Simplified Closed-Form Method to Large Deflection Stability Analysis of Thin Rectangular Plate

*E. I. Adah**
Department of Civil and Environmental Engineering,
University of Calabar, Nigeria
**edwardadah@uncial.edu.ng*

O. M. Ibearugbulem
Department of Civil Engineering,
Federal University of Technology Owerri, Nigeria

ABSTRACT

At the moment, the analyses of postbuckling behavior of thin rectangular plates of constant thickness are generally based on von Karman's large deflection equations which make use of stress functions and trigonometric variables. These equations are coupled, nonlinear partial differential equations of fourth order each. Getting a closed-form solution of these equations is near impossible and tedious. The present work presents a simplified closed-form general equation using a variational approach for large deflection analysis. The approach adopted here is devoid of Airy's stress functions. A new strain-displacement equation is formulated, and the total potential energy equation is minimized. The resulting compatibility equation was solved to obtain the general governing stability equation under large deflection. This governing equation was applied to a plate simply supported all-around using polynomial displacement function and numerical results were obtained. To validate the numerical results obtained, they were compared with the values obtained by Levy whose results are generally acclaimed as exact and with two others. It was observed that the minimum and maximum percentage differences were 0% and 41.56% at stress parameters 3.66 and 21.45 respectively. Also, for deflection to thickness ratio (w/t), the present results showed a close agreement with those of Levy with a minimum and maximum percentage difference of 0.07% and 41.56% at w/t of 0 and 3.376 respectively. Importantly, the present result lies between two other research results. We, therefore, conclude that the present work is adequate

ISSN 1823-5514, eISSN 2550-164X
Received for review: 2020-12-17
Accepted for publication: 2021-08-06
Published: 2022-01-15

and a new simple closed-form approach to understanding and predicting the postbuckling strength of thin rectangular plates.

Keywords: *Buckling and Postbuckling Loads; Strain-Displacement Relations; Membrane Strain; Total Potential Energy; Direct Variation*

Introduction

The investigations of postbuckling behavior of thin rectangular plates of constant thickness are based on von Karman's large deflection equation which makes use of Airy's stress functions and trigonometric variables. These equations are coupled, non-linear partial differential equations of fourth order each. Solving these equations for stresses and postbuckling loads is tedious and time consuming. But the present work using a variational approach, simplifies the large deflection analysis of rectangular thin isotropic plates and provides a clear understanding of the postbuckling behavior of plates without many simplifying assumptions in the derivation of the equations, and offers a quick analysis of plates. Moreso, the use of lightweight plates in aerospace and shipbuilding industries is indispensable and their ability to be folded easily to various shapes makes them relevant. This article will enhance quick analysis and provide reliable data for the advancement of knowledge in this industry.

Literature Review

The von Karman's type nonlinear strain-displacement relations are mostly used in large deflection analysis of rectangular plate [1], given here as:

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2} + \left[\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial u_0}{\partial x}\right] \tag{1}$$

$$\varepsilon_{yy} = \frac{\partial u}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} + \left[\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial v_0}{\partial y}\right] \tag{2}$$

The first term is the bending strain while the second term in the square bracket is the membrane strain. *w* is displacement. The complex nature of the membrane strain of the plate in the x or y directions respectively is the major difficulty associated with large deflection analysis of the plate. A search of available works of literature reveals that little effort had been made to obtain equations for middle plane displacement, u_0, and v_0, along the x and y-axis respectively. In the works of [2]–[6] the authors end up assuming a function for u_0 and v_0. Other authors who assumed functions for u_0 and v_0 are [7]-[9]. Also determining Airy's stress function is another difficult challenge in large deflection analysis. Earlier Scholars also assume Airy's stress functions. However, recent scholars such as [10]-[12] determined the stress functions they used in their various works (post-buckling, free vibration, and pure bending analyses of rectangular plates with large deflection respectively). Their approaches were so involving, and the expressions for stress functions were very lengthy and tedious. Due to these difficulties, most works in this area are based on numerical approaches, especially the finite element method. In order to circumvent the use of Airy's stress function and avoid arriving at the same governing equation introduced by von-Karman, this study presents a simplified closed-form approach to the analysis of a rectangular plate with large deflection. It will provide great relief to analysts and designers of plated structures and saves energy.

Methodology

Derivation of the general stability equation of plate under large deflection

Strain-displacement relations of large deflection analysis of rectangular plates are given in Equations (1) and (2). In large deflection of the plate, it is assumed majorly that the middle surface displacements are not zeros. Consider the membrane terms of Equations (1) and (2):

$$\varepsilon_{xxm} = \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 + \frac{\partial u_0}{\partial x} \tag{3}$$

$$\varepsilon_{yym} = \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 + \frac{\partial v_0}{\partial y} \tag{4}$$

Minimizing Equations (3) and (4) by differentiating with respect to $\frac{\partial}{\partial x}$ and $\frac{\partial}{\partial y}$ respectively yields:

$$\frac{\partial \varepsilon_{xxm}}{\partial(\frac{\partial}{\partial x})} = u_0 + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 = 0 \ \rightarrow \ u_0 = -\frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \tag{5}$$

$$\frac{\partial \varepsilon_{yym}}{\partial(\frac{\partial}{\partial y})} = v_0 + \frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 = 0 \;\; \rightarrow \; v_0 = -\frac{1}{2}\left(\frac{\partial w}{\partial y}\right)^2 \tag{6}$$

