UNIVERSITI TEKNOLOGI MARA
University Publication Centre (UPENA)

ESTEEM

Academic Journal UiTM Pulau Pinang

Volume 4, Number 2 2008 ISSN 1675-7939

ESTEEM Academic Journal is jointly published by the Universiti Teknologi MARA, Pulau Pinang and University Publication Centre (UPENA), Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia.

The views, opinions and technical recommendations expressed by the contributors and authors are entirely their own and do not necessarily reflect the views of the editors, the Faculty or the University.

ESTEEM
Academic Journal UiTM Pulau Pinang

Volume 4, Number 2 | 2008 | ISSN 1675-7939

SOCIAL SCIENCES

Foreword

This is the first time that ESTEEM Academic Journal UiTM Pulau Pinang has come up with 2 publications in a year! Previously, ESTEEM was published once biennially.

For these publications to materialise, I would like to thank Associate Professor Mohd Zaki Abdullah, the Director of UiTM Pulau Pinang for his unflinching support and who always told me, "Go ahead, don't worry about the money!".

Both the Associate Professor Mohd Zaki Abdullah and Dr. Mohamad Abdullah Hemdi, the Deputy Director of Academic Affairs really provided me with a great deal of assistance in ensuring that there are sufficient articles for publishing. Both of them have emphasized the need for lecturers to embark on journal writing. Incidentally this is one of the prerequisites for promotion among the academic staff members of UiTM Pulau Pinang.

I do not think I can run the show alone without the help from the editorial board, reviewers and the cooperation from University Publication Centre (UPENA) of UiTM Malaysia. My special thanks to Mr. Mohd Aminudin Murad for his efficiency in editing articles and to Dr. Khairil Iskandar Othman for speeding up the final stage of printing process.

Since writing is an important criterion in rating a university, I feel it is a great responsibility for me to produce a good journal. Fellow colleagues, let's work closely to put UiTM Pulau Pinang in the final list of Anugerah Kualiti Naib Canselor (AKNC) and Anugerah Kualiti Perdana Menteri (AKPM) by submitting more quality articles to ESTEEM!

Lastly, let me end by thanking all of you for giving your unwavering support to UPENA.

The Chief Editor
November, 2008

ESTEEM, Vol. 4, No. 2, 2008, 15–30

A Study on the Higher Moments of a Biased Estimator

Ng Set Foong
Low Heng Chin
Quah Soon Hoe

ABSTRACT

In this paper, the biased estimator that is derived in Ng, Low, and Quah (2007) is further studied. The expression for the higher moments of the estimator is important if we want to know the whole sampling properties of the estimator. Hence, a general expression for the higher moments of the estimator is derived in this paper. Furthermore, the bias, variance, mean squared error, skewness and kurtosis of the estimator are derived from the higher moments of the estimator.

Keywords: *linear regression, estimator, moments*

Introduction

In linear regression analysis, a linear regression model is constructed to predict a dependent variable from one or several independent variables. The least squares estimator is commonly used to estimate the unknown parameter in the regression model. Besides, a number of biased estimators such as the Ridge Regression Estimator (Hoerl & Kennard, 1970), the Almost Unbiased Generalized Ridge Regression Estimator (Singh, Chaubey, & Dwivedi, 1986), the Liu Estimator (Liu, 1993), and the Iteration Estimator (Trenkler, 1978) have been proposed to estimate the unknown parameter in the regression model.

A biased estimator is derived in Ng, Low, and Quah (2007) as an alternative to the least squares estimator. The biased estimator is shown to be preferable to the least squares estimator in terms of a reduction in mean squared error. Thus, the accuracy of the parameter estimate is increased by using the biased estimator in parameter estimation. In this paper, some further study is done on the estimator that is derived in Ng et al. (2007). The expression for the higher moments of the estimator is important if we want to know the whole sampling properties of the estimator. Hence, a general expression for the higher moments of the estimator is derived in this paper. The bias, variance, mean squared error, skewness, and kurtosis of the estimator are further derived from the higher moments of the estimator.

The Higher Moments of the Estimator

A linear regression model is generally written as

$$\mathbf{Y} = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\varepsilon}, \tag{1}$$

where $\mathbf{Y}$ is a vector of dependent variables, $\mathbf{X}$ is a matrix of p independent variables, $\boldsymbol{\beta}$ is a vector of parameters, and $\boldsymbol{\varepsilon}$ is a vector of errors such that $\boldsymbol{\varepsilon} \sim \mathrm{N}(\mathbf{0}, \sigma^2 \mathbf{I}_n)$ and $\mathbf{I}_n$ is a identity matrix.

