Quest for Research Excellence On Computing, Mathematics and Statistics

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Chapters in Book

The 2nd International Conference on Computing, Mathematics and Statistics (iCMS2015)

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Content

CHAPTER 15
Characterization of p-Groups with a Maximal Irredundant 10-Covering

Rawdah Adawiyah Tarmizi and Hajar Sulaiman

Abstract. A group G is covered by a collection of its proper subgroups if it is equal to the union of the collection. A covering is called irredundant if there is no proper sub-collection is also a covering of G. A covering in which all members are maximal subgroups of G is called maximal. For any integer $n > 2$, a covering with n members is called an n-covering. We say the covering of G as C_n–covering if it is an irredundant maximal core-free n-covering for G. In this paper, we characterize 3-groups having a maximal irredundant 10-covering with core-free intersection and we prove that a group G is a p-group having C_{10}–covering if and only if $G \cong (C_3)^5$ or $G \cong (C_5)^3$.

Keywords: covering groups by subgroups; p-groups; maximal irredundant covering; core-free intersection.

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1 Introduction

Let G be a finite group. If G is non-cyclic, then G can be obtained as a union of its proper subgroups. A covering C of a group G is a collection of proper subgroups of G whose union is the whole group G. We use the term n-covering for C with n members.

A covering C of G is irredundant if no proper sub-collection is also a covering for G, and is called maximal if all its members are maximal subgroups of G. We denote the intersection of members of maximal covering by D. A covering C of G is called core-free if the intersection $D = \bigcap_{M\in C} M$ of C is core-free in G, i.e. $D_G = \bigcap_{g\in G} g^{-1}Dg$ is the trivial subgroup of G. The covering C of G is called a C_n–covering whenever C is an irredundant maximal core-free n-covering for G. We say a group G is a C_n–group if G admits C_n–covering.

It is well known that there is no group can be covered by two proper subgroups. Scorza in [8] was the first to determine the structure of all groups having an irredundant 3-covering with core-free intersection.

Theorem 1.1. (See [8]) Let $\{A_i \mid 1 \le i \le m\}$ be an irredundant covering with core-free intersection D for a group G. Then, $D = 1$ and $G \cong C_2 \times C_2$.

In [7] Greco listed all groups with an irredundant 4-covering with core-free intersection. He also listed all groups with an irredundant 5-covering in which all pairwise intersection are the same.

Bryce et al. in [6] characterized groups with maximal irredundant core-free intersection 5-covering completely. They proved that G is a p-group if and only if G is elementary of order $2^4 = 16$.

Abdollahi et al. in [3] characterized groups with maximal irredundant 6-covering with core-free intersection.

In [4] Abdollahi and Jafarian listed all groups having a maximal irredundant 7-covering with core-free intersection.

Abdollahi et al. in [1] characterized of p-groups with a maximal irredundant n-covering with core-free intersection for $n \in \{7,8,9\}$ completely.

Ataei in [5] characterized nilpotent group having a maximal irredundant 8-covering with core-free intersection.

In [2] Ataei and Sajjad characterized 5-groups with a maximal irredundant 10-covering with core-free intersection.

Theorem 1.2. (See [2]) Let G be a 5-groups such that $|G| \ne 5^4$. Then G is a C_{10}–group, if and only if $G \cong (C_5)^3$.

There are problems of similar nature with slightly different aspects have been studied by many authors ([11], [12], [13], [14], [15]).

Here we characterized 3-groups having a maximal irredundant 10-covering with core-free intersection. Then we prove that *G* is a *p*-group that admits C_{10}–covering if and only if $G \cong (C_3)^5 or \ G \cong (C_5)^3$.

Also, we use the notation as in [2]; for example, C_n the cyclic group of order n, $(C_n)^j$ is the direct product of j copies of C_n, $[n]$ is the set $\{1, \dots, n\}$, and $[n]^m$ is the set of all subsets of $[n]$ of size *m*.