Since minus is a constant that minimizes Equations (3) and (4), it shows that another constant (which is not minus) shall make Equations (3) and (4) not become zeros. However, there is a need to determine the optimum value of that constant when the plate loses its bending stiffness and carries the load with the help of only membrane resistance. In 2017 [13], Ibearugbulem replaced the minus half with an arbitrary constant to obtain Equations (7) and (8).

$$u_0 = c_1\left(\frac{\partial w}{\partial x}\right)^2 \tag{7}$$

$$v_0 = c_1\left(\frac{\partial w}{\partial y}\right)^2 \tag{8}$$

The membrane strains were obtained by substituting Equations (7) and (8) into Equations (3) and (4) yield:

$$\varepsilon_{xxm} = \left(\frac{\partial w}{\partial x}\right)^2 \left[\frac{1}{2} + c_1\right] = c_2\left(\frac{\partial w}{\partial x}\right)^2 \tag{9}$$

$$\varepsilon_{yym} = \left(\frac{\partial w}{\partial y}\right)^2 \left[\frac{1}{2} + c_1\right] = c_2\left(\frac{\partial w}{\partial y}\right)^2 \tag{10}$$

$$c_2 = \frac{1}{2} + c_1 \tag{11}$$

Membrane stress

Assuming that the axial force that generated the deflected shape is only uniaxial. Then, the bending stress is defined as:

$$\sigma_b = \frac{M}{I} y \tag{12}$$

where M is the moment, y is the distance from the middle surface to the extreme fibre and I is the second moment of area. For rectangular cross-section, Equation (12) becomes:

$$\sigma_b = \frac{6M}{bt^2} \tag{13}$$

where b is the breadth and t is the thickness of the section. Consider Figure 1**,** the moment is:

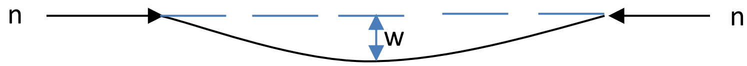

Figure 1: A bent plate with induced in-plane force.

$$M = nt = (\sigma_x\ bt)t = \ \sigma_x\ bt^2 \tag{14}$$

When the plate has lost its bending stiffness and is relying only on membrane resistance, the entire bending stress translates to membrane stress, σ_m. Substituting Equation (14) into Equation (13) yields:

$$\sigma_m = \frac{6\sigma_x\ bt^2}{bt^2} = \ 6\sigma_x \tag{15}$$

Equation (15) is the membrane stress equation.

Strain energy and nonlinear in-plane displacement

The strain energy of a plate is obtained from the membrane strain in Equation (9) and membrane stress in Equation (15) yields:

$$U_m = \frac{1}{2}\ \int_0^a\int_0^b\int_0^z \sigma_m\varepsilon_{xxm}\, dxdydz) = \frac{1}{2}\ \int_0^a\int_0^b\int_0^z c_2\left(\frac{\partial w}{\partial x}\right)^2 6\sigma_x\ \ dxdydz$$
$$= \frac{6\sigma_x\ c_2}{2}\int_0^a\int_0^b\int_0^z\left(\frac{\partial w}{\partial x}\right)^2 dxdydz \tag{16}$$

The external in-plane work at any arbitrary point on the plate per unit length is commonly given as:

$$V_{Nx} = -\frac{\sigma_x}{2}\int_0^a\int_0^b\int_0^z\left(\frac{\partial w}{\partial x}\right)^2 dxdydz \tag{17}$$

Employing the condition of conservation of energy or minimizing the total potential energy arising from the algebraic summation of Equations (16) and (17) gives:

$$c_2 = \frac{1}{6} \tag{18}$$

Substituting Equation (18) into Equation (11) yields:

$$c_1 = -\frac{1}{3} \tag{19}$$

Substituting Equation (19) into Equations (7) and (8) gives:

$$u_0 = -\frac{1}{3}\left(\frac{\partial w}{\partial x}\right)^2 \tag{20}$$

$$v_0 = -\frac{1}{3}\left(\frac{\partial w}{\partial y}\right)^2 \tag{21}$$

Integrating Equations (1) and (2) with respect to x and y respectively yield the nonlinear in-plane displacements:

$$u = -z\frac{\partial w}{\partial x} + \left[\frac{1}{2}\frac{\partial w}{\partial x}^2 + u_0\right] \tag{22}$$

$$v = -z\frac{\partial w}{\partial y} + \left[\frac{1}{2}\frac{\partial w}{\partial y}^2 + v_0\right] \tag{23}$$

Substituting Equations (20) and (21) into Equations (22) and (23) simplifies the nonlinear in-plane displacements as:

$$u = -z\frac{\partial w}{\partial x} + \left[\frac{1}{2}\frac{\partial w}{\partial x}^2 - \frac{1}{3}\frac{\partial w}{\partial x}^2\right] = -z\frac{\partial w}{\partial x} + \frac{1}{6}\frac{\partial w}{\partial x}^2 \tag{24}$$

$$v = -z\frac{\partial w}{\partial y} + \left[\frac{1}{2}\frac{\partial w}{\partial y}^2 - \frac{1}{3}\frac{\partial w}{\partial y}^2\right] = -z\frac{\partial w}{\partial y} + \frac{1}{6}\frac{\partial w}{\partial y}^2 \tag{25}$$

To obtain nonlinear strain displacement relations, differentiate Equations (24) and (25) with respect to x and y respectively to obtain Equations (26) and (27).

