THE ONSET OF STEADY MARANGONI CONVECTION IN A ROTATING FLUID LAYER WITH A PRESCRIBED HEAT FLUX AT ITS LOWER BOUNDARY

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Abstract: The effect of uniform rotation on the onset of steady Marangoni convection in a horizontal fluid layer heated from below is considered theoretically. The fluid layer is bounded below by a rigid plane boundary with a prescribed heat flux and above by a free non-deformable surface subject to a uniform vertical temperature gradient. The theoretical analysis follows the usual small-disturbance approach of perturbation theory and leads, at the marginal state, to a functional relation between the Marangoni and Taylor numbers which is then computed numerically.

Keywords: Surface-tension-driven instability, Marangoni, Convection, Heat Transfer

INTRODUCTION

The onset of surface-tension-gradients-driven (Marangoni) convection in a layer of fluid which is heated (or cooled) from below is a fundamental model problem for several material processing technologies, such as semiconductor crystal growth from melt, in the microgravity environment of space. As Schwabe [7] describes, typically in microgravity surface tension rather than buoyancy forces are the dominant mechanism driving the flow. In his pioneering linear stability theory, Pearson [5] considered both the so-called "conducting" case of a constant temperature rigid lower boundary of the horizontal fluid layer at which no perturbation in temperature is allowed and the so-called "insulating" case of a constant heat flux lower boundary at which no perturbation in the heat flux is allowed. Pearson [5] showed that variation of surface tension with temperature will drive steady Marangoni convection in a fluid layer provided that the non-dimensional Marangoni number, M, (defined in Section 2) is sufficiently large and positive. Since for most fluids surface tension decreases with increasing temperature, this means that steady convection only occurs when the layer is heated sufficiently strongly from below. The most significant limitation of Pearson's [5] work was that it considered only the case of a non-deformable free surface, corresponding to the limit of strong surface tension. Subsequently, Scriven and Sternling [8] extended the work of Pearson [5] to include a deformable free surface with capillary but not gravity waves. Garcia-Ybarra *et al.* [2] and Gouesbet *et al.* [3] performed extensive numerical calculations for the insulating problem with gravity waves at the free surface included.

All of the work mentioned above excluded the effect of rotation of the fluid layer. Vidal and Acrivos [9] were the first to analyze the effect of rotation on Marangoni convection in the conducting case. McConaghy and Finlayson [4] re-examined Vidal and Acrivos' [9] conclusion on the possibility of oscillatory convection. In this work we use the classical linear stability theory to study the effect of rotation on the marginal curves for the onset of steady Marangoni convection in the insulating case. The structure of the paper is as follows. In Sections 2 and 3 we briefly formulate and solve the appropriate linear stability problem, respectively. In Section 4 we present the results of numerical calculations which illustrate the effect of varying the problem parameters on the marginal curves and hence on the critical value of M for the onset of convection. Finally, in Section 5 we summarize the work.

MATHEMATICAL FORMULATION

We wish to examine the stability of a horizontal layer of quiescent fluid of thickness d which is unbounded in the horizontal x- and y-directions. The layer is kept rotating uniformly around a vertical axis with a constant angular velocity Ω. We shall formulate the problem in a general way in which the layer is bounded below by a thermally insulating planar boundary and above by a free deformable surface, subject to a uniform vertical temperature gradient (see Figure 1)

The fluid is Boussinesquian with a mass density ρ assumed to vary linearly on the temperature

$$\rho = \rho_0 \,[1 - \alpha_1\,(T - T_0)]\,, \quad (\alpha_1 > 0), \tag{1}$$

where α_1 is the volume expansion coefficient and T_0 a reference arbitrary temperature. The variations of surface tension γ with the temperature T is assumed in the form

$$\gamma = \gamma_0 - \tau(T - T_0) \tag{2}$$

where γ_0 is a reference value of surface tension and τ is the rate of change of surface tension with temperature.

