CHROMATIC UNIQUENESS OF CERTAIN TRIPARTITE GRAPHS IDENTIFIED WITH A PATH

[1]G.C. Lau and [2]Y.H. Peng
[1]Faculty of Information Technology and Quantitative Science
Universiti Teknologi MARA Cawangan Johor, Segamat, Johor
[2]Department of Mathematics and Institute for Mathematical Research
Universiti Putra Malaysia, 43400 Serdang, Selangor

Abstract: For a graph G, let $P(G)$ be its chromatic polynomial. Two graphs G and H are chromatically equivalent if $P(G) = P(H)$. A graph G is chromatically unique if $P(H) = P(G)$ implies that $H \cong G$. In this paper, we classify the chromatic classes of graphs obtained from $K_{2,2,2} \cup P_m$ $(m \geq 3)$ (respectively, $(K_{2,2,2} - e) \cup P_m$ $(m \geq 5)$ where e is an edge of $K_{2,2,2}$) by identifying the end vertices of the path P_m with any two vertices of $K_{2,2,2}$ (respectively, $K_{2,2,2} - e$). As a by-product of this, we obtained some families of chromatically unique and chromatically equivalent classes of graphs.

Keywords: Chromatic polynomial, Chromatically unique, Chromatically equivalent

INTRODUCTION

Let $P(G; \lambda)$ (or $P(G)$) denote the chromatic polynomial of a simple graph G. Two graphs G and H are *chromatically equivalent* (χ-equivalent), denoted $G \sim H$, if $P(G) = P(H)$. A graph G is *chromatically unique* (χ-unique) if $P(H) = P(G)$ implies that $H \cong G$. The equivalence class determined by G under $\sim$ is denoted by $[G]$. Let $\chi(G)$, $|V(G)|$, $|E(G)|$ be the chromatic number, the number of vertices and the number of edges of G, respectively. Then the *cyclomatic number* of G is $|E(G)| - |V(G)| + 1$.

Let K_n, C_n and P_n denote a complete graph, a cycle and a path, respectively on n vertices. The complete t-partite graph whose t partite sets have $r_1, r_2, \ldots, r_t$ vertices is denoted by $K_{r_1, r_2, \ldots, r_t}$. Let $G \cup_2 H$ denote any graph obtained by overlapping an edge of G and H (or edge-gluing of G and H). It is shown in [1, 2] that $K_{2,s} \cup_2 C_m$ and $K_{2,2,2} \cup_2 C_m$ are χ-unique for all $s \geq 1, m \geq 3$. In this paper, we classify the chromatic classes of graphs obtained from $K_{2,2,2} \cup P_m$ $(m \geq 3)$ (respectively, $(K_{2,2,2} - e) \cup P_m$ $(m \geq 5)$ where e is an edge of $K_{2,2,2}$) by identifying the end vertices of the path P_m with any two vertices of $K_{2,2,2}$ (respectively, $K_{2,2,2} - e$). As a by-product of this, we obtained some families of chromatically unique and chromatically equivalent classes of graphs.

Throughout this paper, all graphs are assumed to be connected unless stated otherwise. Let G be a graph and let A be a subgraph of G. Let $n(A, G)$ denote the number of subgraphs A in G. Let W_n denote the wheel (obtained by joining a vertex to every vertex of C_{n-1}) of order $n \geq 4$. Also let U_n denote the graph obtained from W_n by deleting a spoke of W_n and C_n^* denote a chordless cycle of n vertices. If G has n vertices and m edges, we say G is an (n,m)-graph.

The following are some useful known results needed for determining the chromatic uniqueness of a graph.

Lemma 1 ([3]) *Let G and H be two graphs such that $G \sim H$. Then G and H have the same number of vertices, edges and triangles. If both G and H has no K_4 as subgraph, then $n(C_4^*, G) = n(C_4^*, H)$. Moreover,*

$$-n(C_5^*, G) + n(K_{2,3}, G) + 2n(U_5, G) + 3n(W_5, G) = -n(C_5^*, H) + n(K_{2,3}, H) + 2n(U_5, H) + 3n(W_5, H).$$

Lemma 2 ([6]) *Let G be a graph, then G contains a cut-vertex if and only if $(\lambda - 1)^2 \mid P(G)$.*

Lemma 2 implies that if $H \sim G$, then H is 2-connected if and only if G is also 2-connected.

