Quest for Research Excellence On Computing, Mathematics and Statistics

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Chapters in Book

The 2nd International Conference on Computing, Mathematics and Statistics (iCMS2015)

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Content

CHAPTER 11
Fourth Order Random Differential Equation

D. S. Palimkar and P.R. Shinde

Abstract. In this paper, an existence of random solution is proved for a periodic boundary value problem of fourth order random differential equation through an algebraic random fixed point theorem of Dhage.

Keywords: Random differential equation, periodic boundary value problem, random solution, caratheodory condition.

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1 Introduction

Let R denote the real line and let J = [0, 2π] be a closed and bounded interval in R. Let $C^1(J,R)$) denote the class of real-valued functions defined and continuously differentiable on J. Given a measurable space (Ω, A) and for a given measurable function $x:\Omega\to C^1(J,R)$, consider a fourth order periodic boundary value problem of ordinary random differential equations (PBVP)

$$\begin{aligned} & x''''(t,\omega) = f\,(t, x(t,\omega),\omega) \qquad a.e.\ t\in J \\ & x(0,\omega) = x(2\pi,\omega),\ x'(0,\omega) = x'(2\pi,\omega), \\ & x''(0,\omega) = x''(2\pi,\omega), x'''(0,\omega) = x'''(2\pi,\omega). \end{aligned} \tag{1.1}$$

for all $\omega\in\Omega$, where $f : J\times R\times\Omega\to R$.

By a random solution of the random PBVP (1.1), we mean a measurable function $x:\Omega\to AC^1(J,R)$ that satisfies the equations in (1.1), where $AC^1(J,R)$ is the space of continuous real-valued functions whose first derivative exists and is absolutely continuous on J .

The random PBVP (1.1) is new to the theory of periodic boundary value problems of ordinary differential equations. When the random parameter ω is absent, the random PBVP (1.1) reduces to the classical PBVP of fourth order ordinary differential equation and the study of classical PBVP has been discussed in several papers by many authors for different aspects of the solutions. See for example, Lakshmikantham and Leela[5], ,Nieto [6], Yao [8], and the references therein. In this paper, we discuss the random PBVP (1.1) for existence of solution which generalize several existence results of the classical PBVP proved in the above quoted papers.

2 Auxialary Results

The study of random equations and their solutions have been discussed in Bharucha-Reid [1] which is further applied to different types of random equations such as random differential and random integral equations etc. See Itoh [4], Bharucha-Reid [2] and the references therein.

In this paper, we use the following random nonlinear alternative in proving the main result .

Theorem2.1.(Dhage[3,4]) Let U be a non-empty, open and bounded subset of the separable Banach space E such that $0 \in U$ and let $Q: \Omega \times \bar{U} \to E$ be a compact and continuous random operator. Further suppose that there does not exists an $u \in \partial U$ such that $Q(\omega)x = \alpha x$ for al l $\omega \in \Omega$, where α >1 and ∂U is the boundary of U in E . Then the random equation $Q(\omega)x = x$ has a random solution, i.e., there is a measurable function $\xi : \Omega \to E$ such that $Q(\omega)\xi(\omega) = \xi(\omega)$ for all $\omega \in \Omega$.

A corollary to above theorem in applicable form is

Corollary2.1. Let $B_r(0)$ and $\overline{B_r}(0)$ be the open and closed balls centered at origin of radius r in the separable Banach space E and let $Q: \Omega \times \overline{B_r}(0) \to E$ be a compact and continuous random operator. Further suppose that there does not exists an $u \in E$ with $\|u\| = r$ such that $Q(\omega)u = \alpha u$ for all $\omega \in \Omega$, where $\alpha > 1$. Then the random equation $Q(\omega)x = x$ has a random solution, i.e., there is a measurable function $\xi : \Omega \to \overline{B_r}(0)$ such that Q(ω) ξ (ω)= ξ (ω) forall $\omega \in \Omega$.

The following theorem is used in the study of nonlinear discontinuous random differential equations.

