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Gaussian Integer Solutions of the Diophantine Equation $x^4 + y^4 = z^3$ for $x \neq y$

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The investigation of determining solutions for the Diophantine equation $x^4 + y^4 = z^3$ over the Gaussian integer field, for the specific case of $x \neq y$, is discussed. The discussion includes various preliminary results needed to build the future resolvent theory of the Diophantine equation studied. Our findings show the existence on infinitely many solutions. Since the analytical method used is based on simple algebraic properties, it can be easily generalized to study the behavior and the conditions for existence of solutions to other Diophantine equations, allowing a deeper understanding, even when no general solution is known.

Keywords: Diophantine equation, Gaussian integer, algebraic properties, existence, quartic

1. Introduction

The field of Diophantine equations is ancient, vast, and no general method exists to decide whether a given Diophantine equation has any solution or how many. Many studies were conducted in the past on solving equations in the field of Gaussian integers. For example, Szabó (2004) investigated some fourth-degree Diophantine equations in Gaussian integers, stating that for certain choices of the coefficients a, b, c, the solutions of the Diophantine equation $ax^4 + by^4 = cz^2$ in Gaussian integers satisfy $xy = 0$. Apart from that, Najman (2010) showed that the equation $x^4 \pm y^4 = iz^2$ has only trivial solutions in Gaussian integers. Then, Emory (2012) showed that nontrivial quadratic solutions exist for $x^4 + y^4 = d^2z^4$ when either $d = 1$ or d is a congruent number. Moreover, Izadi et al. (2015) examined solutions in the Gaussian integers for different choices of a, b and c for the Diophantine equation $ax^4+by^4 = cz^2$. The same author, Izadi et al. (2018), then examined a class of fourth-power Diophantine equations of the form $x^4+kx^2y^2+y^4 = z^2$ and $ax^4+by^4 = cz^2$ in the Gaussian integers, where a and b are prime integers. In recent years, Söderlund (2020) discovered that the only primitive non-zero integer solutions to the Fermat quartic $34x^4 + y^4 = z^4$ are $(x, y, z) = (\pm 2, \pm 3, \pm 5)$. The proofs are based on a previously given complete solution to another Fermat quartic, namely $x^4 + y^4 = 17z^4$. Moreover, Jakimczuk (2021) investigated the equation $x^4 - y^4 = z^s$, and showed that if s is an odd prime, then the equation has infinitely many solutions (x, y, z) where $x > y > 0$ and $z > 0$.

In this paper we carry on the investigation of determining solutions for the Diophantine equation $x^4 + y^4 = z^3$ over the Gaussian integer field for the specific case of $x \neq y$, which has remained unsolved. Note that the case $x = y$ has been solved in Ismail et al. (2021).

2. Results and discussion

In this section, we will use elementary algebraic methods to study the behavior of the Diophantine equation $x^4 + y^4 = z^3$ when $x \neq y$. Our interest is to determine which conditions give us non-trivial solutions and which ones produce no solutions of only trivial ones. For simplicity, we will focus only on non-trivial solutions.

The ensuing discussion is supported by the following analysis. Suppose that (a, b, c) is a solution of

$$x^4 + y^4 = z^3 \tag{1}$$

such that $a \neq b$, and $a, b, c \in \mathbb{Z}[i]$. Let

$$a = r + si, \qquad b = t + mi \quad \text{and} \quad c = g + hi, \tag{2}$$

where $r, s, t, m, g, h \in \mathbb{Z}$, and $r \neq t$ or $s \neq m$. Then, replacing on (1), we have

$$\big(r^4 + t^4 - 6(r^2s^2 + t^2m^2) + s^4 + m^4\big) + 4(r^3s - rs^3 + t^3m - tm^3)i = (g^3 - 3gh^2) + (3g^3h - h^3)i,$$

which in turn implies that

$$r^4 + t^4 - 6(r^2s^2 + t^2m^2) + s^4 + m^4 = g^3 - 3gh^2, \tag{3}$$

$$4(r^3s - rs^3 + t^3m - tm^3) = 3g^2h - h^3. \tag{4}$$

Starting from (3) and (4), we will divide our study into four main cases based on possible values for r and s. Each of these cases will the be subdivided into four subcases based in possible values for t and m. Finally, we will further subdivide these into four possibilities based on values for g and h.