Let H be a nonempty graph with two nonadjacent vertices u and v. Let T (respectively, R) be any graph obtained by identifying the end-vertices of a path P_m, $m \geq 3$ with the vertices u and v (respectively, with any two adjacent vertices) of H. That is, $R = H \cup_2 C_m$. Let H^* be the graph obtained from H by identifying the two vertices u and v of H.

Lemma 3 ([5]) $P(T) = P(R) + (-1)^{m-1}P(H^*)$.

We also need the following theorem to prove our main results.

Theorem 1: Let G be a 2-connected graph such that $|E(G)| - |V(G)| = k$. *If G contains a connected subgraph F such that* $|E(F)| - |V(F)| = k - 1$, *then G must be the graph obtained from F by identifying the end vertices of a path* P_m, $m = |V(G)| - |V(F)| + 2$ *with two distinct vertices of F.*

Proof: Since G contains F, we let $Y = G - F$ and assume that there are e edges joining Y to F. Now note that $|E(Y)| = |E(G)| - |E(F)| - e$ and $|V(Y)| = |V(G)| - |V(F)|$ so that

$$|E(Y)| - |V(Y)| = |E(G)| - |V(G)| - (|E(F)| - |V(F)|) - e = 1 - e. \qquad (1)$$

Let $Y_1, Y_2, \ldots, Y_j, j \geq 1$ be the connected components of Y. Suppose there are e_i edges joining F and Y_i, $i = 1, \ldots, j$ so that $e = \sum_{i=1}^{j} e_i$. Let c_i denote the cyclomatic number of Y_i, $i = 1, \ldots, j$. Let $c = \sum_{i=1}^{j} c_i$. Using Equation (1) and from the definition of cyclomatic number, we have

$$c = \sum_{i=1}^{j} c_i = 1 - e + j. \qquad (2)$$

Since G is 2-connected, we have $e \geq 2j$. Hence, Equation (2) implies that $c \leq 1 - j$. Since $c \geq 0$, it follows that $j = 1$ and $c = 0$. Consequently, $e = 2$ and $Y = Y_1$ must be the path P_m, $m = |V(G)| - |V(F)| + 2$ whose end vertices are identified to two distinct vertices of F. □

COMPLETE TRIPARTITE GRAPH $K_{2,2,2}$

In what follow, we let $K^1_{2,2,2}(m)$ (respectively $K^2_{2,2,2}(m)$) denote the graph obtained from $K_{2,2,2} \cup P_m$ by identifying the end vertices of the path P_m, $m \geq 3$ with two adjacent (respectively, non-adjacent) vertices of $K_{2,2,2}$.

Theorem 2: The graph $K^i_{2,2,2}(m)$ *is* χ*-unique for* $m \geq 3$ *and* $i = 1, 2$.

Proof: Suppose $H \sim J \in \{K^i_{2,2,2}(m), i = 1, 2\}$, then H is a 2-connected graph on $m + 4$ vertices and $m + 11$ edges. Since the graphs $K^i_{2,2,2}(m)$, $i = 1, 2$ are χ-unique for $m = 3, 4$ (see [4]), we may assume $m \geq 5$. Note that by Lemma 1, $n(K_3, J) = n(K_3, H) = 8$, $n(C^*_4, J) = n(C^*_4, H) = 3$. Furthermore, $n(W_5, J) = 6$, $n(C^*_5, J) \leq 1$, $n(K_{2,3}, J) = n(U_5, J) = 0$. By Lemma 1, it follows that

$$n(K_{2,3},H) + 2n(U_5,H) + 3n(W_5,H) \geq n(C^*_5,H) + 17 \geq 17. \qquad (3)$$

We first note that a $K_{2,2,2}$ contains six W_5. We now claim that H contains at least six W_5 that forms a $K_{2,2,2}$. Suppose H does not contain a $K_{2,2,2}$. Note that any multipartite graph with a triangle and no K_4 is also a tripartite graph. Since $\chi(H) = 3$, H has a triangle and no K_4. Hence, H is also a tripartite graph. So H must contain (i) a $K_{1,2,2} = W_5$ or (ii) no $K_{1,2,2}$.

Case (i) H contains a $K_{1,2,2}$.

In this case, we note that H contains at most three $K_{1,2,2}$ (that do not overlap on a C_4^*). Otherwise, H must contain at least nine triangles or four C_4^* no matter how all the $K_{1,2,2}$ overlap on each other, a contradiction. By Equation (3), this implies that $n(K_{2,3},H) + 2n(U_5,H) \geq 8$. If $n(K_{2,3},H) \geq 2$, then $n(C_4^*,H) \geq 4$, a contradiction. Therefore, $n(K_{2,3},H) \leq 1$ which implies that $n(U_5,H) \geq 4$. This further implies that $n(C_4^*,H) \geq 4$, a contradiction.