$$\varepsilon_{xx} = -z\frac{\partial^2 w}{\partial x^2} + \frac{1}{6}\left(\frac{\partial w}{\partial x}\right)^2 \tag{26}$$

$$\varepsilon_{yy} = -z\frac{\partial^2 w}{\partial y^2} + \frac{1}{6}\left(\frac{\partial w}{\partial y}\right)^2 \tag{27}$$

The in-plane shear strain with the x-y plane is:

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 2\left[-z\frac{\partial^2 w}{\partial x \partial y} + \frac{1}{6}\left(\frac{\partial w}{\partial x}\right)\left(\frac{\partial w}{\partial y}\right)\right] \tag{28}$$

Equations (26) to (28) are the new nonlinear strain-displacement relations. The total potential energy of a thin rectangular plate is given as Equation (29):

$$\begin{aligned}\Pi = \frac{1}{2} \int_0^a \int_0^b \int_0^z \left(\sigma_{xx}\, \varepsilon_{xx} + \sigma_{yy}\, \varepsilon_{yy} + \tau_{xy}\gamma_{xy}\right) dxdydz \\ - \frac{N_x}{2}\int_0^a \int_0^b \left(\frac{\partial w}{\partial x}\right)^2 dxdy\end{aligned} \tag{29}$$

The constitutive relations are:

$$\sigma_x = \frac{E}{1 - v^2}\left(\varepsilon_{xx} + v\varepsilon_{yy}\right);$$

$$\sigma_y = \frac{E}{1 - v^2}\left(\varepsilon_{yy} + v\varepsilon_{xx}\right) \tag{30a, b}$$

$$\tau_{xy} = \frac{E(1 - v)}{2(1 - v^2)}\gamma_{xy} \tag{30c}$$

Substituting Equation (30) into Equation (29) yield Equation (31):

$$\begin{aligned}\Pi = \frac{E}{2(1 - v^2)} \int_0^a \int_0^b \int_0^z \Bigg[\varepsilon_{xx}{}^2 + 2v\varepsilon_{xx}\, \varepsilon_{yy} + \varepsilon_{yy}{}^2 \\ + (1 - v)\frac{\gamma_{xy}{}^2}{2}\Bigg] dxdydz \\ - \frac{N_x}{2}\int_0^a \int_0^b \left(\frac{\partial w}{\partial x}\right)^2 dxdy\end{aligned} \tag{31a}$$

$$\Pi = \frac{E}{2(1-v^2)} \int_0^a \int_0^b \int_0^z \left[\varepsilon_{xx}{}^2 + 2v\varepsilon_{xx}\,\varepsilon_{yy} + \frac{\gamma_{xy}{}^2}{2} - v\frac{\gamma_{xy}{}^2}{2} + \varepsilon_{yy}{}^2 \right] dxdydz - \frac{N_x}{2}\int_0^a \int_0^b \left(\frac{\partial w}{\partial x}\right)^2 dxdy \qquad (31b)$$

Substituting Equations (26), (27), and (28) into Equation (31) and carry out closed domain integration with respect to z yield Equation (32):

$$\Pi = \frac{D}{2}\int_0^a \int_0^b \left[\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + 2\left(\frac{\partial^2 w}{\partial x \partial y}\right)^2 + \left(\frac{\partial^2 w}{\partial y^2}\right)^2\right] dxdy + \frac{gD}{2*36}\int_0^a \int_0^b \left[\left(\frac{\partial w}{\partial x}\right)^4 + 2\left(\frac{\partial w}{\partial x}\right)^2\left(\frac{\partial w}{\partial y}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^4\right] dxdy - \frac{N_x}{2}\int_0^a \int_0^b \left(\frac{\partial w}{\partial x}\right)^2 dxdy \qquad (32a)$$

$$\Pi D = \frac{Et^3}{12(1-v^2)}, \qquad g = \frac{12}{t^2}, \quad gD = \frac{Et}{(1-v^2)} \qquad (33)$$

In non-dimensional parameters,

$$x = aR, \;\; y = bQ, \;\; 0 \le R \le 1, 0 \le R \le 1 \qquad (34)$$

Substituting Equation (34) into Equation (32a) give**:**

$$\Pi = \frac{bD}{2a^3}\int_0^1 \int_0^1 \left[\left(\frac{\partial^2 w}{\partial R^2}\right)^2 + \frac{2}{Ƨ^2}\left(\frac{\partial^2 w}{\partial R \partial Q}\right)^2 + \frac{1}{Ƨ^4}\left(\frac{\partial^2 w}{\partial Q^2}\right)^2\right] dRdQ + \frac{gD}{2a^3*36}\int_0^a \int_0^b \left[\left(\frac{\partial w}{\partial R}\right)^4 + \frac{2}{S^2}\left(\frac{\partial w}{\partial R}\right)^2\left(\frac{\partial w}{\partial Q}\right)^2 + \frac{1}{Ƨ^4}\left(\frac{\partial w}{\partial Q}\right)^4\right] dRdQ - \frac{SN_x}{2}\int_0^1 \int_0^1 \left(\frac{\partial w}{\partial R}\right)^2 dRdQ \qquad (35)$$

where, $\quad Aspect\ ratio \quad Ƨ = \frac{b}{a} \qquad (36)$

Let rewrite Equation (32a) in this form:

$$\Pi = \frac{D}{2}\int_0^a\int_0^b\left[\frac{\partial^3}{\partial x^3}\frac{\partial w^2}{\partial x} + 2\frac{\partial^3}{\partial x\partial y^2}\frac{\partial w^2}{\partial x} + \frac{\partial^3}{\partial y^3}\frac{\partial w^2}{\partial y}\right]dxdy$$
$$+ \frac{gD}{2*36}\int_0^a\int_0^b\left[\frac{\partial^2}{\partial x^2}\left(\frac{\partial w^2}{\partial x}\right)^2 + 2\frac{\partial}{\partial x}\frac{\partial}{\partial x}\frac{\partial}{\partial y}\frac{\partial}{\partial y}(w^2)^2\right.$$
$$\left. + \frac{\partial^2}{\partial y^2}\left(\frac{\partial w^2}{\partial y}\right)^2\right]dxd - \frac{N_x}{2}\int_0^a\int_0^b\left(\frac{\partial w}{\partial x}\right)^2 dxdy \qquad (32b)$$