Case 1. **(**$r = 0$**,** $s = 0$**,** $t = 0$**,** $m \neq 0$**,** $g \neq 0$**,** $h = 0$**)**
From (3) we have $m^4 = g^3$. (Notice that (4) is automatically satisfied under this case.) It follows that $|m| = g^{3/4}$, with m an integer. This implies that $g = u^4$ for some integer u. Thus, $|m| = |u|^3$, or, equivalently, $m = u^3$. Hence, $(m, g) = (u^3, u^4)$. By letting $u = \pm 1, \pm 2, \pm 3, \ldots, \pm k, \ldots$, where k is an integer, we obtain infinitely many solutions for (m, g). In turn, this leads us to infinitely many solutions for (a, b, c) of the form

$$\boxed{(a, b, c) = (0, n^3 i, n^4)}.$$

Case 2. **(**$r = 0$**,** $s = 0$**,** $t \neq 0$**,** $m \neq 0$**,** $g \neq 0$**,** $h = 0$**)**
From (4), we obtain $4(t^3m - tm^3) = 0$, which we can rewrite as $4tm(t^2 - m^2)$. Since $t, m \neq 0$, we must have $t^2 - m^2 = 0$, which implies $|t| = |m|$ or, equivalently, $t = \pm m$. Upon replacing on (3) we obtain

$$-4m^4 = g^3. \tag{5}$$

We can clearly see that $g < 0$ and $2 \mid g$. Then, let

$$g = -2^{\alpha} v, \tag{6}$$

where $v \wedge 2 = 1$. Replacing on (5) yields $-4m^4 = -2^{3\alpha}v^3$, which implies

$$|m| = 2^{\frac{3\alpha - 2}{4}} v^{\frac{3}{4}}. \tag{7}$$

Since m is an integer, then $3\alpha \equiv 2 \pmod 4$, which is an equation whose only solutions are of the form $\alpha = 4k + 2$ for $k \in \mathbb{Z}$. On the other hand, once again due to m being an integer, there must exist an integer u such that $v = u^4$. Then, replacing on (6), we have $g = -2^{4k+2}u^4$, and replacing on (7), we have $|m| = 2^{3k+1}|u|^3$.

Therefore, this case leads to $(t, m, g) = (\pm 2^{3k+1}u^3, \pm 2^{3k+1}u^3, -2^{4k+2}u^4)$ and $(t, m, g) = (\pm 2^{3k+1}u^3, \mp 2^{3k+1}u^3, -2^{4k+2}u^4)$. In turn, this lead to

$$\boxed{(a, b, c) = \big(0, 2^{3k+1}u^3(1 \pm i), -2^{4k+2}u^4\big)},$$

for $k, u \in \mathbb{Z}$ and $k > 0$.

Case 3. **(**$r = 0$**,** $s \neq 0$**,** $t = 0$**,** $m \neq 0$**,** $g \neq 0$**,** $h = 0$**)**
From (3), we obtain $s^4 + m^4 = g^3$. (Under these conditions, (4) is automatically satisfied.) Since s, m and g are all integers, from Theorem 1.2 and Theorem 1.3 in Ismail and Mohd Atan (2013), the triplet $(x, y, z) = (s, m, g)$ is a solution to the equation $x^4 + y^4 = z^3$ if and only if $s = m = 4n^3$ and

$g = 8n^4$ (which contradicts the hypothesis that $a \neq b$), or $s = un^{3k-1}$, $m = vn^{3k-1}$ and $g = n^{4k-1}$, where $n = u^4 + v^4$ and for any integer k. It follows from (2) that

$$\boxed{(a, b, c) = (un^{3k-1}i, vn^{3k-1}i, n^{4k-1})},$$

where $u \neq v$.