The estimator that is derived in Ng et al. (2007) is given by

$$\hat{\boldsymbol{\beta}}_c = [\mathbf{I} - (\boldsymbol{\lambda} + c\mathbf{I})^{-1}(c-1)]\hat{\boldsymbol{\beta}}, \tag{2}$$

where $c > 1$ and $\hat{\boldsymbol{\beta}}$ is the least squares estimator of parameter $\boldsymbol{\beta}$.

The estimator, $\hat{\boldsymbol{\beta}}_c$, can be generalized as follows:

$$\hat{\boldsymbol{\beta}}_g = [\mathbf{I} - (\boldsymbol{\lambda} + \mathbf{G})^{-1}(\mathbf{G} - \mathbf{I})]\hat{\boldsymbol{\beta}}, \tag{3}$$

where matrix $\mathbf{G} = \mathrm{diag}(c_j)$ is a diagonal matrix of biasing factors c_j, $c_j > 1$, $j = 1, 2, ..., p$.

Suppose $(\hat{\beta}_g)_j$ is the j-th element of the vector $\hat{\boldsymbol{\beta}}_g$. Then, a feasible version of $(\hat{\beta}_g)_j$ is given by

$$(\hat{\beta}_{\hat{g}})_j = \left[1 - \frac{\hat{c}_j - 1}{\lambda_j + \hat{c}_j}\right]\hat{\beta}_j, \tag{4}$$

where $\hat{c}_j = \dfrac{\hat{\sigma}^2(\lambda_j + 1) + \lambda_j \hat{\beta}_j^2}{\lambda_j \hat{\beta}_j^2}$, $j = 1, 2, ..., p$, which is obtained from minimizing the mean squared error of $(\hat{\beta}_g)_j$. Thus, $(\hat{\beta}_{\hat{g}})_j$ is a known value because λ_j, $\hat{\beta}_j$, and $\hat{\sigma}^2$ can be obtained from observed data and hence $(\hat{\beta}_{\hat{g}})_j$ can be used for parameter estimation.

By using Binomial Expansion, the r-th moment of $(\hat{\beta}_{\hat{g}})_j$, $j = 1, 2, ..., p$, $r = 1, 2, 3, ...$, is given by

$$\mathrm{E}\left[((\hat{\beta}_{\hat{g}})_j)^r\right]=\mathrm{E}\left[\left(1-\frac{\hat{c}_j-1}{\lambda_j+\hat{c}_j}\right)^r\left(\hat{\beta}_j\right)^r\right]$$
$$=\mathrm{E}\left[\left(1-\vartheta_j\right)^r\left(\hat{\beta}_j\right)^r\right]$$
$$=\mathrm{E}\left[\left(\sum_{m=0}^{r}{}^rC_m(-1)^m(\vartheta_j)^m\right)\left(\hat{\beta}_j\right)^r\right]$$
$$=\sum_{m=0}^{r}{}^rC_m(-1)^m\,\mathrm{E}\left[(\vartheta_j)^m\left(\hat{\beta}_j\right)^r\right], \tag{5}$$

where ${}^rC_m=\dfrac{r!}{m!(r-m)!}$, $r=1, 2, 3, \ldots, m=0, 1, \ldots, r$,

$\vartheta_j=\dfrac{\hat{c}_j-1}{\lambda_j+\hat{c}_j}$ and $\hat{c}_j=\dfrac{\hat{\sigma}^2(\lambda_j+1)+\lambda_j\hat{\beta}_j^2}{\lambda_j\hat{\beta}_j^2}$.

Let $G(m,r)=\mathrm{E}\left[(\vartheta_j)^m\left(\hat{\beta}_j\right)^r\right]$ for even numbers of *r*, that is, $r=2, 4, 6, \ldots$, and $J(m,r)=\mathrm{E}\left[(\vartheta_j)^m\left(\hat{\beta}_j\right)^r\right]$ for odd numbers of *r*, that is, $r=1, 3, 5, \ldots$.

In order to solve $G(m,r)=\mathrm{E}\left[(\vartheta_j)^m\left(\hat{\beta}_j\right)^r\right]$ for even numbers of *r* and $J(m,r)=\mathrm{E}\left[(\vartheta_j)^m\left(\hat{\beta}_j\right)^r\right]$ for odd numbers of *r*, the probability density function of $\hat{\beta}_j$ and the probability density function of ϑ_j should be first obtained. Note that $\hat{\beta}_j$ is a normally distributed variable with $\hat{\beta}_j\sim\mathrm{N}(\beta_j,\frac{\sigma^2}{\lambda_j})$. Let

$$\tau_j=\frac{\sqrt{\lambda_j}\hat{\beta}_j}{\sigma}. \tag{6}$$

In fact, $\tau_j\sim\mathrm{N}(\mu_j,1)$. τ_j is also a normally distributed random variable with a mean of $\mu_j=\dfrac{\sqrt{\lambda_j}\beta_j}{\sigma}$ and variance 1. Therefore, instead of obtaining the probability density function of $\hat{\beta}_j$, we will obtain the probability density function of τ_j because $\hat{\beta}_j$ is related to τ_j.

