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A SEMI ANALYTICAL ITERATIVE METHOD FOR SOLVING THE EMDEN-FOWLER EQUATIONS

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Emden-Fowler equations are frequently encountered in physical and natural sciences. However, the Emden-Fowler solution is complicated due to singularity behaviour at the origin. Emden-Fowler equations are solved in this paper using a semi analytical iterative method (SAIM) that combines analytical and numerical computation. This method provides an exact solution to the Emden-Fowler equations, and transformation approaches are implemented to overcome the difficulties associated with singularity behaviour. As a result, the exact solutions of Emden-Fowler equations are successfully obtained using the SAIM. This shows that the SAIM is very effective and reliable, and that no restrictive assumptions for nonlinear terms are required. Furthermore, comparisons with the exact solutions and the solutions obtained by the variational iterative method (VIM) show the efficiency of SAIM in solving equations with singularity.

Keywords: Initial value problems, Emden-Fowler equation, semi analytical iterative method, singularity behaviour

1. Introduction

This article focused on the study of singular initial value problems (IVPs) in second order ordinary differential equations (ODEs). Many problems in mathematical physics and astrophysics can be modelled by the so-called singular IVPs of the Emden-Fowler equation,

$$y'' + \frac{k}{x}y' + af(x)g(y) = h(x), \quad 0 < x, \tag{1}$$

subject to the conditions

$$y(0) = \alpha, \qquad y'(0) = 0, \tag{2}$$

where α is a constant and $f(x)$, $g(y)$ and $h(x)$ are some given function of x and y, respectively. Meanwhile, k is a constant called shape factor where $k > 0$. For this study, we have chosen $k = 2$.

The Emden-Fowler equations were used to model several phenomena in mathematical physics and astrophysics, such as the theory of stellar structure, the thermal behaviour of spherical cloud of gas, isothermal gas sphere and thermionic currents. The solution of Emden-Fowler (1) and (2) is challenging due to singularity behaviour at the origin.

Several approaches were applied to find solutions for this type of equations and to overcome the difficulty of the singular point, such as Adomian decompostion method (ADM) (Wazwaz, 2005; Abdullah Alderremy et al., 2019; Alaqel and Hasan, 2020), variational iterative method (VIM) (Wazwaz, 2007, 2011, 2013, 2015; Wazwaz and Khuri, 2015; Olayiwola, 2019), homotopy analysis method (HAM) (Bataineh et al., 2007, 2009; Al-Hayani et al., 2017), homotopy perturbation method (HPM) (Chowdhury and Hashim, 2007, 2009) etc.

This paper focused on solving three examples of Emden-Fowler equations using a semi analytical iterative method (SAIM). The SAIM, proposed by Temimi and Ansari (2011a,b) in order to overcome the nonlinear problem in differential equations. Many researchers has successfully applied the SAIM for solving linear and nonlinear ordinary and partial differential equations such as the duffing equation

(Al-Jawary and Al-Razaq, 2016), thin flow problem (Al-Jawary, 2017), Fokker-Planck's equation (Al-Jawary et al., 2017), nonlinear ODEs arising in physics (Al-Jawary et al., 2020), chemisty problems (Al-Jawary and Raham, 2017), differential algebraic equations (Al-Jawary and Hatif, 2018), nonlinear Burgers and advection–diffusion equations (Al-Jawary et al., 2018) etc.

Recently Selamat et al. (2019, 2020) applied SAIM to solve two types of Blasius equations and Klein-Gordon equation, respectively. Meanwhile, Latif et al. (2020) approximate the solution of Newell-Whitehead-Segel equation using the SAIM. Comparison of the efficiency and accuracy of SAIM with other methods can be seen in Abed and AL-Jawary (2021) and Al-Jawary et al. (2020).