Case 4. ($r = 0$, $s \neq 0$, $t \neq 0$, $m \neq 0$, $g \neq 0$, $h = 0$)
From (3) and (4) we obtain

$$t^4 - 6t^2m^2 + m^4 + s^4 = g^3 \tag{8}$$

$$4t^3m - 4tm^3 = 0, \tag{9}$$

respectively. We can rewrite (9) as $4tm(t^2 - m^2) = 0$. Since $t, m \neq 0$, we must have $|t| = |m|$. Substituting in (8) yields

$$s^4 - 4m^4 = g^3. \tag{10}$$

There are two possibilities to be considered here:

(i) $|s| = |m|$,

(ii) $|s| \neq |m|$.

Under (i) we have the following theorem, which states the form of solutions to the equation $s^4 4m^4 = g^3$ when $|s| = |m|$.

Theorem 1. *The solutions to the equation $x^4 - 4y^4 = z^3$, when $|x| = |y|$, are given by $x = s$, $y = m$ and $z = g$, where*

$$(s, m, g) = (9n^3, 9n^3, -27n^4),$$
$$(s, m, g) = (9n^3, -9n^3, -27n^4).$$

Proof. Let $x = s$ and $y = m$ such that $s = m$, and let $z = g$ be a solution to $x^4 - 4y^4 = z^3$. We see that

$$-3m^4 = g^3. \tag{11}$$

This clearly implies that $g \equiv 0 \pmod 3$ and g is negative. Let $g = -3^e u$, where $3 \wedge u = 1$ and $e > 1$. Thus, from (11) wee that

$$-3m^4 = -3^{3e}u^3,$$

from which we obtain

$$|m| = 3^{\frac{3e-1}{4}} u^{\frac{3}{4}} \quad \text{or} \quad m = \pm 3^{\frac{3e-1}{4}} u^{\frac{3}{4}}. \tag{12}$$

Since m is an integer, we must have that $\frac{3e-1}{4}$ is an integer and there exists an integer v such that $u = v^4$. Thus, $3e - 1 \equiv 0 \pmod 4$, which on simplifying gives $e = 3 + 4j$ for some integer j. It follows from (12) that

$$m = \pm 3^{2+3j} v^3. \tag{13}$$

By (11) and (13), we obtain $g^3 = -3(3^{2+3j}v^3)^4$, which on simplifying gives $g = -3^3(3^j v)^4$. Let $n = 3^j v$. Then, we will have $g = -27n^4$, which from (13) gives $m = \pm 9n^3$. Therefore, $s = \pm 9n^3$. Hence, considering that $|s| = |m|$ (or $s = \pm m$), we have

$$(s, m, g) = (9n^3, 9n^3, -27n^4),$$
$$(s, m, g) = (9n^3, -9n^3, -27n^4),$$

as asserted, with $n = 3^j v \in \mathbb{Z}$. $\square$

Now, remembering that $|t| = |m|$, we have the following solutions for the system (8)–(9) under the condition $|s| = |m|$:

$$(s, t, m, g) = (9n^3, 9n^3, \pm 9n^3, -27n^4),$$
$$(s, t, m, g) = (9n^3, -9n^3, \pm 9n^3, -27n^4).$$

This, in turn, gives us the following solutions to our original Diophantine equation:

$$\boxed{(a, b, c) = \left(9n^3 i, 9n^3(1 \pm i), -27n^4\right)},$$
$$\boxed{(a, b, c) = \left(9n^3 i, -9n^3(1 \pm i), -27n^4\right)}.$$

Next, under (ii), we will show that (10) has no solutions when $|s| \neq |m|$. First, we state the following result.