Theorem2.2.(Carath'eodory) Let $Q: \Omega \times E \to E$ be a mapping such that $Q(x, .)$ is measurable for all $x \in E$ and $Q(\omega, .)$ is continuous for all $\omega \in \Omega$. Then the map $(\omega, x) \to Q(\omega, x)$ is jointly measurable.

The following lemma appears in Nieto [6] and is useful in the study of second order periodic boundary value problems of ordinary differential equations.

Lemma2.1. For any real number m >0 and $\sigma \in L^1(J, R)$, x is a solution to the differential equation

$$x''''(t)+m^2x(t)=\sigma(t) \quad a.e.\ t\in Jm$$
$$x(0)=x(2\pi),x'(0)=x'(2\pi), \tag{2.1}$$
$$x''(0)=x''(2\pi),x''(0)=x''(2\pi).$$

if and only if it is a solution of the integral equation

$$x(t) = \int_0^{2\pi} G_m(t,s)\sigma(s)ds \tag{2.2}$$

where

$$G_m(t,s) = \frac{1}{2m(e^{2m\pi}-1)}\left[e^{m(t-s)}+e^{m(2\pi-t+s)}\right], \quad if\ \ 0\le s\le t\le 2\pi,$$
$$= \frac{1}{2m(e^{2m\pi}-1)}\left[e^{m(s-t)}+e^{m(2\pi-s+t)}\right], \quad if\ \ 0\le t< s\le 2\pi$$

(2.3)

Notice that the Green's function G_m is continuous and nonnegative on $J\times J$ and the numbers

$$\alpha=\min\left\{\left|G_m(t,s)\right|: t,s\in\left[0,2\pi\right]\right\}=\frac{e^{m\pi}}{m(e^{2m\pi}-1)}$$

and

$$\beta=\max\left\{\left|G_m(t,s)\right|: t,s\in\left[0,2\pi\right]\right\}=\frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}$$

exist for all positive real number m .

We need the following definitions .

Definition2.1.A function $f:J\times R\times\Omega\to R$ is called random Carath'edory if

(i) the map $(t,\omega) \to f(t,x,\omega)$ is jointly measurable for all $x \in R$, and

(ii) the map $x \to f(t,x,\omega)$ is continuous for all $t \in J$ and $\omega \in \Omega$.

Definition2.2.A function $f : J \times R \times \Omega \to R$ is called random L^1-Carath'eodory if

(iii) for each real number r>0 there is a measurable and bounded function $q_r : \Omega \to L^1(J,R)$ such that

$\left| f(t,x,\omega) \right| \le q_r(t,\omega) \quad a.e.\ t \in J$ for all $\omega \in \Omega$ and $x \in R$ with $|x| \le r$.

3 Existence Results

For a given real number $m > 0$, consider the random PBVP,

$$\begin{aligned} x''''(t,\omega) + m^2 x(t,\omega) &= f_m(t, x(t,\omega), \omega) \quad a.\ e. t \in J \\ x(0,\omega) = x(2\pi,\omega),\ & x'(0,\omega) = x'(2\pi,\omega), \\ x''(0,\omega) = x''(2\pi,\omega),\ & x'''(0,\omega) = x'''(2\pi,\omega). \end{aligned} \tag{3.1}$$

for all $\omega \in \Omega$, where the function $f_m : J \times R \times \Omega \to R$ is defined by

$$f_m(t,x,\omega) = f(t,x,\omega) + m^2 x$$

Note: The random PBVP (1.1) is equivalent to the random PBVP (3.1) and therefore, a random solution to the PBVP (3.1) is the random solution to the PBVP (1.1) and vice versa.

The random PBVP (3.1) is equivalent to the random integral equation,

$$x(t,\omega) = \int_0^{2\pi} G_m(t,s) f_m(s, x(s,\omega), \omega) ds \tag{3.2}$$

for all $t \in J$ and $\omega \in \Omega$,where the function $G_m(t,s)$ is defined by (2.3).

Consider the following set of assumptions as

(A_1) The function f_m is random Carath'eodory on $J \times R \times \Omega$.