2. Basic Idea of SAIM

To illustrate the basic idea of SAIM, we consider the following differential equation (Temimi and Ansari, 2011a,b)

$$L(u(x)) + N(u(x)) + g(x) = 0, \tag{3}$$

along with boundary conditions

$$B\left(u, \frac{du}{dx}\right) = 0, \tag{4}$$

where x denotes the independents variable, $u(x)$ is an unknown function, L is a linear operator, N is a nonlinear operator and B, is a boundary operator.The main requirement here is that L be the linear part of the DE but it is possible to take some linear parts and add them to N as needed. By assuming that $u_0(x)$ is an initial guess of the solution to the problem $u(x)$ and is the solution of the equation

$$L(u_0(x)) + g(x) = 0, \qquad B\left(u_0, \frac{du_0}{dx}\right) = 0. \tag{5}$$

To generate the next iterative, we solve the following problem.

$$L(u_1(x)) + g(x) + N(u_0(x)) = 0, \qquad B\left(u_1, \frac{du_1}{dx}\right) = 0. \tag{6}$$

and thus we have a simple iterative procedure which is effectively the solution of a linear set of problems i.e.,

$$L(u_{n+1}(x)) + g(x) + N(u_n(x)) = 0, \qquad B\left(u_{n+1}, \frac{du_{n+1}}{dx}\right) = 0. \tag{7}$$

Then, the solution for the problem (3) with (4) is given by

$$u(x) = \lim_{n\to\infty} u_n(x) \tag{8}$$

3. Emden-Fowler Initial Value Problems

In this section we will study three Emden-Fowler initial value models with singular behaviour at $x = 0$. To overcome the singularity we defined the following transformation formulae:

$$\begin{aligned} u(x) &= xy(x) \\ u_x(x) &= xy_x(x) + y(x) \\ u_{xx}(x) &= xy_{xx}(x) + 2y_x(x) \end{aligned} \tag{9}$$

that will carry out (1) to

$$u_{xx} + \alpha x f(x) g(u/x) = h(x) \tag{10}$$

It is clear that (10) does not include the singularity at $x = 0$ as in (1). Comparing (10) with (3) gives the following

$$\begin{aligned} L(u(x)) &= u_{xx} \\ N(u(x)) &= \alpha x f(x) g(u/x) \\ f(x) &= h(x) \end{aligned} \tag{11}$$

with initial condition

$$u(0) = 0, \qquad u'(0) = \alpha, \tag{12}$$

By using SAIM, the initial problem with initial condition that needs to be solved is

$$L(u_0(x)) = u_{xx}(x) = h(x), \quad u_0(0) = 0, \qquad u_0'(0) = \alpha, \tag{13}$$

where

$$u_0(x) = \int\int h(x)\, dxdx \tag{14}$$

For the second iterative, we need to solve the following equation

$$L(u_1(x)) + N(u_0(x)) = h(x), \quad u_1(0) = 0, \qquad u_1'(0) = \alpha, \tag{15}$$

or,

$$\begin{aligned} u_1(x) &= \int\int -N(u_0(x)) + h(x)\, dxdx \\ &= \int\int -\alpha x f(x) g(u_0(x)/x) + h(x)\, dxdx \end{aligned} \tag{16}$$

The general form of iterative function to the solution that needs to be solved is given by

$$L(u_{n+1}(x)) + N(u_n(x)) = h(x), \quad u_{n+1}(0) = 0, \qquad u_{n+1}'(0) = \alpha, \tag{17}$$

where

$$\begin{aligned} u_{n+1}(x) &= \int\int -N(u_n(x)) + h(x)\, dxdx \\ &= \int\int -\alpha x f(x) g(u_n(x)/x) + h(x)\, dxdx \end{aligned} \tag{18}$$

3.1 Example 1

Consider the homogeneous nonlinear Emden-Fowler equation

$$y'' + \frac{2}{x} y' + \left(3 - x^2\right) y = 0 \tag{19}$$

with initial conditions

$$y(0) = 1, \quad y'(0) = 0. \tag{20}$$

Using the transformation (9) into (19) and (20) gives

$$u''(x) + \left(-x^2 + 3\right) u(x) = 0 \tag{21}$$

$$u(0) = 0, \quad u'(0) = 1. \tag{22}$$

In view of SAIM, we have the following form

$$L(u) = u''(x), \quad N(u) = \left(-x^2 + 3\right) u(x), \quad g(x) = 0. \tag{23}$$

The initial problem that need to be solved is

$$L(u_0(x)) = u''(x) = 0, \quad \text{with} \quad u(0) = 0, \quad u'(0) = 1 \tag{24}$$

The next iterative to the solution can be obtained by solving

$$L(u_{n+1}(x)) = \left(-x^2+3\right) u_n(x) \quad \text{with} \quad u(0)=0, \quad u'(0)=1 \tag{25}$$