Lemma 1. *Let u and v be integers such that $u \wedge v = 1$, and let $(u^2 - 2v^2) \wedge (u^2 + 2v^2) = d$. We have that if u is odd, then $d = 1$; if u is even, then $d = 2$.*

Proof. Let $(u^2 - 2v^2) \wedge (u^2 + 2v^2) = d$. Then, there exist s and t such that

$$u^2 - 2v^2 = ds \quad \text{and} \quad u^2 + 2v^2 = dt.$$

Suppose first that u is odd. Then, d is odd since both $u^2 - 2v^2$ and $u^2 + 2v^2$ are odd. Also,

$$2u^2 = d(s + t) \quad \text{and} \quad 4v^2 = d(t - s).$$

Since $d \wedge 2 = 1$, we must have that $d \mid u^2$ and $d \mid v^2$. We conclude that $d = 1$ since $u \wedge v = 1$.

Suppose next that u is even. Let $u = 2^e w$, where e is a positive integer and $2 \wedge w = 1$. Then,

$$u^2 - 2v^2 = (2^e w)^2 - 2v^2 \quad \text{and} \quad u^2 + 2v^2 = (2^e w)^2 + 2v^2,$$

from which we see that

$$u^2 - 2v^2 = 2(2^{2e-1}w^2 - v^2) \quad \text{and} \quad u^2 + 2v^2 = 2(2^{2e-1}w^2 + v^2).$$

Now, since $u \wedge v = 1$, it follows that v is odd and $w \wedge v = 1$, and by a similar method as above, it can be proved that

$$(2^{2e-1}w^2 - v^2) \wedge (2^{2e-1}w^2 + v^2) = 1.$$

Thus, we can clearly see that

$$(u^2 - 2v^2) \wedge (u^2 + 2v^2) = 2(2^{2e-1}w^2 - v^2) \wedge 2(2^{2e-1}w^2 + v^2) = 2.$$

Therefore, $(u^2 - 2v^2) \wedge (u^2 + 2v^2) = 1$ when u is odd, and $(u^2 - 2v^2) \wedge (u^2 + 2v^2) = 2$ when u is even, as asserted. □

We now have the following lemma which states the nonexistence of solutions for (10) under certain particular conditions.

Lemma 2. *There are no integer solutions to $x^4 - 4y^4 = z^3$ such that $x \wedge y = 1$, x is odd, and $y \neq 0$.*

Proof. Suppose there exist integers u, v and g such that $u^4 - 4v^4 = g^3$, with $u \wedge v = 1$, u odd, and $v \neq 0$. Then,

$$(u^2 - 2v^2)(u^2 + 2v^2) = g^3.$$

Since u is odd, by Lemma 1 we have $(u^2 - 2v^2) \wedge (u^2 + 2v^2) = 1$, so $(u^2 + 2v^2)$ and $(u^2 - 2v^2)$ are coprime factors of g^3. Let $g = ab$, such that $u^2 + 2v^2 = a^3$ and $u^2 - 2v^2 = b^3$. Then $a \wedge b = 1$. We can readily see that

$$a^3 + b^3 = 2u^2, \tag{14}$$

$$a^3 - b^3 = 4v^2. \tag{15}$$

From Cohen (2002), given that $a \wedge b = 1$, equation (14) has only two disjoint parameterized solutions where u is odd, according to the following cases (up to exchange of u and v).

(a) For $s, t \in \mathbb{Z}$ such that $s \wedge t = 1$, s is odd and $s \not\equiv t \pmod 3$,

$$\begin{cases} a = (s^2 + 2t^2)(5s^2 + 8ts + 2t^2), \\ b = -(s^2 + 4ts - 2t^2)(3s^2 + 4ts + 2t^2), \\ u = \pm(s^2 - 2ts - 2t^2)(7s^4 + 20ts^3 + 24t^2s^2 + 8t^3s + 4t^4). \end{cases}$$

By replacing in (15), we have

$$v^2 = 2s(19s^4 - 4s^3t + 8st^3 + 4t^4)(s^4 + 4s^3t + 16s^2t^2 + 24st^3 + 12t^4)(s^2 + st + t^2)(s + 2t). \tag{16}$$

Since v is an integer, at least one of the parameterized factors in (15) must be even. We can readily see that $s^2 + st + t^2$ must be even, since the remaining factors are odd. Upon rewriting $s^2 + st + t^2 = s^2 + t(s + t)$, we can see that s and $t(s + t)$ have the same parity. Thus, $t(s + t)$ should be odd, implying that t and $t + s$ are odd. However, this is not possible since $t + s$ would then be the sum of two odd numbers, making it even. This is a contradiction, so this case has no solutions.