Minimizing Equation (32b) with respect to w, u_0, v_0 based on the differential part, gives the governing equation and two displacement compatibility equations as presented in Equations (37), (38), and (39) respectively. Minimizing Equation 32b with respect to w gives:

$$\frac{\partial \Pi}{\partial w} = D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\partial y^2} + \frac{\partial^4 w}{\partial y^4}\right]$$
$$+ \frac{gD}{36}\left[\left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial y}\right)^2\frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2\frac{\partial^2 w}{\partial y^2}\right.$$
$$\left. + \left(\frac{\partial w}{\partial y}\right)^2\frac{\partial^2 w}{\partial y^2}\right] - N_x\frac{\partial^2 w}{\partial x^2} = 0 \qquad (37)$$

Minimizing Equation 32b with respect to (dw^2/dx) gives:

$$\frac{\partial \Pi}{\partial\left(\frac{\partial w^2}{\partial x}\right)} = \frac{D}{2}\frac{\partial}{\partial x}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] + \frac{gD}{36}\frac{\partial}{\partial x}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = 0$$

For this equation to be true each term must be zero. Hence, for a nontrivial solution:

$$\frac{gD}{36}\frac{\partial}{\partial x}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = 0$$

Also, for nontrivial solution:

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = 0 \qquad (38)$$

Minimizing Equation 32b with respect to (dw^2/dy) gives:

$$\frac{\partial \Pi}{\partial\left(\frac{\partial w^2}{\partial y}\right)} = \frac{D}{2}\frac{\partial}{\partial y}\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right] + \frac{gD}{36}\frac{\partial}{\partial y}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = 0$$

For this equation to be true each term must be zero. Hence, for a nontrivial solution:

$$\frac{gD}{36}\frac{\partial}{\partial y}\left[\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2\right] = 0$$

Also, for a nontrivial solution:

$$\left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 = 0 \qquad (39)$$

From Equations (38) and (39):

$$\left(\frac{\partial w}{\partial x}\right)^2 = -\left(\frac{\partial w}{\partial y}\right)^2 \qquad (40)$$

The strains of the middle surface of the plate are:

$$\varepsilon_{xo} = \frac{\partial uo}{\partial x} = -\frac{1}{3}\left(\frac{\partial w}{\partial x}\right)^2 \qquad (41)$$

$$\varepsilon_{yo} = \frac{\partial uo}{\partial y} = -\frac{1}{3}\left(\frac{\partial w}{\partial y}\right)^2 \qquad (42)$$

Substituting Equation (40) into Equation (41) yields:

$$\varepsilon_{xo} = \frac{\partial uo}{\partial x} = \frac{1}{3}\left(\frac{\partial w}{\partial y}\right)^2 \qquad (43)$$

Comparing Equation (42) and Equation (43) yields:

$$\varepsilon_{xo} = -\varepsilon_{yo} \qquad (44)$$

Substitute Equation (40) into Equation (37) yields:

$$D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] + \frac{gD}{36}\left[\left(\frac{\partial w}{\partial x}\right)^2 \frac{\partial^2 w}{\partial x^2} - \left(\frac{\partial w}{\partial x}\right)^2 \frac{\partial^2 w}{\partial x^2} + \left(\frac{\partial w}{\partial x}\right)^2 \frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial w}{\partial x}\right)^2 \frac{\partial^2 w}{\partial y^2}\right] - N_x \frac{\partial^2 w}{\partial x^2} = 0$$

That is:

$$D\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right] - N_x \frac{\partial^2 w}{\partial x^2} = 0 \qquad (45)$$

The approximate and non-intractable solution of Equation (45) is in the polynomial form and given as:

$$w = A(a_0 + a_1 R + \frac{a_2}{2!}R^2 + \frac{a_3}{3!}R^3 + \frac{a_4}{4!}R^4)(b_0 + b_1 Q + \frac{b_2}{2!}Q^2 + \frac{b_3}{3!}Q^3 + \frac{b_4}{4!}Q^4) \qquad (46)$$

Equation (46) can be reduced to:

$$w = Ah \qquad (47)$$

where;

$$h = (a_0 + a_1 R + \frac{a_2}{2!}R^2 + \frac{a_3}{3!}R^3 + \frac{a_4}{4!}R^4)(b_0 + b_1 Q + \frac{b_2}{2!}Q^2 + \frac{b_3}{3!}Q^3 + \frac{b_4}{4!}Q^4) \qquad (48)$$