Thus, the first few iterative solution are

$$\begin{aligned}
u_0(x) &= x \\
u_1(x) &= x - \frac{x^3}{2} + \frac{x^5}{20} \\
u_2(x) &= x - \frac{x^3}{2} + \frac{x^5}{8} - \frac{x^7}{840} + \frac{x^9}{1440} \\
u_3(x) &= x - \frac{x^3}{2} + \frac{x^5}{8} - \frac{x^7}{48} + \frac{x^9}{420} - \frac{59\,x^{11}}{369600} + \frac{x^{13}}{224640} \\
&\vdots \\
u_n(x) &= x - \frac{x^3}{2} + \frac{x^5}{8} - \frac{x^7}{48} + \frac{x^9}{384} - \frac{x^{11}}{3840} + \cdots
\end{aligned} \tag{26}$$

Recall that,

$$u(x) = \lim_{n\to\infty} u_n \tag{27}$$

that gives the exact solution of (21) by

$$u(x) = xe^{-\frac{x^2}{2}} \tag{28}$$

where the noise terms vanish in the limit.
Using (9) gives the exact solution of (19) by (Wazwaz and Khuri, 2015),

$$y(x) = e^{-\frac{x^2}{2}} \tag{29}$$

3.2 Example 2

We consider the inhomogeneous Emden-Fowler equation

$$y'' + \frac{2}{x}y' - \left(6+4x^2\right) y = \left(6-6x^2-4x^4\right) \tag{30}$$

with initial conditions

$$y(0)=1, \quad y'(0)=0. \tag{31}$$

Using the transformation (9) into (30) and (31) gives

$$u''(x) + \left(6+4x^2\right) u(x) = 6x - 6x^3 - 4x^5 \tag{32}$$

$$u(0)=0, \quad u'(0)=1 \tag{33}$$

In view of SAIM, we distributed (32) as,

$$L(u) = u''(x), \quad N(u) = \left(6+4x^2\right) u(x) - \left(6-6x^3-4x^5\right), \quad g(x)=0, \tag{34}$$

thus the initial problem is

$$L(u_0(x)) = u_0''(x) = 0 \quad \text{with} \quad u(0)=0, \quad u'(0)=1 \tag{35}$$

The subsequent problem can be obtained from the iterative problem generating relation

$$L(u_{n+1}(x)) + \left(6+4x^2\right) u_n(x) - \left(6x-6x^3-4x^5\right) = 0. \tag{36}$$

Then, we obtained the following successive solutions,

$$\begin{aligned}
u_0(x) &= x \\
u_1(x) &= x + 2x^3 - \frac{1}{10}x^5 - \frac{2}{21}x^7 \\
u_2(x) &= x + 2x^3 + \frac{1}{2}x^5 + \frac{17}{210}x^7 - \frac{17}{1260}x^9 - \frac{4}{1155}x^{11} \\
u_3(x) &= x + 2x^3 + \frac{1}{2}x^5 + \frac{1}{6}x^7 + \frac{29}{840}x^9 + \frac{17}{7700}x^{11} + ... \\
u_4(x) &= x + 2x^3 + \frac{1}{2}x^5 + \frac{1}{6}x^7 + \frac{1}{24}x^9 + \frac{367}{46200}x^{11} + ... \\
u_5(x) &= x + 2x^3 + \frac{1}{2}x^5 + \frac{1}{6}x^7 + \frac{1}{24}x^9 + \frac{1}{120}x^{11} + ... \\
&\vdots
\end{aligned} \tag{37}$$

Therefore, we obtained the general equation for finding the next consecutive iteration,

$$u_n(x) = x + 2x^3 + \frac{1}{2}x^5 + \frac{1}{6}x^7 + \frac{1}{24}x^9 + \frac{1}{120}x^{11} + ... \tag{38}$$

or,

$$y_n(x) = 1 + 2x^2 + \frac{1}{2}x^4 + \frac{1}{6}x^6 + \frac{1}{24}x^8 + \frac{1}{120}x^{10} + ... \tag{39}$$