We shall first obtain the probability density function of τ_j and then obtain the probability density function of ϑ_j. The probability density function of τ_j is given by

$$f_{\tau_j}(\tau_j) = \frac{\exp(-\frac{(\tau_j-\mu_j)^2}{2})}{\sqrt{2\pi}}$$

$$= \frac{\exp(-\frac{\tau_j^2}{2} - \frac{\mu_j^2}{2} + \tau_j\mu_j)}{\sqrt{2\pi}}$$

$$= \frac{\exp(-\frac{\tau_j^2}{2})\exp(-\frac{\mu_j^2}{2})\exp(\tau_j\mu_j)}{\sqrt{2\pi}}, \quad (7)$$

where $\tau_j = \frac{\sqrt{\lambda_j}\hat{\beta}_j}{\sigma}$, $-\infty < \tau_j < \infty$ and $\mu_j = \frac{\sqrt{\lambda_j}\beta_j}{\sigma}$.

Before obtaining the probability density function of ϑ_j, we re-write the variable $\vartheta_j = \frac{\hat{c}_j - 1}{\lambda_j + \hat{c}_j}$ as follows:

$$\vartheta_j = \frac{\hat{c}_j - 1}{\lambda_j + \hat{c}_j}$$

$$= \frac{\frac{\hat{\sigma}^2(\lambda_j+1)+\lambda_j\hat{\beta}_j^2}{\lambda_j\hat{\beta}_j^2} - 1}{\lambda_j + \frac{\hat{\sigma}^2(\lambda_j+1)+\lambda_j\hat{\beta}_j^2}{\lambda_j\hat{\beta}_j^2}}$$

$$= \frac{\hat{\sigma}^2}{\lambda_j\hat{\beta}_j^2 + \hat{\sigma}^2}$$

$$= \frac{1}{\frac{(\frac{\lambda_j\hat{\beta}_j^2}{\sigma^2})/1}{(\frac{v\hat{\sigma}^2}{\sigma^2})/v} + 1}$$

$$= \frac{1}{\varpi_j + 1}, \quad (8)$$

where $\hat{c}_j = \dfrac{\hat{\sigma}^2(\lambda_j + 1) + \lambda_j \hat{\beta}_j^2}{\lambda_j \hat{\beta}_j^2}$, $v = n - p$, $\varpi_j = \dfrac{(\frac{\lambda_j \hat{\beta}_j^2}{\sigma^2})/1}{(\frac{v\hat{\sigma}^2}{\sigma^2})/v}$ and $j = 1, 2, ..., p$.

Note that ϖ_j is a ratio of the variable $\dfrac{\lambda_j \hat{\beta}_j^2}{\sigma^2}$ divided by 1 to the variable $\dfrac{v\hat{\sigma}^2}{\sigma^2}$ divided by v. In order to obtain the probability distribution of ϖ_j, the probability distribution of the variables, $\dfrac{\lambda_j \hat{\beta}_j^2}{\sigma^2}$ and $\dfrac{v\hat{\sigma}^2}{\sigma^2}$, should be first identified. Since $\hat{\beta}_j$ is a normally distributed variable with $\hat{\beta}_j \sim \mathrm{N}(\beta_j, \dfrac{\sigma^2}{\lambda_j})$, $\dfrac{\sqrt{\lambda_j}(\hat{\beta}_j - \beta_j)}{\sigma}$ is a normally distributed variable with $\dfrac{\sqrt{\lambda_j}(\hat{\beta}_j - \beta_j)}{\sigma} \sim \mathrm{N}(0,1)$. Hence, $\left(\dfrac{\sqrt{\lambda_j}(\hat{\beta}_j - \beta_j)}{\sigma} + \dfrac{\sqrt{\lambda_j}\beta_j}{\sigma}\right)^2 = \dfrac{\lambda_j \hat{\beta}_j^2}{\sigma^2}$ has a non-central Chi-squared distribution with 1 degree of freedom and non-central parameter, $\theta_j = \left(\dfrac{\sqrt{\lambda_j}\beta_j}{\sigma}\right)^2 = \dfrac{\lambda_j \beta_j^2}{\sigma^2}$. Thus, the probability distribution of the variable $\dfrac{\lambda_j \hat{\beta}_j^2}{\sigma^2}$ is a non-central Chi-squared distribution with 1 degree of freedom and non-central parameter, $\theta_j = \dfrac{\lambda_j \beta_j^2}{\sigma^2}$. Note that $\dfrac{v\hat{\sigma}^2}{\sigma^2}$ has an ordinary Chi-squared distribution with $v = n - p$ degrees of freedom. From these two independent Chi-squared variables, $\dfrac{\lambda_j \hat{\beta}_j^2}{\sigma^2}$ and $\dfrac{v\hat{\sigma}^2}{\sigma^2}$, it is found that the variable $\varpi_j = \dfrac{(\frac{\lambda_j \hat{\beta}_j^2}{\sigma^2})/1}{(\frac{v\hat{\sigma}^2}{\sigma^2})/v}$ has a non-central F distribution with 1 degree of freedom for the numerator, v degrees of freedom for the denominator and a non-central parameter, $\theta_j = \dfrac{\lambda_j \beta_j^2}{\sigma^2}$. The probability density function of ϖ_j is given by