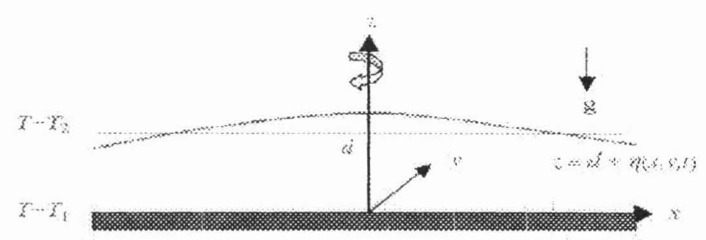


Figure 1: Sketch of the one-layer model

The deformation of the interface is represented by the relation

$$z = d + \eta(x,y,t) \tag{3}$$

wherein $\eta(x,y,t)$ is an *a-priori* unknown deformation with respect to the mean thickness d.

In the reference state, the fluid is at rest with respect to the rotating axes and heat propagates only by conduction. When motion sets in, the velocity $\mathbf{v} = (u,v,w)$, pressure p and temperature T fields obey the usual balance equations of mass, momentum and energy (cf. Chandrasekhar [1]),

$$\nabla \cdot \mathbf{v} = 0 \tag{4}$$

$$\rho_0\left[\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\cdot\nabla)\mathbf{v} + 2\Omega\times\mathbf{v}\right] = -\nabla p + \mu\nabla^2\mathbf{v} - \rho\mathbf{g}\mathbf{e}_z \tag{5}$$

$$\frac{\partial T}{\partial t} + (\mathbf{v}\cdot\nabla)T = \kappa\ \nabla^2 T \tag{6}$$

where $\mathbf{g} = (0,0,-g)$ is the gravitational field, $\mathbf{e}_z=(0,0,1)$ is a unit vector in the z-direction, μ is the viscosity, κ is the thermal diffusivity and $\nabla^2 = \partial^2/\partial x^2 + \partial^2/\partial y^2 + \partial^2/\partial z^2$ is the Laplacian operator.

At the deformably free surface, at $z = d + \eta(x,y,t)$, the boundary conditions comprise of the kinematic, the heat flux, the two shear stress and the normal stress conditions which are given by, respectively,

$$\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} + v\frac{\partial \eta}{\partial y} = w\,, \tag{7}$$

$$k\nabla T\cdot\mathbf{n} + hT = 0, \tag{8}$$

$$2\mu D_{nt} = \frac{\partial \gamma}{\partial T}\nabla T\cdot\mathbf{t}\,, \tag{9}$$

$$(p_a - p) + 2\mu D_{nn} = \gamma\nabla\cdot\mathbf{n} \tag{10}$$

where h is the heat transfer coefficient, k is the thermal conductivity of the fluid, p_a is the pressure of the atmosphere, D_{ij} is the rate of strain tensor, **t** and **n** denote tangential and normal unit vectors, respectively. At the lower, rigid and plane, boundary we have the condition of continuity of velocity between the solid and the fluid. This lower boundary is subject to a prescribed heat flux.

We introduce infinitesimal disturbances to the governing equations and boundary conditions by setting

$$(u, v, w, \rho, p, T) = (0,0,0,\bar{\rho},\bar{p},\bar{T}) + (u', v', w', \rho', p', \theta'), \qquad (11)$$

where the primed quantities represent the perturbed variables. A set of scales d, d^2, κ, ΔT is chosen for distance, time and temperature, respectively. The perturbed quantities in normal mode forms are

$$\begin{bmatrix} w' \\ \theta' \\ \zeta' \\ \eta' \end{bmatrix} = \begin{bmatrix} W(z) \\ \Theta(z) \\ K(z) \\ E \end{bmatrix} e^{i(a_x x + a_y y)+\sigma t} \qquad (12)$$

where a_x and a_y are wavenumbers of disturbances in the x and y directions, respectively. W, Θ, K and E are amplitudes of vertical velocity, temperature, vertical vorticity and deflection of the free upper surface, respectively. The growth parameter σ is in general a complex variable denoted by $\sigma = \sigma_r + i\sigma_i$, where σ_r is the growth rate of the instability and σ_i is the frequency. If $\sigma_r > 0$, the disturbances grow and the system becomes unstable. If $\sigma_r < 0$, the disturbances decay and the system becomes stable. When $\sigma_r = 0$, the instability of the system, at the marginal state, sets in stationarily, provided $\sigma_i = 0$, or oscillatorily, provided $\sigma_i \neq 0$.