Thus,

$$y_n(x) = x^2 + \sum_{m=0}^{kn} \frac{1}{m!}x^{2m} \tag{40}$$

for some positive integer k. Taking the limit of the latter equation and recalling (8) gives the exact solution of (30) (Wazwaz and Khuri, 2015),

$$y_n(x) = x^2 + e^{x^2}. \tag{41}$$

3.3 Example 3

We next consider the homogeneous Emden-Fowler equation

$$y'' + \frac{2}{x}y' - \left(4x^2 - 6\right)y = 0 \tag{42}$$

with boundary conditions

$$y(0) = 1, \quad y(1) = e^{-1} \tag{43}$$

Using the transformation (9) into (42) and (43) gives

$$u''(x) - \left(4x^2 - 6\right)u(x) = 0 \tag{44}$$

where,

$$L(u) = u''(x), \quad N(u) = 4x^2 - 6, \quad g(x) = 0 \tag{45}$$

with the boundary conditions

$$u(0) = 0, \quad u(1) = e^{-1}. \tag{46}$$

We select $y'(0) = b$, then gives the initial condition $u'(0) = 1$.

Applying the SAIM, yields the following successive solutions

$$\begin{aligned}
u_0(x) &= x \\
u_1(x) &= x - x^3 + \frac{x^5}{5} \\
u_2(x) &= x - x^3 + \frac{x^5}{2} - \frac{13x^7}{105} + \frac{x^9}{90} \\
u_3(x) &= x - x^3 + \frac{x^5}{2} - \frac{x^7}{6} + \frac{4x^9}{105} - \frac{54x^{11}}{11350} + \frac{x^{13}}{3510} \\
u_4(x) &= x - x^3 + \frac{x^5}{2} - \frac{x^7}{6} + \frac{x^9}{24} - \frac{47x^{11}}{5575} + ... \\
u_5(x) &= x - x^3 + \frac{x^5}{2} - \frac{x^7}{6} + \frac{x^9}{24} - \frac{x^{11}}{120} + \frac{2489x^{13}}{1801800} - \frac{1189x^{19}}{6306300} + ... \\
&\vdots
\end{aligned} \tag{47}$$

Therefore, we obtained the $n-th$ solution as

$$u_n(x) = x - x^3 + \frac{x^5}{2} - \frac{x^7}{6} + \frac{x^9}{24} - \frac{x^{11}}{120} + ... \tag{48}$$

or,

$$y_n(x) = 1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6} + \frac{x^8}{24} - \frac{x^{10}}{120} + ... \tag{49}$$

which is equivalent to,

$$y_n(x) = \sum_{m=0}^{kn} \frac{(-1)^m}{m!} x^{2m}, \tag{50}$$

for some positive integer k. By taking the limit for the last equation will gives the exact solution (Wazwaz and Khuri, 2015),

$$y(x) = e^{-x^2} \tag{51}$$

4. Conclusion

In this paper, a semi analytical iterative method has been successfully implemented to obtain the exact solution for initial value problems of Emden-Fowler equations. The results obtained by SAIM are identical with resulted of Wazwaz and Khuri (2015) using VIM. The obtained results reveal that SAIM is simpler and shorter in its computational procedures and time than VIM, where VIM needs to compute the Lagrange multiplier via integration in order to develop the correction functional for the Endem-Fowler equation. This demonstrates the reliability and rapid convergence of the SAIM without any restricted assumptions. However, in this article, the examples only focused on Emden-Fowler equations with shape factor, $k = 2$. For other work, it's recommended to study the model where $k \neq 2$.

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References

Abdullah Alderremy, A., Elzaki, T. M., and Chamekh, M. (2019). Modified adomian decomposition method to solve generalized emden–fowler systems for singular ivp. *Mathematical problems in Engineering*, 2019.

Abed, S. M. and AL-Jawary, M. A. (2021). Efficient iterative methods for solving the sir epidemic model. *Iraqi Journal of Science*, pages 613–622.

Al-Hayani, W., Alzubaidy, L., and Entesar, A. (2017). Analytical solution for the time-dependent emden-fowler type of equations by homotopy analysis method with genetic algorithm. *Applied Mathematics*, 8(5):693–711.

Al-Jawary, M. (2017). A semi-analytical iterative method for solving nonlinear thin film flow problems. *Chaos, Solitons & Fractals*, 99:52–56.

Al-Jawary, M., Adwan, M., and Radhi, G. (2020). Three iterative methods for solving second order nonlinear odes arising in physics. *Journal of King Saud University-Science*, 32(1):312–323.

Al-Jawary, M. and Al-Razaq, S. (2016). A semi analytical iterative technique for solving duffing equations. *International Journal of pure and applied Mathematics*, 108(4):871–885.