Substitute Equation (47) into Equation (35) yield:

$$\Pi \ = \frac{bDA^2}{2a^3} \int_0^1 \int_0^1 \left[\left(\frac{\partial^2 h}{\partial R^2}\right)^2 + \frac{2}{S^2}\left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 + \frac{1}{S^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2 \right] dRdQ$$
$$+ \ \frac{bgDA^4}{2a^3 * 36} \int_0^1 \int_0^1 \left[\left(\frac{\partial h}{\partial R}\right)^4 + \frac{2}{S^2}\left(\frac{\partial h}{\partial R}\right)^2 \left(\frac{\partial h}{\partial Q}\right)^2 \right.$$
$$\left. + \frac{1}{S^4}\left(\frac{\partial h}{\partial Q}\right)^4 \right] dR - \frac{SN_x \ A^2}{2} \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial R}\right)^2 dRdQ \qquad (49)$$

Minimizing Equation (49) with respect to A yields:

$$\frac{\partial \Pi}{\partial A} \ = \frac{bDA}{a^3} \int_0^1 \int_0^1 \left[\left(\frac{\partial^2 h}{\partial R^2}\right)^2 + \frac{2}{\text{Ƨ}^2}\left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 + \frac{1}{\text{Ƨ}^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2 \right] dRdQ$$
$$+ \ \frac{bgDA^3}{18a^3} \int_0^1 \int_0^1 \left[\left(\frac{\partial h}{\partial R}\right)^4 + \frac{2}{\text{Ƨ}^2}\left(\frac{\partial h}{\partial R}\right)^2 \left(\frac{\partial h}{\partial Q}\right)^2 \right.$$
$$\left. + \frac{1}{\text{Ƨ}^4}\left(\frac{\partial h}{\partial Q}\right)^4 \right] dRdQ - \text{Ƨ}N_x \ A \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial R}\right)^2 dRdQ = 0 \ (50)$$

Multiply Equation (50) by $\frac{a^3}{bD}$:

$$\int_0^1 \int_0^1 \left[\left(\frac{\partial^2 h}{\partial R^2}\right)^2 + \frac{2}{\text{Ƨ}^2}\left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 + \frac{1}{\text{Ƨ}^4}\left(\frac{\partial^2 h}{\partial Q^2}\right)^2 \right] dRdQ$$
$$+ \ \frac{gA^2}{18} \int_0^1 \int_0^1 \left[\left(\frac{\partial h}{\partial R}\right)^4 + \frac{2}{\text{Ƨ}^2}\left(\frac{\partial h}{\partial R}\right)^2 \left(\frac{\partial h}{\partial Q}\right)^2 \right.$$
$$\left. + \frac{1}{\text{Ƨ}^4}\left(\frac{\partial h}{\partial Q}\right)^4 \right] dRdQ - N_x \ \frac{a^2}{D} \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial R}\right)^2 dRdQ$$
$$= 0 \qquad (51)$$

In symbolized form, Equation (51) becomes:

$$\left[k_{bx} + \frac{2k_{bxy}}{\text{Ƨ}^2} + \frac{k_{by}}{\text{Ƨ}^4}\right] + \frac{gA^2}{18}\left[k_{mx} + \frac{2k_{mxy}}{\text{Ƨ}^2} + \frac{k_{my}}{\text{Ƨ}^4}\right] - N_x \ \frac{a^2}{D} \ k_{Nx}$$
$$= 0 \qquad (52)$$

where:

$$k_{bx} = \int_0^1 \int_0^1 \left(\frac{\partial^2 h}{\partial R^2}\right)^2 dRdQ; \qquad k_{bxy} = \int_0^1 \int_0^1 \left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 dRdQ$$

$$k_{by} = \int_0^1 \int_0^1 \left(\frac{\partial^2 h}{\partial Q^2}\right)^2 dRdQ; \qquad k_{mx} = \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial R}\right)^4 dRdQ$$

$$k_{mxy} = \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial R}\right)^2 \left(\frac{\partial h}{\partial Q}\right)^2 dRdQ; \qquad k_{my} = \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial Q}\right)^4 dRdQ$$

$$k_{Nx} = \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial R}\right)^2 dRdQ \qquad (53a-g)$$

Subscripts b and m denote bending and membrane parts respectively. From Equation (52):

$$K_{bT} + \frac{gA^2}{18} K_{mT} - N_x \ \frac{a^2}{D} \ k_{Nx} \ = 0 \qquad (54)$$

where:

$$K_{bT} = \left[k_{bx} + \frac{2k_{bxy}}{Ƨ^2} + \frac{k_{by}}{Ƨ^4}\right]; \ \ K_{mT} = \left[k_{mx} + \frac{2k_{mxy}}{Ƨ^2} + \frac{k_{my}}{Ƨ^4}\right] \qquad (55)$$

Substitute Equation (33) into Equation (54) yield Equation (56):

$$K_{bT} + \frac{2}{3}\left(\frac{A}{t}\right)^2 K_{mT} = 12(1 - v^2) * K_{Nx} \frac{N_x \ a^2}{Et^3} \qquad (56)$$

$$K_{bT} + \frac{2}{3}\left(\frac{A}{t}\right)^2 K_{mT} = 12(1 - v^2) * K_{Nx} \frac{\sigma_x \ a^2}{Et^2} \qquad (57)$$

where;

$$\sigma_x \ = \frac{N_x}{t} \qquad (58)$$

Equation (57) can also be written as Equations (58) and (59):

$$\frac{\sigma_x \ a^2}{Et^2} = \frac{1}{12(1 - v^2)k_{Nx}}\left[K_{bT} + \frac{2}{3}\left(\frac{A}{t}\right)^2 \ K_{mT}\right] \qquad (59) *$$

$$\left(\frac{A}{t}\right)^2 = 18(1 - v^2) * \frac{k_{Nx}}{K_{mT}} * \frac{\sigma_x \ a^2}{Et^2} - \ 1.5 \frac{K_{bT}}{K_{mT}} \qquad (60) *$$

Equation (59) is the general large deflection stability equation for the thin rectangular plate.