2 Preliminaries

We will require the following lemma as an aid tool to obtain our results.

Lemma 2.1. (See [9], Proposition 2.5) *Let G be a finite p-group for a prime p, with a maximal irredundant n-covering. Then either* $n \geq \frac{3(p+1)}{2}$ *or* $n = p + 1$.

Lemma 2.2. (See [1], Theorem 1.3) *Let G be a finite 2-group and n be a positive integer. Then G admits a*C_{n+1}*–covering if and only if n is even and* $G \cong (C_2)^n$.

Lemma 2.3. (See [6], Lemma 2.2). *Let* $\Gamma = \{A_i \mid 1 \leq i \leq m\}$ *be an irredundant covering of a group G whose intersection of the members is D.*

(a) *If p is a prime, x a p-element of G and* $|\{i \mid x \in A_i\}| = n$*, then either* $x \in D$ *or* $p \leq m - n$.

(b) $\bigcap_{j \neq i} A_j = D \ \forall \ i \in \{1,2, \dots, m\}$.

(c) *If* $\bigcap_{i \in S} A_i = D$ *whenever* $|S| = n$, *then* $|\bigcap_{i \in T} A_i : D| \leq m - n + 1$ *whenever* $|T| = n - 1$.

(d) *If* Γ *is maximal and U is an abelian minimal normal subgroup of G, then if*$|\{i \mid U \subseteq A_i\}| = n$, *either* $U \subseteq D$ *or*$|U| \leq m - n$.

Lemma 2.4. (See [1], Lemma 3.2). *Let G be a finite p-group having a* C_n*–covering* $\{M_i \mid i = 1, \dots, n\}$. *Then*

(a) $p \leq n - 1$.

(b) *If s the integer such* $1 \leq s \leq n - 2$ *and* $p = n - s$, *then* $\bigcap_{i \in S} M_i = 1$ *for every subset S of* $\{1,2, \dots, n\}$*with* $|S| \geq s + 1$.

(c) $if\ n = p + 1$, *then* $G \cong \left(C_p\right)^2$.

Lemma 2.5. (See [1], Lemma 3.3). *Let* $G = \left(C_p\right)^d \ for\ d \geq 2$ *and p is a prime number. Suppose that G has* C_n*–covering* $\{M_i \mid i = 1, \dots, n\}$. *Let* $T \subseteq \{1,2, \dots, n\}$.

(a) *If* $|T| = n - p$, *then* $|\bigcap_{i \in T} M_i| = 1$ *or p.*

(b) *If* $|T| = 2$, *then* $|\bigcap_{i\in T} M_i| = p^{\,d-2}$.
(c) $\bigcap_{i\in T} M_i = 1$ *for some T of size* d.
(d) $If\ \ \bigcap_{i\in S} M_i = 1$ *whenever*$|S| = d$, *then* $p \leq |\bigcap_{i\in T} M_i| \leq n - d + 1$ *whenever* $|T| = d - 1$.

3 Result

In this section we characterize *p*-groups with a maximal irredundant 10-covering.

Theorem 3.1. *Let G be a* 3*-group such that* $|G| \neq 3^4$. *Then G is a* C_{10}*–group, if and only if* $G \cong (C_3)^5$.

Proof. Suppose that *G* is a 3-group, $G = \bigcup_{i=1}^{10} M_i$ and $D = \bigcap_{i=1}^{10} M_i$. Since the Frattini subgroups of *G*, $\phi(G) = G^{'}G^3 \leq D$, we have *D* is a normal subgroup of *G*. Therefore *D* = 1 and *G* is an elementary abelian 3-group.By Lemma 2.5(b)

$$|G\colon M_i| = 3 \text{ and } \left|G\colon M_i \cap M_j\right| = 3^2 = 9 \text{ for all distinct } i,j \in [10] \qquad \textbf{(15)}$$

and by lemma 2.4(b) implies that

$$\forall\, S \subseteq [10] \text{ with } S \geq 10 - 3 + 1 = 8,\ \bigcap_{i\in S} M_i = 1 \qquad \textbf{(2)}$$

It follows that $|G| \leq 3^8$. Since an elementary abelian group of order 3^2 has only four maximal subgroups, we have $|G| \geq 3^3$.