(A_2) There exists a measurable and bounded function $\gamma : \Omega \to L^2(J,R)$ and a continuous and non-decreasing function $\psi : R_+ \to (0,\infty)$ such that $|f_m(t,x,\omega)| \le \gamma(t,\omega)\psi(|x|)$ $a.e. t \in J$ for all $\omega \in \Omega$ and $x \in R$.

4 Main Existence Results

Theorem 4.1. Assume that the hypotheses (A_1)-(A_2) hold. Suppose that there exists a real number $r > 0$ such that

$$r > \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \|\gamma(\omega)\|_{L^1} \psi(r) \tag{4.1}$$

for all $\omega \in \Omega$.Then the random PBVP (1.1) has a random solution defined on J.

Proof: Set $E = C(J,R)$ and define a mapping $Q : \Omega \times E \to E$ by

$$Q(\omega)x(t) = \int_0^{2\pi} G_m(t,s) f_m(s, x(s,\omega), \omega) ds$$

for all $t \in J$ and $\omega \in \Omega$.Since the map $t \to G_m(t,s)$ is continuous on J, $Q(\omega)$ defines a mapping $Q : \Omega \times E \to E$. Define a closed ball $\bar{B}_r(0)$ in E centered at origin 0 of radius r, where the real number r satisfies the inequality (3.3). We, show that Q satisfies all the conditions of Corollary2.1on $\bar{B}_r(0)$.

First, we show that Q is a random operator on $\bar{B}_r(0)$. Since $f_m(t,x,\omega)$ is random Caratheodory, the map $\omega \to f_m(t,x,\omega)$ is measurable in view of Theorem 2.2. Similarly, the product $G_m(t,s) f_m(s, x(s,\omega), \omega)$ of a continuous and a measurable function is again measurable. Further, the integral is a limit of a finite sum of measurable functions, therefore, the map

$$\omega \to \int_0^{2\pi} G_m(t,s) f_m(s,x(s,\omega),\omega)ds = Q(\omega)x(t)$$

is measurable. As a result, Q is a random operator on $\Omega \times \bar{B}_r(0)$ into E.

Next, we, show that the random operator $Q(\omega)$ is continuous on $\bar{B}_r(0)$. Let $\{x_n\}$ be a sequence of points in $\bar{B}_r(0)$ converging to the point x in $\bar{B}_r(0)$. Then ,it is enough to prove that $\lim_{n\to\infty} Q(\omega)x_n(t) = Q(\omega)x(t)$ for all $t \in J$ and $\omega \in \Omega$. By dominated convergence theorem, we obtain,

$$\lim_{n\to\infty} Q(\omega)x_n(t) = \lim_{n\to\infty} \int_0^{2\pi} G_m(t,s) f_m(s,x_n(s,\omega),\omega)ds$$

$$= \lim_{n\to\infty} \int_0^{2\pi} G_m(t,s) \lim_{n\to\infty}\left[f_m(s,x_n(s,\omega),\omega)\right] ds$$

$$= \int_0^{2\pi} G_m(t,s)\left[f_m(s,x(s,\omega),\omega)\right] ds$$

$$= Q(\omega)x(t)$$

for all $t \in J$ and $\omega \in \Omega$. This shows that $Q(\omega)$ is a continuous random operator on $\bar{B}_r(0)$.

Now, we show that $Q(\omega)$ is a compact random operator on $\bar{B}_r(0)$. To finish, it is enough to prove that $Q(\omega)(\bar{B}_r(0))$ is uniformly bounded and equi-continuous set in E for each $\omega \in \Omega$. Since the map $\omega \to \gamma(t,\omega)$ is bounded and $L^2(J,R) \subset L^1(J,R)$,by hypothesis (A_2),there is constant c such that $\|\gamma(\omega)\|_{L^1} \leq c$ for all $\omega \in \Omega$ .Let $\omega \in \Omega$ be fixed. Then for any $x : \Omega \to \bar{B}_r(0)$,one has

$$|Q(\omega)x(t)| \leq \int_0^{2\pi} G_m(t,s)\,|f_m(s, x(s,\omega), \omega)|\,ds$$

$$\leq \int_0^{2\pi} G_m(t,s)\gamma(s,\omega)\psi(|x(s,\omega)|)ds$$

$$\leq \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)}\left(\int_0^{2\pi} \gamma(s,\omega)ds\right)\psi(r) \ \leq K_1$$

for all $t \in J$,where $K_1 = \dfrac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} c\psi(r)$.