Numerical Application to SSSS plates using Polynomial Analysis

The shape profile, h, for SSSS plate is given as:

$$h = h_x * h_y = (R - 2R^3 + R^4)(Q - 2Q^3 + Q^4) \quad (61)$$

$$where, h_x = (R - 2R^3 + R^4); \;\; h_y = (Q - 2Q^3 + Q^4) \quad (62)$$

Evaluation of stiffness are as follows:

$$k_{bx} = \int_0^1 \int_0^1 \left(\frac{\partial^2 h}{\partial R^2}\right)^2 dRdQ = \int_0^1 \left(\frac{\partial^2 h_x}{\partial R^2}\right)^2 \partial R * \int_0^1 {h_y}^2 \partial Q$$
$$= 0.2361905$$

$$k_{bxy} = \int_0^1 \int_0^1 \left(\frac{\partial^2 h}{\partial R \partial Q}\right)^2 dRdQ = \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^2 \partial R * \int_0^1 \left(\frac{\partial h_y}{\partial Q}\right)^2 \partial Q$$
$$= 0.2359184$$

$$k_{by} = \int_0^1 \int_0^1 \left(\frac{\partial^2 h}{\partial Q^2}\right)^2 dRdQ = \int_0^1 {h_x}^2 \partial R * \int_0^1 \left(\frac{\partial^2 h_y}{\partial Q^2}\right)^2 \partial Q$$
$$= 0.23619048$$

$$k_{mx} = \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial R}\right)^4 dRdQ = \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^4 \partial R * \int_0^1 {h_y}^4 \partial Q = 0.001299769$$

$$k_{mxy} = \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial R}\right)^2 \left(\frac{\partial h}{\partial Q}\right)^2 dRdQ$$
$$= \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^2 . {h_x}^2 \partial R * \int_0^1 \left(\frac{\partial^2 h_y}{\partial Q^2}\right)^2 . {h_y}^2 \partial Q$$
$$= 0.000138178$$

$$k_{my} = \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial Q}\right)^4 dRdQ = \int_0^1 {h_x}^4 \partial R * \int_0^1 \left(\frac{\partial^2 h_y}{\partial Q^2}\right)^4 \partial Q$$
$$= 0.001299769$$
$$k_{Nx} = \int_0^1 \int_0^1 \left(\frac{\partial h}{\partial R}\right)^2 dRdQ = \int_0^1 \left(\frac{\partial h_x}{\partial R}\right)^2 \partial R * \int_0^1 {h_y}^2 \partial Q$$
$$= 0.023900227$$

Table 1: Summary of stiffness values

	k_{bx}	k_{bxy}	k_{by}	k_{mx}	k_{mxy}	k_{my}	k_{Nx}
SSSS	0.23619	0.235918	0.2361905	0.0012998	0.000138	0.0012998	0.023900

Substituting these values in Equation (59), yield the results as presented in Table 2.

Results and Discussion

This work has formulated a new membrane stress equation given in Equation (15) which aided in the determination of the constant C_1. Also, the new nonlinear strain-displacement relation has been formulated as given in Equations (26) to (28), which is different from those in the literature. Furthermore, a simplified stress parameter equation has been derived as Equation (59). This is the governing linear/nonlinear stability equation. This equation is unique and applicable to all boundary conditions of plates. At w/t equals zero, the equation reduces to the critical load. Table 1 presents the stiffness values of the plate simply supported all-round (SSSS).

The numerical results obtained from this work for buckling/postbuckling load coefficients for an SSSS plate used as a case study are presented in Table 2. These results are for the deflection to thickness ratio (w/t) of 0 to 4 and aspect ratio (b/a) of 0.5 to 2. This is because it was noticed from the results that the plate claimed to possess some fictitious high strength within the aspect ratio of 0.1 to 0.5. This may not be true and may lead to failure if considered for design. Therefore, for large deflection analysis, the applicable aspect ratio is from 0.5 upward.

Table 2: Numerical Values of Coefficient of Buckling/Postbuckling Load, η, for given values of $\frac{w}{t}$ for SSSS plate

w/t	A/t	$N_x = \eta \frac{D}{a^2}, \mathcal{Z} = \frac{b}{a}, v = 0.3, \quad 1.3 \leq \mathcal{Z} \leq 2.0;$							
		0.5	0.6	0.7	0.8	0.9	1	1.1	1.2
		η							
0	0	246.97	140.97	91.33	64.86	49.32	39.51	32.95	28.36
0.25	2.56	251.21	143.19	92.66	65.75	49.98	40.03	33.39	28.75
0.5	5.12	263.94	149.82	96.65	68.44	51.97	41.61	34.72	29.91
0.75	7.68	285.15	160.88	103.31	72.93	55.28	44.24	36.92	31.84
1	10.24	314.85	176.36	112.62	79.21	59.91	47.92	40.02	34.56
1.25	12.8	353.03	196.27	124.60	87.28	65.87	52.65	43.99	38.04
1.5	15.36	399.69	220.60	139.23	97.14	73.16	58.43	48.85	42.30
1.75	17.92	454.85	249.35	156.53	108.80	81.77	65.27	54.59	47.34
2	20.48	518.48	282.53	176.49	122.25	91.70	73.16	61.22	53.15
2.25	23.04	590.60	320.13	199.11	137.50	102.96	82.10	68.73	59.73
2.5	25.6	671.21	362.12	224.39	154.54	115.54	92.09	77.12	67.09
2.75	28.16	760.30	408.61	252.34	173.37	129.45	103.13	86.40	75.23
3	30.72	857.87	459.48	282.94	194.00	144.69	115.23	96.56	84.14

