$$|\Psi| < 1 \tag{83}$$

The error growth factor can be controlled by (83). Also setting $z = \lambda h$ in (82) and simplifying further, the stability region of the scheme satisfies:

$$\Psi = \left[1 + \frac{1}{120}\left(120z + 60z^2 + 20z^3 + 5z^4 + z^5\right)\right] = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120} \tag{84}$$

The stability functions of FCM, TCM and SCM using Dahlquist's Test Equation are summarized in the Table 1.

Table 1. The stability functions for FCM, TCM and SCM using Dahlquist's Test Equation

Method	**Stability Function**
FCM	$1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \frac{z^4}{24} + \frac{z^5}{120}$
TCM	$1 + z + \frac{z^2}{2} + \frac{z^3}{6}$
SCM	$1 + z + \frac{z^2}{2}$

The stability regions for the stability functions as in Table 1 represented in unshaded area for FCM, TCM and SCM are displayed in Figure 1, Figure 2 and Figure 3 respectively.

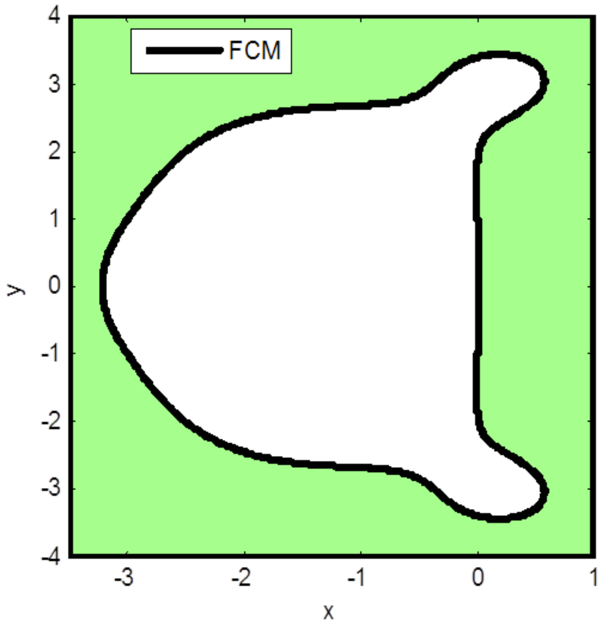

Figure 1. The Stability Region (Un-shaded) for FCM

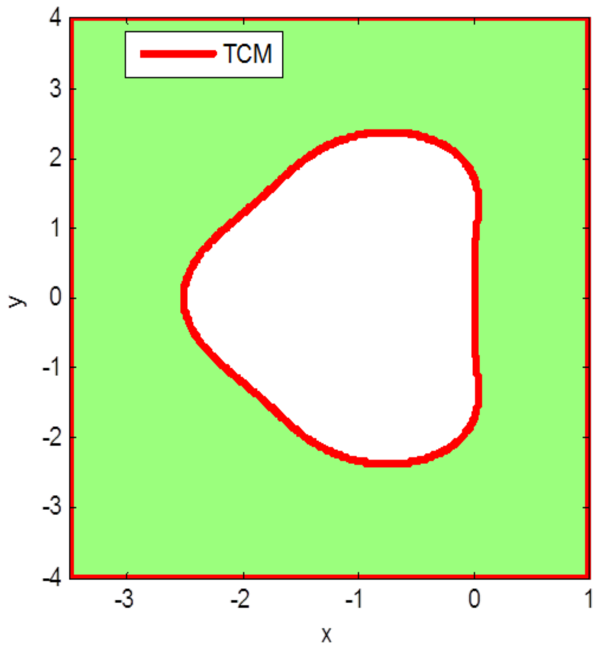

Figure 2. The Stability Region (Un-shaded) for TCM

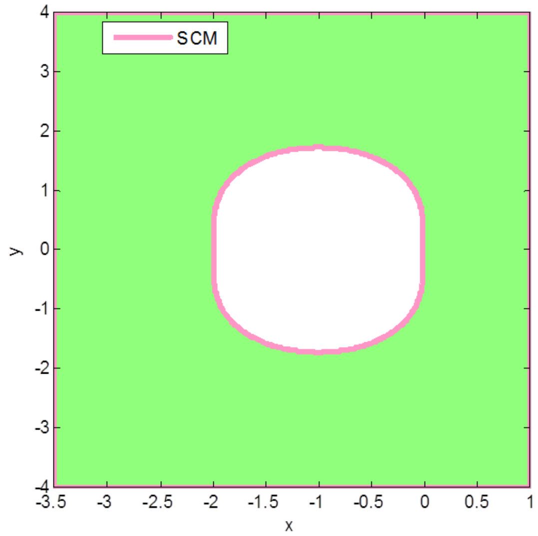

Figure 3. The Stability Region (Un-shaded) for SCM

5. Implementation of the Scheme, Discussion of Results and Concluding Remarks

This section presents an illustrative example, discussion of results and conclusion as follows:

5.1 Implementation of the Scheme on Initial Value Problem of First Order Ordinary Differential Equation

Consider the initial value problem of the form:

$$y' = y, y(0) = 1,$$

whose analytical solution is obtained as: $y(x) = e^x$.

The comparative study of the results generated via FCM, TCM and SCM against exact solution ('YXN') in the interval of integration $x \in [0,2]$ with $h = 0.1$ is shown in Figure 4.