(b) For $s, t \in \mathbb{Z}$ such that $s \wedge t = 1$, $s \not\equiv t \pmod 2$ and $3 \nmid t$,

$$\begin{cases} a = (3s^2 - 2ts + t^2)(3s^2 + 6ts + t^2), \\ b = (3s^2 - 6ts + t^2)(3s^2 + 2ts + t^2), \\ u = \pm(3s^2 - t^2)(9s^4 + 18t^2s^2 + t^4). \end{cases}$$

By replacing in (15), we have

$$v^2 = 2st(81s^4 - 6s^2t^2 + t^4)(3s^4 - 2s^2t^2 + 3t^4)(3s^2 + t^2). \tag{17}$$

First, we will prove that all the parameterized factors of (17) are pairwise coprime. Indeed, we know that $s \wedge t = 1$, and it is evident that s does not divide any other parameterized factors, nor does t. Then, we only need to prove that the parenthesized factors are pairwise coprime. As an example, we will prove the second equality; the other two have similar proofs.

Let $d = (81s^4 - 6s^2t^2 + t^4) \wedge (3s^2 + t^2)$, and suppose $d \neq 1$. Then, there exist integers α and β such that

$$81s^4 - 6s^2t^2 + t^4 = d\alpha, \tag{18}$$

$$3s^2 + t^2 = d\beta. \tag{19}$$

Let us notice that (19) and the hypotheses impose certain restrictions on d. Indeed, we must have $2 \nmid d$, because the left-hand-side of the equation is odd; also, $3 \nmid d$ because $3 \nmid t$; finally, $d \nmid t^2$ because otherwise we would have $d \mid s^2$, which contradicts that $s \wedge t = 1$.

By multiplying (19) by $-27s^2$ and adding (19), we have

$$t^2(t^2 - 33s^2) = d(\alpha - 27s^2\beta).$$

From this, we must have $d \mid (t^2 - 33s^2)$. Then, there exists an integer γ such that $t^2 - 33s^4 = d\gamma$. From this and (19), we have $36s^2 = d(\beta - \gamma)$, which leads to a contradiction in light of the restrictions imposed by (19). Thus, we must have $d = 1$ in this case.

Therefore, the parameterized factors on (17) are pairwise coprime. We must conclude that all those factors are squares, except for the one that is even, i.e, s or t. In particular, we must have that

$$3s^4 - 2s^2t^2 + 3t^4 = r^2 \tag{20}$$

for some integer r. Since $s \not\equiv t \pmod 2$, there exists an integer k such that $s - t = 2k + 1$ or, equivalently, $s = 2k + t + 1$. By replacing on (20), we obtain

$$\begin{aligned} r^2 &= 48k^4 + 96k^3t + 64k^2t^2 + 16kt^3 + 4t^4 + 96k^3 + 144k^2t + 64kt^2 \\ &\quad + 8t^3 + 72k^2 + 72kt + 16t^2 + 24k + 12t + 3. \end{aligned}$$

We can readily see that the left-hand-side of this equation has the form $4n + 3$, for some integer n, i.e., $4n + 3 = r^2$. However, $r^2 \not\equiv 3 \pmod 4$ for all $r \in \mathbb{Z}$, which leads to a contradiction, so this case has no solutions.

Therefore, we conclude that there are no integer solutions to $x^4 - 4y^4 = z^3$ with $x \wedge y = 1$, x odd, and $y \neq 0$. □

We now prove the following result, which states the nonexistence of solutions to (10) when $x \wedge y = 1$ and x is even. Notice that these conditions automatically imply that $y \neq 0$. Thus, this result is "complementary" to the previous lemma in the sense that we are considering exactly the same hypothesis, except for the fact that x is now even.