To validate this mathematical model, the numerical results of this work for SSSS were compared with the works of [3], [10], [14]. The values of the stress parameter and w/t of this present work were compared with those obtained by [3], for Poisson ratio of 0.316 in Tables 3 and 4. The reason for comparing with [3] is that his results have been acclaimed as the exact solution to the von Karman large deflection equation [10], and his pattern of result presentation, that is in terms of stress parameter and w/t, is the same with ours. This makes it easier to compare. The deflection to thickness ratio (w/t) values for the given values of stress parameter $\frac{\sigma_x a^2}{Et^2}$ indicates that the present values agreed closely with those of [3] from the start with a minimum and maximum percentage difference of 0% and 41.56% at stress parameter $(\frac{\sigma_x a^2}{Et^2})$ of 3.66 and 41.56 respectively, and are lower bound to those of [3]. Even though the percentage difference is a bit high at higher values of w/t, this new equation is considered to have predicted adequately the postbuckling behavior of the plate. The divergence may be attributed to the differences in approach and simplifying assumptions made by Levy. Because his results are even higher than the other two research works as well.

Table 3: Deflection for given values of stress parameter

Ƨ = b/a = 1, $\boldsymbol{\nu}$ = 0.316		Present	Levy (1942)	
$\frac{\sigma_x\ a^2}{Et^2}$	A/t	w/t	w/t	% diff.
3.66	0.5833839	0.056971087	0	0
3.72	2.9012933	0.283329422	0.25	13.33177
3.96	6.3817056	0.623213437	0.498	25.14326
4.34	9.5854864	0.936082660	0.743	25.9869
4.87	12.7761437	1.247670287	0.984	26.79576
5.51	15.7919690	1.542184473	1.220	26.40856
6.30	18.8609553	1.841890169	1.450	27.02691
7.22	21.8994409	2.138617278	1.673	27.83128
8.24	24.8374180	2.425529102	1.889	28.40281
9.38	27.7554226	2.710490490	2.101	29.00954
10.61	30.5932245	2.987619581	2.303	29.72729
11.99	33.4920916	3.270712072	2.498	30.93323
13.48	36.3634865	3.551121733	2.687	32.15935
14.97	39.0241734	3.810954430	2.871	32.73962
16.79	42.0463253	4.106086456	3.044	34.89115
18.77	45.1047663	4.404762338	3.212	37.13457
21.45	48.9409851	4.779393077	3.376	41.5697

Also, Figure 1 showed a comparison of values of buckling/postbuckling load coefficient from the present study with those available in works of literature [10][14]. It is observed that the present work lies in between two other results [10][14]. This further proves the adequacy of this new model. It's also indicated that postbuckling load increases as w/t increase. The results further indicated a gradual increase in the strength of a plate beyond the yield point; this agreed with plate behavior under in-plane load, unlike column. The present equation is conservative when compared to those of [3] and [14]. This provides much reliability to its usage. Based on the fact that this model predicts results that are lower than those of [3] and [14], and higher than those of [10], therefore the somehow large percentage difference observed as the w/t increases when compared with [3] is not a matter to worry about, since the predicted post buckling strength of the plate is not above the ones predicted by Levy. Besides the predicted strength by [10] are even much lower than those of this work. The approach used is simple and the governing model applies to all boundary conditions using both polynomial and trigonometric displacement shape functions. The use of the new model is easy and the process can be reproduced easily too. The present approach is devoid of so many simplifying assumptions which makes the results more adequate. This will ease the complex problem of large deflection of plate analysis.

Table 4: Stress Parameter for given values of deflection

Ƨ = b/a = 1, $\boldsymbol{\nu}$ = 0.316		Present	Levy (1942)	
w/t	A/t	$\frac{\sigma_x\ a^2}{Et^2}$	$\frac{\sigma_x\ a^2}{Et^2}$	% diff.
0	0	3.65747186	3.66	0.06907487
0.25	2.56	3.70615428	3.72	0.37219672
0.498	5.09952	3.85064683	3.96	2.76144377
0.743	7.60832	4.08747318	4.34	5.81859033
0.984	10.07616	4.41166462	4.87	9.41140415
1.220	12.4928	4.81681453	5.51	12.5804984
1.450	14.848	5.29514854	6.3	15.9500231
1.673	17.13152	5.83761015	7.22	19.1466737
1.889	19.34336	6.43690401	8.24	21.8822328
2.101	21.51424	7.09577581	9.38	24.3520702
2.303	23.58272	7.78870816	10.61	26.5908749
2.498	25.57952	8.51792801	11.99	28.9580649
2.687	27.51488	9.28124113	13.48	31.1480628
2.871	29.39904	10.07781953	14.97	32.6798962
3.044	31.17056	10.87488319	16.79	35.2299988
3.212	32.89088	11.69353304	18.77	37.7009427
3.376	34.57024	12.53510179	21.45	41.5612970