The governing equations of the perturbed state in dimensionless forms, assuming the Boussinesq approximation, are

$$(D^2 - a^2)(D^2 - a^2 - \sigma P_r^{-1})W - T_a^* DK = a^2 R^* \Theta \qquad (13)$$
$$D^2 - a^2 - \sigma P_r^{-1})K = -DW \qquad (14)$$
$$(D^2 - a^2)\Theta = -W \qquad (15)$$

subject to

$$W - \sigma E = 0, \qquad (16)$$
$$C_r^*[(D^2 - 3a^2 - \sigma P_r^{-1})DW - T_a^* K] - a^2(a^2 + B_o^*)E = 0, \qquad (17)$$
$$(D^2 + a^2)W + a^2 M^*(\Theta - E) = 0, \qquad (18)$$
$$D\Theta + B_i^*(\Theta - E) = 0, \qquad (19)$$
$$DK = 0, \qquad (20)$$

evaluated on the undisturbed position of the upper free surface $z = \pi$, and

$$W = 0, \qquad (21)$$
$$D\Theta = 0, \qquad (22)$$
$$K = 0, \qquad (23)$$
$$DW = 0, \qquad (24)$$

evaluated on the lower rigid boundary $z = 0$, where the operator $D = d/dz$ denotes differentiation with respect to the vertical coordinate z and $a = (a_x^2 + a_y^2)^{1/2}$ is the horizontal wave number of the disturbance. The starred dimensionless numbers are defined by $R^* = R/\pi^4$, $M^* = M/\pi^2$, $T_a^* = T_a/\pi^4$, $C_r^* = \pi C_r$, $B_i^* = B_i/\pi$, $B_o^* = B_o/\pi^2$, where the Rayleigh number, $R = \alpha g \Delta T d^3/\nu\kappa$, where ν is the kinematic viscosity, the Marangoni number, $M = \gamma\Delta d/\rho_0\nu\kappa$, the Taylor number, $T_a = 4\,\Omega^2 d^4/\nu^2$, the capillary number, $C_r = \rho_0\nu\kappa/\gamma_0 d$, the Biot number, $B_i = hd/k$, the Bond number, $B_o = \rho g d^2/\gamma$, and the Prandtl number, $P_r = \nu/\kappa$. The Rayleigh number R accounts for buoyancy destabilizing effect. The number M accounts for surface tension destabilizing effect. The Taylor number Ta represents the square of the ratio between Coriolis and frictional forces. The capillary number C_r shows an idea of the rigidity of the upper free surface of the fluid layer. The Biot number B_i represents the heat flux flow through the interface, and the physical parameter Bond number B_o is the ratio between gravity effect in keeping the surface flat and the effect of surface tension in making a meniscus. The Prandtl number, P_r, stands for the ratio between thermal and heat diffusivities.

SOLUTION OF THE LINEARIZED PROBLEM

Combining equations (13) – (15) then gives a single linear eighth-order ordinary differential equation for Θ,

$$(D^2 - a^2 - \sigma)\,[(D^2 - a^2)(D^2 - a^2 - \sigma P_r^{-1})^2 + T_a^* D^2\,]\,\Theta + a^2 R^*(D^2 - a^2 - \sigma P_r^{-1})\,\Theta = 0. \qquad (25)$$

Equation (25) together with the boundary conditions (16) – (24) constitute a linear eigenvalue problem for the unknown temporal exponent σ. Relation (17) gives the expression for the surface deflection E in terms of the other quantities. In the general case $\sigma \neq 0$ we seek solutions in the forms

$$W(z)=ACe^{\xi z}, \quad K(z)=BC\,e^{\xi z}, \quad \Theta(z)=Ce^{\xi z}, \qquad (27)$$

where the complex quantities A, B and C and the exponent ξ are to be determined. Substituting these forms into the equations (13) – (15) and eliminating A, B and C we obtain an eighth-order algebraic equation for ξ, namely

$$(\xi^2 - a^2 - \sigma)\,[(\xi^2 - a^2)(\xi^2 - a^2 - \sigma P_r^{-1})^2 + T_a^* \xi^2\,]\,\Theta + a^2 R^*(\xi^2 - a^2 - \sigma P_r^{-1})\,\Theta = 0,$$