Lemma 3. *There are no integer solutions to $x^4 - 4y^4 = z^3$ with $x \wedge y = 1$ and x even.*

Proof. Suppose $x = u$, $y = v$ and $z = g$ satisfy the equation $x^4 - 4y^4 = z^3$, with $u \wedge v = 1$ and u an even integer. By lemma 1, we have

$$(u^2 - 2v^2) \wedge (u^2 + 2v^2) = 2.$$

Therefore, we must have

$$\left(\frac{u^2 - 2v^2}{2}\right) \wedge \left(\frac{u^2 + 2v^2}{2}\right) = 1.$$

Let $u = 2^e w$, with $e \geq 1$ and $2 \wedge w = 1$. Then,

$$(2^e w)^4 - 4v^4 = g^3,$$

from which

$$\left((2^e w)^2 + 2v^2\right)\left((2^e w)^2 - 2v^2\right) = g^3.$$

That is,

$$4\left(2^{2e-1}w^2 + v^2\right)\left(2^{e-1}w^2 - v^2\right) = g^3. \tag{21}$$

It can be clearly seen that g is even. Hence, let

$$g = 2^f p_1^{e_1} p_2^{e_2} \dots p_k^{e_k}.$$

be the prime power decomposition of g, where $2 \wedge p_i = 1$ for $i = 1, 2, \dots, k$, and $f > 0$. Dividing both sides of (21) by 4 we obtain

$$\left(2^{2e-1}w^2 + v^2\right)\left(2^{e-1}w^2 - v^2\right) = 2^{3f-2} p_1^{3e_1} p_2^{3e_2} \dots p_k^{3e_k}. \tag{22}$$

Since $f > 0$, then $3f - 2 \geq 1$. Hence, the integer on the right-hand-side of (22) is even. Clearly, since v is odd, both factors on the left-hand-side of (22) are odd, and thus the integer of the right-hand-side should also be odd. Therefore, we have a contradiction. We conclude that there are no integer solutions $x = u$, $y = v$ and $z = g$ to the equation $x^4 - 4y^4 = z^3$ such that $u \wedge v = 1$ and u is even. □

The following result shows the nonexistence of solutions to (10) such that s and m are coprime.

Lemma 4. *There exist no integer solutions to the equation $x^4 - 4y^4 = z^3$ with $x \wedge y = 1$ and $y \neq 0$.*

Proof. Direct consequence of Lemma 2 and Lemma 3. □

Finally, we have the following theorem which states the nonexistence of solutions when $|x| \neq |y|$, i.e., the main result for (ii).

Theorem 2. *The equation $x^4 - 4y^4 = z^3$ has no integer solutions with $|x| \neq |y| \neq 0$.*

Proof. We will prove by contradiction. Suppose there exists a solution $x = s$, $y = m$ and $z = g$ to this equation, with $|x| \neq |y| \neq 0$. We then have $s^4 - 4m^4 = g^3$ with $s \neq m$. Let $d = s \wedge m$, $u = \frac{s}{d}$ and $v = \frac{m}{d}$. Then $u \wedge v = 1$ and $v \neq 0$. Since $d \mid s$ and $d \mid m$, we have $d^4 \mid g^3$. That is,

$$u^4 - 4v^4 = \frac{g^3}{d^4}, \tag{23}$$

where $\frac{g^3}{d^4}$ is an integer. Let $w = \frac{g^3}{d^4}$. Then, $wd^4 = g^3$, and thus $g = w^{\frac{1}{3}} d^{\frac{4}{3}}$. Since g is an integer, there exist h and k such that $w = h^3$ and $d = k^3$. Replacing in (23), we have $u^4 - 4v^4 = h^3$. Thus, (u, v, h) is a solution to the equation $x^4 - 4y^4 = z^3$ with $u \wedge v = 1$. This contradicts Lemma 4.