Al-Jawary, M. and Hatif, S. (2018). A semi-analytical iterative method for solving differential algebraic equations. *Ain Shams Engineering Journal*, 9(4):2581–2586.

Al-Jawary, M. A., Azeez, M. M., and Radhi, G. H. (2018). Analytical and numerical solutions for the nonlinear burgers and advection–diffusion equations by using a semi-analytical iterative method. *Computers & Mathematics with Applications*, 76(1):155–171.

Al-Jawary, M. A., Radhi, G. H., and Ravnik, J. (2017). Semi-analytical method for solving fokker-planck's equations. *Journal of the Association of Arab Universities for Basic and Applied Sciences*, 24:254–262.

Al-Jawary, M. A. and Raham, R. K. (2017). A semi-analytical iterative technique for solving chemistry problems. *Journal of King Saud University-Science*, 29(3):320–332.

Alaqel, S. and Hasan, Y. Q. (2020). Solutions of emden-fowler type equations of different order by adomian decomposition method. *Adv. Math., Sci. J*, 9:969–978.

Bataineh, A. S., Noorani, M., and Hashim, I. (2007). Solutions of time-dependent emden–fowler type equations by homotopy analysis method. *Physics Letters A*, 371(1-2):72–82.

Bataineh, A. S., Noorani, M. S. M., and Hashim, I. (2009). Homotopy analysis method for singular ivps of emden–fowler type. *Communications in Nonlinear Science and Numerical Simulation*, 14(4):1121–1131.

Chowdhury, M. and Hashim, I. (2009). Solutions of emden–fowler equations by homotopy-perturbation method. *Nonlinear Analysis: Real World Applications*, 10(1):104–115.

Chowdhury, M. S. and Hashim, I. (2007). Solutions of time-dependent emden–fowler type equations by homotopy-perturbation method. *Physics Letters A*, 368(3-4):305–313.

Latif, B., Selamat, M. S., Rosli, A. N., Yusoff, A. I., and Hasan, N. M. (2020). The semi analytics iterative method for solving newell-whitehead-segel equation. *Mathematics and Statistics*, 8(2):87–94.

Olayiwola, M. O. (2019). Solutions of emden-fowler type equations by variational iteration method. *Çankaya Üniversitesi Bilim ve Mühendislik Dergisi*, 16(2):1–9.

Selamat, M. S., Halmi, N. A., and Ayob, N. A. (2019). A semi analytic iterative method for solving two forms of blasius equation. *Journal of Academia*, 7(2):76–85.

Selamat, M. S., Muhidzir, A., Abd Razak, N. N., and Hassanuddin, Z. A. (2020). Semi analytical iterative method for solving klein-gordon equation. *Gading Journal of Science and Technology (e-ISSN: 2637-0018)*, 3(01):10–18.

Temimi, H. and Ansari, A. R. (2011a). A new iterative technique for solving nonlinear second order multi-point boundary value problems. *Applied Mathematics and Computation*, 218(4):1457–1466.

Temimi, H. and Ansari, A. R. (2011b). A semi-analytical iterative technique for solving nonlinear problems. *Computers & Mathematics with Applications*, 61(2):203–210.

Wazwaz, A.-M. (2005). Adomian decomposition method for a reliable treatment of the emden–fowler equation. *Applied Mathematics and Computation*, 161(2):543–560.

Wazwaz, A.-M. (2007). The variational iteration method for solving linear and nonlinear systems of pdes. *Computers & Mathematics with Applications*, 54(7-8):895–902.

Wazwaz, A.-M. (2011). A reliable treatment of singular emden–fowler initial value problems and boundary value problems. *Applied mathematics and computation*, 217(24):10387–10395.

Wazwaz, A.-M. (2013). A reliable iterative method for solving the time-dependent singular emden-fowler equations. *Central European Journal of Engineering*, 3(1):99–105.

Wazwaz, A.-M. (2015). Solving two emden-fowler type equations of third order by the variational iteration method. *Applied Mathematics & Information Sciences*, 9(5):2429.

Wazwaz, A.-M. and Khuri, S. A. (2015). The variational iteration method for solving the volterra integro-differential forms of the lane-emden and the emden-fowler problems with initial and boundary value conditions. *Open Engineering*, 5(1).

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