with eight distinct roots $\xi_1, \ldots, \xi_8$. Denoting the values of A, B and C corresponding to ξ_i for $i = 1, \ldots, 8$ by A_i, B_i and C_i we can use equations (14) and (15) to determine A_i and B_i to be

$$A_i = -(\xi_i^2 - a^2 - \sigma), \qquad B_i = -\frac{\xi_i A_i}{\xi_i^2 - a^2 - \sigma P_r^{-1}}, \qquad (28)$$

for $i = 1, \ldots, 8$. The boundary conditions (16) – (24) can be used to determine the eight unknowns $C_1, \ldots, C_8$ (up to an arbitrary multiplier), and the general solution to the stability problem is therefore

$$W(z) = \sum_{i=1}^{8} A_i C_i e^{\xi_i z}, \qquad K(z) = \sum_{i=1}^{8} B_i C_i e^{\xi_i z}, \qquad \Theta(z) = \sum_{i=1}^{8} C_i e^{\xi_i z} \qquad (29)$$

Imposing boundary conditions (16) – (24) yields a linear system $\mathbf{PA} = \mathbf{0}$, where $\mathbf{A} = [A_1, \ldots, A_8]^T$. In general, the 8×8 coefficient matrix $\mathbf{P}$ (whose entries depend on a, M, R, σ, C_r, T_a, P_r, B_o and B_i) is complex and may be rather complicated, and so, in general, it has to be calculated either numerically or symbolically using a symbolic algebra package. In this work we use both approaches. We use a FORTRAN 77 program employing the Numerical Algorithms Group (NAG) routine F03ADF to evaluate the determinant of $\mathbf{P}$ using LU factorization with partial pivoting. A modification of Powell's [6] hybrid algorithm, which is a combination of Newton's method and the method of steepest descent, implemented using NAG routine C05NBF is then used to find the eigenvalues of $\mathbf{P}$ by solving the two non-linear equations obtained from the real and imaginary parts of the determinant of $\mathbf{P}$.

MARGINAL STABILITY CURVE

In this work we shall concentrate on the problem of the onset of steady Marangoni convection in a horizontal fluid layer with non-deformable free surface, i.e. we set $\sigma = 0$, $R = 0$ and $C_r = 0$. The marginal stability curves in the (a,M) plane on which $\sigma_r = 0$ separate regions of unstable modes with $\sigma_r > 0$ from those of stable modes with $\sigma_r < 0$. The critical Marangoni number, denoted by M_c, for the onset of convection is the global minimum of M over $a > 0$. The corresponding critical wave number is denoted by a_c.

For steady convection, the dispersion relation $F(a, M, T_a, B_i) = 0$ takes the linear form $D_1 + MD_2 = 0$, where D_1 and D_2 are two 6×6 determinants which depend on the whole set of parameters of the problem except M. Given any set of values for T_a, B_i, we can determine the Marangoni number as a function of the wave number a.

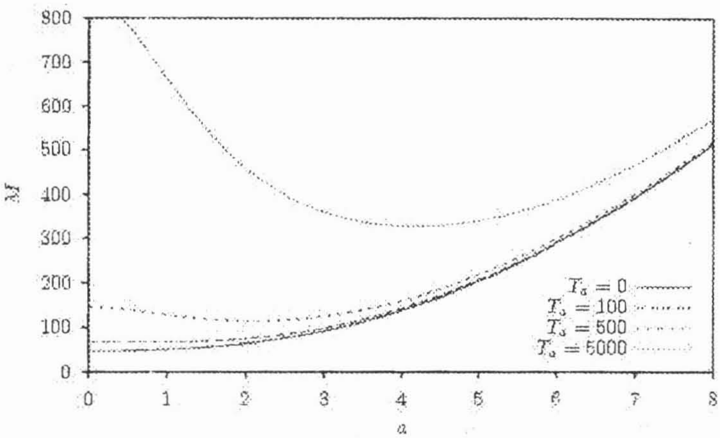
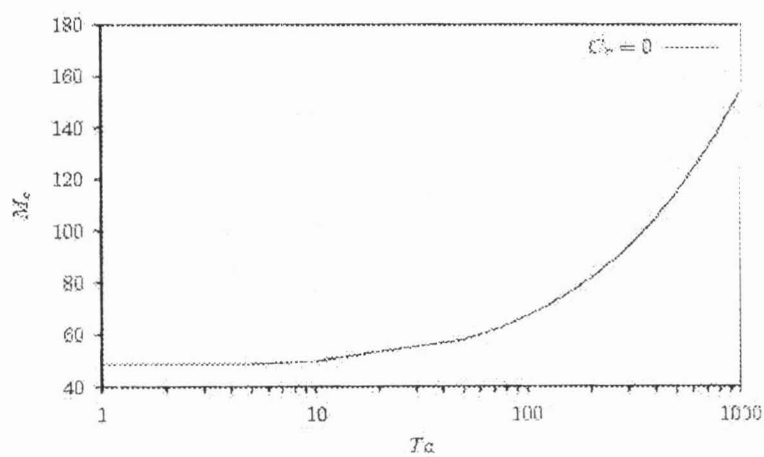