Let $|G| = 3^3$, so that $G \cong (C_3)^3$. We have used the following function run in GAP [16] to proof the claim. The inputs of the function are a group and a number of covering, and the outputs are all combinations of maximal irredundant 10-covering with core free intersection of *G*, and if there is no such covering for *G*, then the list is empty.

```
f:=function(G,p) local S,M,n,i,C,T,Q,R;
n:=Size(G);
M:=MaximalSubgroups(G);
C:=Combinations(M,p);
S:=[];
for i in [1..Size(C)] do if Size(Union(C[i]))=n then
Add(S,C[i]);
fi;od;
T:=[];
for i in [1..Size(S)] do if
Size(Core(G,Intersection(S[i])))=1 then Add(T,S[i]);
```

```
fi;od;
R:=[];
for i in [1..Size(T)] do Q:=Combinations(T[i],p-1);
if (n in List(Q,i->Size(Union(i))))=false then
Add(R,T[i]);
fi;od;
return R;
end;
```

finally, we found that the list is empty, and therefore $|G| \neq 3^3$. Assume that $|G| = 3^5$, so that $G \cong (C_3)^5$. By GAP in [16], we can check that if $G = \langle a, b, c, d \rangle$, then the set

$$C = \{\langle a,b,c,d\rangle, \langle a,b,c,e\rangle, \langle a,b,d,e\rangle, \langle a,c,d,e\rangle, \langle b,c,d,e\rangle, \langle ac^{-1},b,d,e\rangle, \langle a,b,ce,d\rangle, \langle a^{-1}b, a^{-1}c, a^{-1}d, a^{-1}e\rangle, \langle a,b,c,de\rangle, \langle ae, a^{-1}b, a^{-1}c, a^{-1}d\rangle\}$$

of maximal subgroups form a C_{10}–covering for G.

Let $|G| = 3^6$. By part (c) of Lemma 2.5, we assume that there exist $S \in [10]^6$ such that $\left|\bigcap_{i\in S} M_i\right| = 1$. Since the covering is irredundant, therefore there exist $S \in [10]^2$ such that for all $T \in [10]^6, O = \bigcap_{i\in T} M_i \nleq \bigcap_{i\in S} M_i$. Therefore,

$$|G| = 3^6 = \left|G\colon \bigcap_{i\in 1}^{8} M_i\right| = \left|G\colon \bigcap_{i\in 1}^{6} M_i\right| \left|G\colon \bigcap_{i\in 1}^{2} M_i\right| = |G\colon O|3^2$$
$$|G\colon O| = 3^4$$
$$|O| = 3^2,$$

which is a contradiction by $\left|\bigcap_{i\in 1}^{6} M_i\right| = 1$.

If $|G| = 3^7$. Then Lemma 2.5 implies that $\bigcap_{i\in T} M_i = 1$ for at least $T \in [10]^7$. Therefore, we assume that there exist $S \in [10]^7$ such that $\left|\bigcap_{i\in S} M_i\right| = 1$. Since the covering is irredundant, therefore there exist $j \in [10]$ such that for all $L \in [10]^7$, $N = \bigcap_{i\in L} M_i \nleq M_j$. Therefore

$$|G| = 3^7 = \left|G\colon \bigcap_{i\in 1}^{8} M_i\right| = |G\colon N|\left|G\colon M_j\right| = |G\colon N|3$$
$$|G\colon N| = 3^6$$
$$|N| = 3,$$

which is a contradiction by $|\bigcap_{i\in 1}^{7} M_i| = 1$.