Therefore, we conclude there are no integer solutions $x = s$, $y = m$ and $z = g$ to the equation $x^4 - 4y^4 = z^3$ with $|s| \neq |m|$. □

Corollary 1. *The equation $x^4 - 4y^4 = z^3$ has integer solutions with $|x| \neq |y|$ if and only if $y = 0$. In that case, $x = n^3$ and $y = n^4$ for $n \in \mathbb{Z}$.*

Proof. Notice that, given the assertion of the previous theorem, it is enough to prove that there exist solutions when $y = 0$. Indeed, suppose $y = 0$. Then, $x^4 = z^3$, which implies

$$x = z^{\frac{3}{4}}. \tag{24}$$

Since x is an integer, there exists an integer n such that $z = n^4$. Replacing in (24), we have $x = n^3$. □

Remark 1. Although the previous corollary shows there exist solutions for (10) with $|s| \neq |m|$, we do not need to consider them under the context of the case we are currently studying (i.e., **Case 2.4.3**), because one of the corresponding conditions is $m \neq 0$.

2.1 Symmetrical cases

The cases we have studied in the previous discussion were carefully chosen as representatives of a simple algebraic analysis that can be performed on any Diophantine equation. There exist other cases that we have left out, whose solutions can be readily found by exploiting symmetries in their equations. In order to avoid redundancy, we have left out these case until now. Here, we present a small summary of the results that are obtained when the symmetric equations are exploited, and we further apply our method to solve them. Table 1 shows the results of this procedure.

3. Conclusions

In this work, we have studied the algebraic properties of the Diophantine equation $x^4 + y^4 = z^4$ in Gaussian integers, for $x \neq y$. Our main focus has been on studying some of the conditions that give rise to non-trivial solutions, and study their particular forms. Our findings show the existence on infinitely many solutions.

Since the analytical method we used in this study is based on simple algebraic properties, it can be easily generalized to study the behavior and the conditions for existence of solutions to other Diophantine equations, allowing a deeper understanding, even when no general solution is known.

Future work on this subject will be focused on obtaining a general solution to the equation.

Table 1: Solutions to cases considering symmetrical cases.

Conditions	Symmetrical to	Solutions (a, b, c)
$r = 0, s = 0, t \neq 0, m = 0, g \neq 0, h = 0$	Case 1	$(0, n^3, n^4)$
$r = 0, s \neq 0, t = 0, m = 0, g \neq 0, h = 0$	Case 1	$(n^3 i, 0, n^4)$
$r \neq 0, s = 0, t = 0, m = 0, g \neq 0, h = 0$	Case 1	$(n^3, 0, n^4)$
$r \neq 0, s \neq 0, t = 0, m = 0, g \neq 0, h = 0$	Case 2	$(2^{3k+1}u^3(1 \pm i), 0, -2^{4k+2}u^4)$
$r = 0, s \neq 0, t \neq 0, m = 0, g \neq 0, h = 0$	Case 3	$(4n^3 i, 4n^3, 8n^4)$ $(un^{3k-1}i, vn^{3k-1}, n^{4k-1})$
$r \neq 0, s = 0, t = 0, m \neq 0, g \neq 0, h = 0$	Case 3	$(4n^3, 4n^3 i, 8n^4)$ $(un^{3k-1}, vn^{3k-1}i, n^{4k-1})$
$r \neq 0, s = 0, t \neq 0, m = 0, g \neq 0, h = 0$	Case 3	$(un^{3k-1}, vn^{3k-1}, n^{4k-1})$
$r \neq 0, s = 0, t \neq 0, m \neq 0, g \neq 0, h = 0$	Case 4	$(9n^3, 9n^3(1 \pm i), -27n^4)$ $(9n^3, -9n^3(1 \pm i), -27n^4)$
$r \neq 0, s \neq 0, t = 0, m \neq 0, g \neq 0, h = 0$	Case 4	$(9n^3(1 \pm i), 9n^3 i, -27n^4)$ $(-9n^3(1 \pm i), 9n^3 i, -27n^4)$
$r \neq 0, s \neq 0, t \neq 0, m = 0, g \neq 0, h = 0$	Case 4	$(9n^3(1 \pm i), 9n^3, -27n^4)$ $(-9n^3(1 \pm i), 9n^3, -27n^4)$

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