Case (ii) H contains no $K_{1,2,2}$.

This implies that $n(K_{2,3},H) + 2n(U_5,H) \geq 17$. By the observation in Case (i), H must contain at least four C_4^*, a contradiction. Therefore, H contains a $K_{2,2,2}$ as subgraph. Since H is a 2-connected $(m + 4, m + 11)$-graph and $K_{2,2,2}$ is a 2-connected (6,12)-graph, by Theorem 1 and Lemma 3, $K_{2,2,2}^i(m)$ is χ-unique for $m \geq 3$ and $i = 1, 2$. The proof is thus complete. □

Remark. The chromatic uniqueness of $K_{2,2,2}^1(m)$, $m \geq 3$ has also been established by the authors in [2].

$K_{2,2,2}$ WITH AN EDGE DELETED

Let G_1 and G_2 be graphs, each containing a complete subgraph K_p with p vertices. If G is the graph obtained from G_1 and G_2 by identifying the two subgraphs K_p, then G is called a *K_p-gluing* of G_1 and G_2.

Let $G^{(0)}$ be a given graph which is K_p-gluing of some graphs, say G_1 and G_2. Forming another K_p-gluing of G_1 and G_2, we obtain a new graph $G^{(1)}$. Note that $G^{(1)}$ may not be isomorphic to $G^{(0)}$. Clearly, $G^{(1)}$ is a K_p-gluing of some graphs, say H_1 and H_2. Note that H_1 and H_2 may not be G_1 and G_2. Forming another K_p-gluing of H_1 and H_2, we obtain another graph $G^{(2)}$. The process of forming $G^{(1)}$ from $G^{(0)}$ (or $G^{(2)}$ from $G^{(1)}$) is called an *elementary operation*. A graph H is called a *relative* of G if H can be obtained from G by applying a finite sequence of elementary operations. Note that if H is a relative of G, then $H \sim G$.

Let $K_{2,2,2} - e$ denote the graph obtained by deleting an edge of $K_{2,2,2}$. Consider a graph H obtained from $G \cup P_m$ by identifying the two end vertices of P_m to two different vertices of G, where G is either $(K_{2,2,2} - e)$ or $K_{1,2,3}$. Then the graph H must be one of the graphs $G_i(m)$, $1 \leq i \leq 7$ (or their relatives) as shown in Figure 1.

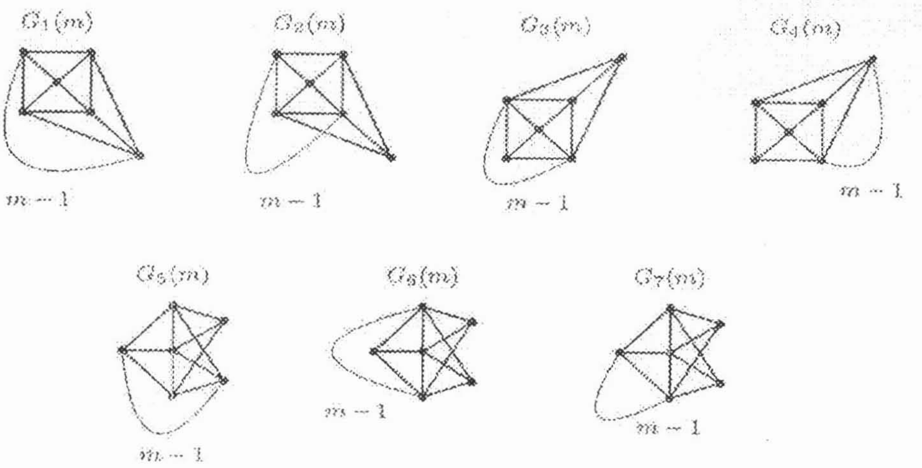

Figure 1: Graphs obtained from $K_{2,2,2} - e$ or $K_{1,2,3}$

Note that for $m = 3, 4$, the graphs G_i are χ-unique if and only if $i = 1, m = 4$ (see [4]). Thus, we only consider the graphs G_i for $m \geq 5$.

Theorem 3: $P(G_i) \neq P(G_j)$ for $1 \leq i < j \leq 4$. Also, $P(G_2) = P(G_5)$, $P(G_3) = P(G_6)$ and $P(G_4) = P(G_7)$.