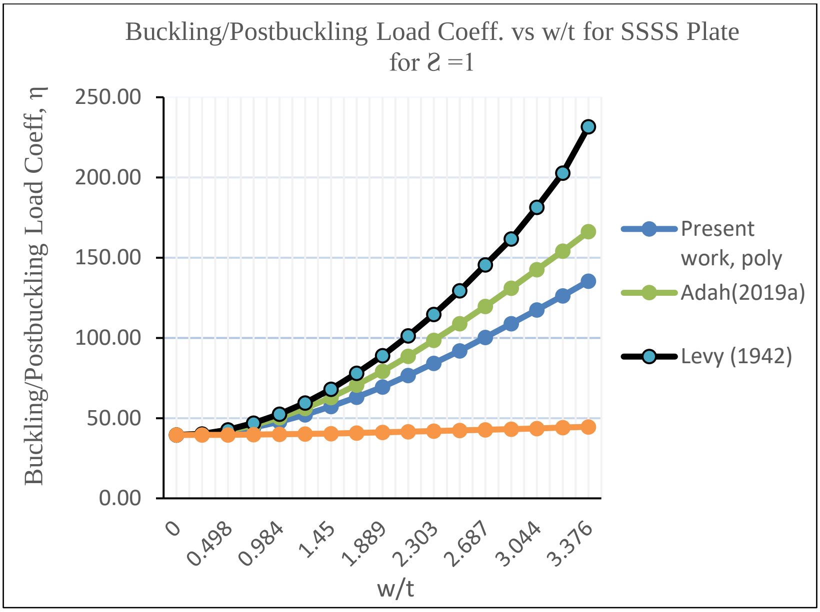

Figure 1: The relationship between buckling/postbuckling load coefficient with w/t for SSSS plate.

Conclusion and Recommendations

From the above analysis, the following conclusions are made.

i. That the new nonlinear strain-displacements relations have been derived. The general governing stability equation under large deflection has been derived too.

ii. That the simplified approach adopted in this work to derive and solved the postbuckling problem is straightforward and devoid of so many assumptions and Airy's stress function. This is a new and different approach from the existing attempts in literature which are based on von Karman's large deflection equation and gives a better understanding of the postbuckling behavior of thin rectangular isotropic plates.

iii. That the derived equation is adequate for predicting the postbuckling strength of rectangular plates and applies to various boundary conditions of the plate and various ductile plate materials.

Therefore, the recommendation that the present general buckling/postbuckling equation for the analysis of thin plates is adequate and should be used for easy, faster, and accurate analysis of postbuckling strength of thin rectangular plates. Further, this simplified approach should be extended to orthotropic and anisotropic plates, as well as for thick plates.

References

[1] M. R. Elsami, “Buckling and Postbuckling of Beams, Plates, and Shells” Springer International Publishing AG. ISBN 978-3-319-62368-9, 2018.
[2] S. Levy, “Square plate with clamped edges under normal pressure producing large deflections” Technical notes: National Advisory Committee for Aeronautics (NACA), N0. 740, 1942.
[3] Levy S, “Bending of rectangular plates with large deflections” Technical notes: National Advisory Committee for Aeronautics (NACA), N0. 846, 1942.
[4] S. Manuel, “Analytical results for postbuckling behavior of plates in compression and in shear” National Aeronautics and Space Administration (NASA), NASA Technical Memorandum 85766 S, 1984.
[5] F. Bloom and D. Coffin, “Thin Plate Buckling and Postbuckling” London: Chapman & Hall/CRC, 2001.
[6] E. Byklum and J. Amdahl, “A simplified method for elastic large deflection analysis of plates and stiffened panels due to local buckling” *Thin-Walled Structures*, vol. 40, no. 11, pp. 925–953, 2002.
[7] H. Tanriöver and E. Senocak, "Large deflection analysis of unsymmetrically laminated composite plates: analytical-numerical type approach", *International Journal of Non-linear Mechanics*, vol. 39, pp. 1385–1392, 2004.
[8] S. A. M. GhannadPour and M. M. Alinia, “Large deflection behavior of functionally graded plates under pressure loads” Composite Structures, vol. 75, pp. 67–71, 2006.
[9] I. Shufrin, O. Rabinovitch and M. Eisenberger, “A semi-analytical approach for the nonlinear large deflection analysis of laminated rectangular plates under general out-of-plane loading” *International Journal of Non-Linear Mechanics*, vol. 43, pp. 328–340, 2008.
[10] O. A. Oguaghamba, “Analysis of buckling and postbuckling loads of Isotropic thin rectangular plates” PhD Thesis: Department of Civil Engineering Federal University of Technology, Owerri Nigeria, 2015.
[11] P. D. Onodagu, “Nonlinear Dynamic Analysis of Thin Rectangular Plates using Ritz Method" PhD Thesis: Department of Civil Engineering Federal University of Technology, Owerri Nigeria, 2018.
[12] J. I. Enem, “Geometrically Nonlinear Analysis of Isotropic Rectangular Thin Plates Using Ritz Method” PhD Thesis: Department of Civil Engineering Federal University of Technology, Owerri Nigeria, 2018.

[13] O. M. Ibearugbulem, Lecture Note on Large deflection analysis of continuums. Department of Civil Engineering. Federal University of Technology, Owerri, Nigeria. Unpublished, 2017.
[14] E. I. Adah, "Buckling and Postbuckling Analysis of SSSS Rectangular Plates using Modified Iyengar's Equation" *IOSR Journal of Mechanical and Civil Engineering*, vol. 16, no. 2, pp. 35-40, 2019.