This shows that $Q(\omega)(\bar{B}_r(0))$ is a uniformly bounded subset of E for each $\omega \in \Omega$.

Next ,we show that $Q(\omega)$)($\bar{B}_r(0)$) is an equi-continuous set in E .Le $x \in \bar{B}_r(0)$ be arbitrary. Then for any $t_1, t_2 \in J$, one has

$$|Q(\omega)x(t_1) - Q(\omega)x(t_2)| \leq$$

$$\int_0^{2\pi} |G_m(t_1,s) - G_m(t_2,s)|\,|f_m(s, x(s,\omega), \omega)|ds$$

$$\leq \int_0^{2\pi} |G_m(t_1,s) - G_m(t_2,s)|\,\gamma(s,\omega)\,\psi(r)ds$$

$$\leq \int_0^{2\pi} (|G_m(t_1,s) - G_m(t_2,s)|^2\,ds)^{\frac{1}{2}} \ \left(\int_0^{2\pi} |\gamma(s,\omega)|^2\,ds\right)^{\frac{1}{2}} \psi(r)$$

Hence for all $t_1, t_2 \in J$, $|Q(\omega)x(t_1) - Q(\omega)x(t_2)| \to 0$ as $t_1 \to t_2$, uniformly for all $x \in \bar{B}_r(0)$ .Therefore, $Q(\omega)(\bar{B}_r(0))$ is an equi-continuous set in E. As $Q(\omega)(\bar{B}_r(0))$ is uniformly bounded and equi-continuous, it is compact by Arzel -Ascoli theorem for each $\omega \in \Omega$. Consequently, $Q(\omega)$ is a completely continuous random operator on $\bar{B}_r(0)$.

Finally, we prove that there does not exist an $u \in E$ with $\|u\| = r$ such that $Q(\omega)u(t) = \alpha u(t,\omega)$ for all $t \in J$ and $\omega \in \Omega$, where $\alpha > 1$. Suppose not. Then there exists an element $u \in E$ satisfying $Q(\omega)u(t) = \alpha u(t,\omega)$ for some $\omega \in \Omega$ .Let $\lambda = \dfrac{1}{\alpha}$ .Then $\lambda < 1$ and $\lambda Q(\omega)u(t) = u(t,\omega)$ for some $\omega \in \Omega$ .Now for this $\omega \in \Omega$,one has

$$\begin{aligned} \left|\mathrm{u}(\mathrm{t},\omega)\right| &\le \lambda \left|Q(\omega)\mathrm{u}(\mathrm{t})\right| \\ &\le \int_0^{2\pi} G_m(t,s)\left| f_m(s,u(s,\omega),\omega)\right| ds \\ &\le \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \int_0^{2\pi} \gamma(s,\omega)\psi(\|u(\omega)\|)ds \\ &\le \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \|\gamma(\omega)\|_{L^1}\, \psi(\|u(\omega)\|) \end{aligned}$$

for all $t \in J$. Taking supremum over t in the above inequality yields,

$$\|u(\omega)\| \le \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \|\gamma(\omega)\|_{L^1}\, \psi(\|u(\omega)\|)$$

or

$$r \le \frac{e^{2m\pi}+1}{2m(e^{2m\pi}-1)} \|\gamma(\omega)\|_{L^1}\, \psi(r)$$

for some $\omega \in \Omega$. This contradicts to the condition (3.3).
Thus, all the conditions of Corollary 2.1 are satisfied. Hence the random equation $Q(\omega)x(t) = x(t,\omega)$ has a random solution in $\bar{B}_r(0)$), i.e., there is a measurable function $\xi : \Omega \to \bar{B}_r(0)$ such that $Q(\omega)\xi(t) = \xi(t,\omega)$ for all $t \in J$ and $\omega \in \Omega$. As a result, the random PBVP (1.1) has a random solution defined on J. This completes the proof **.**

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