Proof: We first note that $P(K_{2,2,2} - e) = \lambda(\lambda - 1)(\lambda - 2)(\lambda^3 - 8\lambda^2 + 23\lambda - 23) = P(K_{1,2,3})$. By Lemma 3, we have

(1) $P(G_1) = P(G_4) + (-1)^{m-1}P(K_4)P(K_4)/P(K_3) = P(G_4) + (-1)^{m-1}\lambda(\lambda - 1)(\lambda - 2)(\lambda - 3)^2$.
(2) $P(G_2) = P(G_4) + (-1)^{m-1}P(W_5) = P(G_4) + (-1)^{m-1}\lambda(\lambda - 1)(\lambda - 2)(\lambda^2 - 5\lambda + 7)$.
(3) $P(G_3) = P(G_4) + (-1)^{m-1}P(K_3)P(K_3)P(K_3) / P(K_2)P(K_2) = P(G_4) + (-1)^{m-1}\lambda(\lambda - 1)(\lambda - 2)^3$.
(4) $P(G_4) = P(K_{2,2,2} - e)P(C_m)/P(K_2) = P(K_{1,2,3})P(C_m) / P(K_2) = P(G_7)$.
(5) $P(G_5) = P(G_7) + (-1)^{m-1}P(W_5)$.
(6) $P(G_6) = P(G_7) + (-1)^{m-1}P(K_3)P(K_3)P(K_3) / P(K_2)P(K_2)$. □

Theorem 4: For $m \geq 5$, the graphs $G_1(m)$ is χ-unique and $H \in [G_i(m)]$, $i = 2, 3, 4$ if and only if $H = G_i$ or G_{i+3} (or their relatives).

Proof: Let G be a graph as defined in the theorem. Suppose $H \sim G$, then H is a 2-connected graph on $m + 4$ vertices and $m + 10$ edges. Note that by Lemma 1, $n(K_3, G) = n(K_3, H) = 6$, $n(C_4^*, G) = n(C_4^*, H) = 3$. Furthermore, $n(U_5, G_i) = n(W_5, G_i) = 2$, $n(K_{2,3}, G_i) = 0$, $n(C_5^*, G_i) \leq 1$ for $i = 1, 2, 3, 4$ whereas $n(U_5, G_i) = 0$, $n(W_5, G_i) = 3$, $n(K_{2,3}, G_i) = 1$, $n(C_5^*, G_i) \leq 1$ for $i = 5, 6, 7$. By Lemma 1, it follows that

$$n(K_{2,3}, H) + 2n(U_5, H) + 3n(W_5, H) \geq n(C_5^*, H) + 9 \geq 9. \qquad (4)$$

We claim that H has exactly two or three W_5. Suppose otherwise. Then H must have (i) at least four W_5 or (ii) at most one W_5.

Case (i) H has at least four W_5.

In this case, we note that H contains at least seven K_3 no matter how the W_5 overlap on each other, a contradiction.

Case (ii) H has at most one W_5. We consider two cases.

<u>Subcase (a)</u> If H has exactly one W_5, by Equation (4), this implies that $n(K_{2,3}, H) + 2n(U_5, H) \geq 6$. If $n(K_{2,3}, H) \geq 2$, then $n(C_4^*, H) \geq 5$, a contradiction. Therefore, $n(K_{2,3}, H) \leq 1$ which implies that $n(U_5, H) \geq 3$. Since all the U_5 cannot be subgraphs of the W_5 (see Theorem 2 of [3]), this further implies that $n(C_4^*, H) \geq 4$, a contradiction.

<u>Subcase (b)</u> If H has no W_5, by Equation (4), this implies that $n(K_{2,3}, H) + 2n(U_5, H) \geq 9$. By the observation in Subcase (a), H must contain at least four C_4^*, a contradiction.

Therefore, H contains two or three W_5 as subgraph. In either case, all the W_5 must overlap on a $K_{1,1,2}$ (a C_4 with a chord) to form a $K_{2,2,2} - e$ or a $K_{1,2,3}$. Otherwise, H has at least seven K_3, a contradiction. Since H is a 2-connected $(m + 4, m + 10)$-graph and both $K_{2,2,2} - e$ and $K_{1,2,3}$ are 2-connected (6,11)-graph, by Theorems 1 and 3, $G_1(m)$ is
χ-unique and $H \in [G_i(m)]$, $i = 2, 3, 4$ if and only if $H = G_i$ or G_{i+3} (or their relatives) for $m \geq 5$.

The proof is thus complete. □

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