Figure 2: Numerically-calculated marginal curves for the onset of steady Marangoni convection plotted as function of a in the case $C_r = 0$ and $B_i = 0$ for several values of T_a

Figure 2 shows typical marginal stability curves for the onset of steady Marangoni convection for various values of the Taylor number Ta in the case $C_r = 0$ and $B_i = 0$. As a validation of cur algorithm, we recover the marginal curve obtained by Pearson [5] for the pure Marangoni problem without rotation, $T_a = 0$, having the critical value $M_c = 48$ at $a_c = 0$ in the case $B_i = 0$. Figure 2 clearly shows that the marginal curves are shifted upwards as T_a increases, i.e. the effect of rotation is to stabilize the layer. Physically, rotation introduces vorticity into the fluid which then causes the fluid to move in the horizontal planes with higher velocity. The velocity of the fluid perpendicular to the planes reduces, thus the onset of convection is inhibited (Chandrasekhar [1]).

(a)



(b)

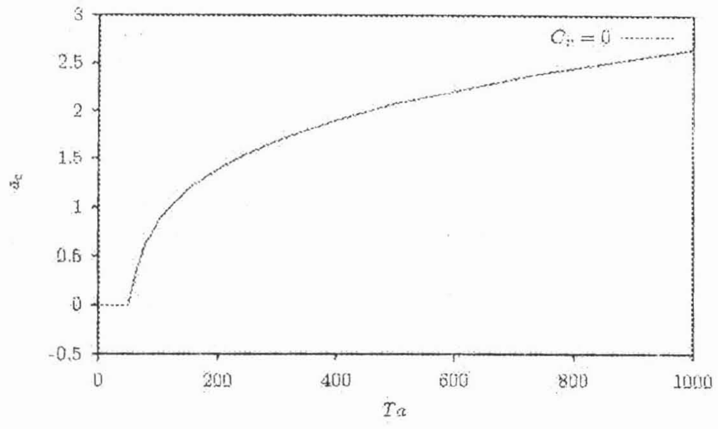


Figure 3. Numerically-calculated (a) M_c and (b) a_c as functions of T_a for $C_r = 0$ in the case $B_i = 0$.

The critical Marangoni number determines, via the definition of the Marangoni number, the critical temperature difference $\triangle T_c$ required for a particular fluid layer perturbed with disturbances of wave number a to be in the marginal state, which means as much as being just on the verge of instability. At $\triangle$T just above $\triangle T_c$ the entire infinite fluid layer should change spontaneously from the state of rest to convective motions. In Figures 3(a) and 3(b) we plot the numerically-calculated values of M_c and a_c, respectively, as functions of T_a in the case $C_r = 0$ and $B_i = 0$. Clearly M_c and a_c are monotonically increasing functions of T_a which show that rotation stabilize the layer and the cell size gets bigger.

CONCLUSIONS

In this work we used classical linear stability theory to investigate the effect of rotation on the onset of steady Marangoni convection in a horizontal planar layer of fluid heated from below with a prescribed heat flux at its lower boundary. The results showed the stabilizing effect of rotation.

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