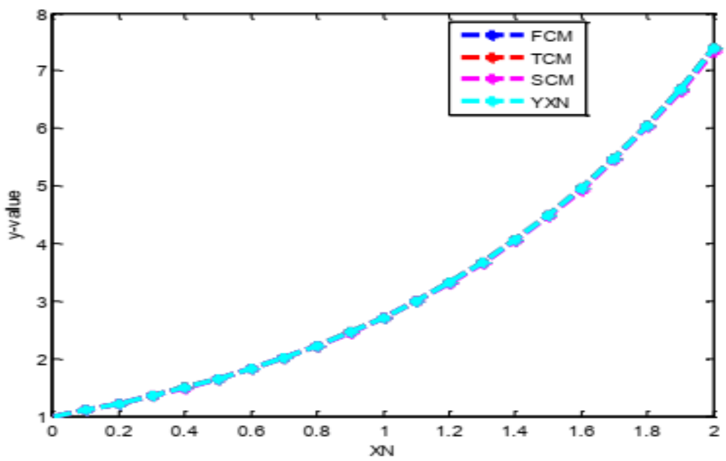

Figure 4. The Comparative Study of the Results generated via FCM, TCM and SCM

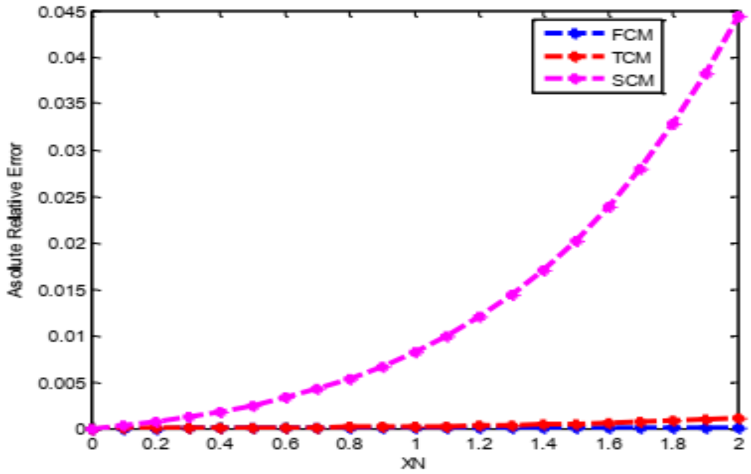

Figure 5. The Comparative Study of the Absolute Relative Errors Incurred via FCM, TCM and SCM

The comparative study of the absolute relative errors generated via FCM, TCM and SCM in the interval of integration $x \in [0,2]$ with $h = 0.1$ is shown in Figure 5. The comparative study of the final absolute relative errors generated via the FCM in the context of TCM and SCM by varying the step length $h =$ with $x \in [0,1]$ is shown in Table 2.

Table 2. The Comparative Study of the Final Absolute Relative Errors generated via FCM, TCM and SCM with varying Step Length (h).

h	**FCM**	**TCM**	**SCM**
2^{-1}	0.00007701	0.01729036	0.13734107
2^{-2}	0.00000298	0.00276121	0.04397290
2^{-3}	0.00000010	0.00039062	0.01247877
2^{-4}	0.00000000	0.00005196	0.00332373
2^{-5}	0.00000000	0.00000670	0.00085754
2^{-6}	0.00000000	0.00000085	0.00021778
2^{-7}	0.00000000	0.00000011	0.00005487
2^{-8}	0.00000000	0.00000001	0.00001377
2^{-9}	0.00000000	0.00000000	0.00000345
2^{-10}	0.00000000	0.00000000	0.00000086

The plots of Table 2 are displayed in Figure 6. It can be observed that the SCM produced the highest absolute relative error as compared to TCM and FCM.

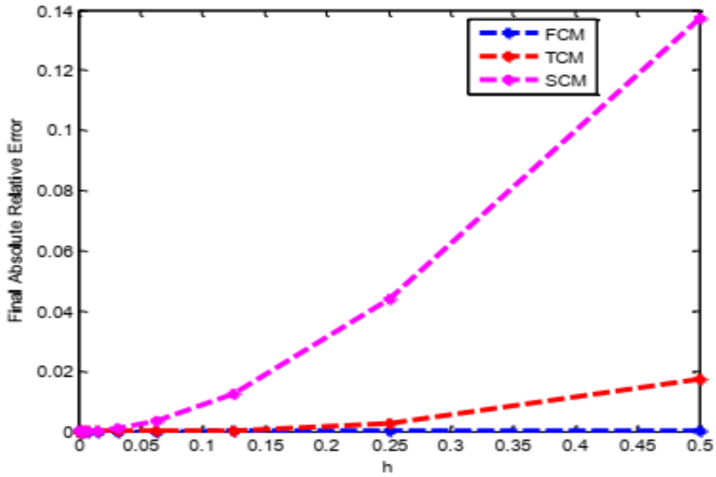

Figure 6. The Final Absolute Relative Errors using Table 2

5.2 Discussion of Results and Concluding Remarks

In this paper, the stability analysis of FCM in the context of TCM and SCM for first order ODEs is presented. The stability functions for FCM, TCM and SCM are captured in Table 1 using the Dahlquist's test equation. The stability regions of FCM, TCM and SCM were plotted in Figure 1, Figure 2, and Figure 3 respectively. The comparative study of the results generated via FCM, TCM and SCM is presented in Figure 4. It is clearly seen from Figure 4 that FCM performs better than TCM and SCM. It is observed from Figure 5 that the absolute relative error curve of FCM shows that the scheme follows the curve of the exact solution elegantly. By varying the step length (h), the accuracy and convergence of the FCM, TCM and SCM in terms of the final absolute relative errors are shown in Table 2. It is also observed from the Figure 6 that FCM is more stable and converges faster to the exact solution than its counterparts for every first order decrease in the step length. Hence, FCM is a good approach to be included in the class of explicit one-step methods for the solution of initial value problems in ODEs. Finally, all the computations were carried out with the aid of MATLAB R2014a, 8.3.0.552, 32 bit (win 32) in double precision. The methodology can be extended for the solution of higher order ordinary differential equations.

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