Now assume that $|G| = 3^8$. Then Lemma 2.5(c) implies that

$$\left|\bigcap_{i\in T} M_i\right| = 3 \text{ for every } T \in [10]^7. \qquad (3)$$

Then it follows from (1) that for every $\in [10]^3$, we have $\left|G: \bigcap_{i\in K} M_i\right| = 3^5$or 3^6. Now we prove that $\left|\bigcap_{i\in K} M_i\right| = 3^5$ for all $K \in [10]^3$. Suppose, for contradiction, that there exist $L \in [10]^3$ such that $\left|\bigcap_{i\in L} M_i\right| = 3^6$. Let $L' \in [10]^4$such that $L \cap L^{'} = \emptyset$. Then it follows from (1) and (3) that
$\left|\bigcap_{i\in L\cup L"} M_i\right| = \left|\bigcap_{i\in L'\cup L"} M_i\right| = 3$ for every $L"$ is a proper subgroup of L of size 2. Since $|L" \cup L'| = 6$, it follows that $|G| \leq 3^6$, which is a contradiction. Therefore,

$$\left|\bigcap_{i\in K} M_i\right| = 3^5 \text{ for all } K \in [10]^3. \qquad (4)$$

By (1), $\left|\bigcap_{i\in T} M_i\right| \in \{3^4 \text{or } 3^5\}$ for all $T \in [10]^4$, we prove that $\left|\bigcap_{i\in T} M_i\right| = 3^4$ for all $T \in [10]^4$. Suppose, for a contradiction that there exists $L \in [10]^4$ such that $\left|\bigcap_{i\in L} M_i\right| = 3^5$. Let $L' \in [10]^3$ such that $L \cap L^{'} = \phi$. Then (1) and (3) imply that
$\left|\bigcap_{i\in L\cup L"} M_i\right| = \left|\bigcap_{i\in L'\cup L"} M_i\right| = 3$ for every $L" \subset L$ of size 3. Since $|L" \cup L'| = 6$, it follows that $|G| \leq 3^6$, which is a contradiction. We conclude that,

$$\left|\bigcap_{i\in T} M_i\right| = 3^4 \text{ for all } T \in [10]^4. \qquad (5)$$

By a similar argument as in the previous, we can prove that

$$\forall V \in [10]^5, \left|\bigcap_{i\in V} M_i\right| = 3^3 \qquad (6)$$

and

$$\forall W \in [10]^6, \left|\bigcap_{i\in W} M_i\right| = 3^2. \qquad (7)$$

Now using (1) – (7), it follows from the inclusion-exclusion principle that

$$\left|\bigcup_{i=1}^{10} M_i\right| = \binom{10}{1}3^7 - \binom{10}{2}3^6 + \binom{10}{3}3^5 - \binom{10}{4}3^4 + \binom{10}{5}3^3 - \binom{10}{16}3^2 + \binom{10}{7}3 - \binom{10}{8} + \binom{10}{9} - \binom{10}{10} = 6453,$$

which is not 3^8, a final contradiction.

Theorem 3.2. *Let G be a C_{10}–group. Then G is a p-group for a prime p if and only if* $G \cong (C_3)^5$ *or* $G \cong (C_5)^3$.

Proof. Let G be a p-group with aC_{10}–covering $\{M_i | i \in [10]\}$. Since the Frattini subgroups of G, $\phi(G) = G'G^p \leq D$, we have D is a normal subgroup of G. Therefore D = 1 and G is an elementary abelian p-group. By Lemma 2.1 implies that $p \leq 5$ and it follows from Lemma 2.2 that $p \neq 2$. Therefore $p = 3$ or $p = 5$.
If $p = 5$, then by Theorem 1.2 it follows that $G \cong (C_5)^3$.If $p = 3$, then Theorem 3.1 implies that $G \cong (C_3)^5$.

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