$$f_{\varpi_j}(\varpi_j \mid 1, v; \theta_j, 0) = \frac{1}{v}\sum_{t=0}^{\infty} \frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t \left(\frac{\varpi_j}{v}\right)^{\frac{1}{2}+t-1} \Gamma(\frac{1+v}{2}+t)}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{v}{2})(1+\frac{\varpi_j}{v})^{\frac{1+v}{2}+t}(t!)}$$

$$= \sum_{t=0}^{\infty} \frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t \varpi_j^{t-\frac{1}{2}} \Gamma(\frac{1+v}{2}+t) v^{\frac{v}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{v}{2})(v+\varpi_j)^{\frac{1+v}{2}+t}(t!)}, \qquad (9)$$

where $\varpi_j > 0$, $\theta_j = \dfrac{\lambda_j \beta_j^2}{\sigma^2}$, $v = n - p$ and $t = 0, 1, 2, \ldots$.

From $\vartheta_j = \dfrac{1}{\varpi_j + 1}$ (Equation 8), the variabl*e* ϖ can be expressed as $\varpi_j = \dfrac{1-\vartheta_j}{\vartheta_j}$. Thus, the probability density function of ϑ_j is given by

$$f_{\vartheta_j}(\vartheta_j) = f_{\varpi_j}\left(\frac{1-\vartheta_j}{\vartheta_j}\right)\left|\frac{d}{d\vartheta_j}\left(\frac{1-\vartheta_j}{\vartheta_j}\right)\right|$$

$$= \sum_{t=0}^{\infty} \frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t \left(\frac{1-\vartheta_j}{\vartheta_j}\right)^{t-\frac{1}{2}} \Gamma(\frac{1+v}{2}+t) v^{\frac{v}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{v}{2})(v+\frac{1-\vartheta_j}{\vartheta_j})^{\frac{1+v}{2}+t}(t!)\vartheta_j^2}$$

$$= \sum_{t=0}^{\infty} \frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t (1-\vartheta_j)^{t-\frac{1}{2}} \Gamma(\frac{1+v}{2}+t) v^{\frac{v}{2}} \vartheta_j^{\frac{v}{2}-1}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{v}{2})(v\vartheta_j + 1 - \vartheta_j)^{\frac{1+v}{2}+t}(t!)}, \qquad (10)$$

where $0 < \vartheta_j < 1$, $\theta_j = \dfrac{\lambda_j \beta_j^2}{\sigma^2}$, $v = n - p$ and $t = 0, 1, 2, \ldots$.

Now, the probability density function of $\tau_j = \dfrac{\sqrt{\lambda_j}\hat{\beta}_j}{\sigma}$ and the probability density function of ϑ_j are obtained. We shall next derive the r-th moment of $(\hat{\beta}_{\hat{g}})_j$. Theorem 1 gives the r-th moment of $(\hat{\beta}_{\hat{g}})_j$, where $j = 1, 2, \ldots, p$ and r is an even number, that is $r = 2, 4, 6, \ldots$. Theorem 2 gives the r-th moment of $(\hat{\beta}_{\hat{g}})_j$, where $j = 1, 2, \ldots, p$ and r is an odd number, that is $r = 1, 3, 5, \ldots$.

Theorem 1. For even numbers of r, that is $r = 2d$, $d = 1, 2, 3, \ldots$, the r-th moment of $(\hat{\beta}_{\hat{g}})_j$, $j = 1, 2, \ldots, p$, is defined by

$$\mathrm{E}\left[((\hat{\beta}_{\hat{g}})_j)^r\right] = \sum_{m=0}^{r} {}^rC_m (-1)^m G(m,r),$$

where ${}^rC_m = \dfrac{r!}{m!(r-m)!}$,

$r = 2d = 2, 4, 6, \ldots$, $d = 1, 2, 3, \ldots$, $m = 0, 1, \ldots, r$,

$$G(m,r) = G(m,2d) = 2^d \left(\tfrac{\sigma^2}{\lambda_j}\right)^d \sum_{t=0}^{\infty} \sum_{q=0}^{\infty} h_t(\theta_j) h_q(\mu_j) K_{t,q}(m,d),$$

$$h_t(\theta_j) = \frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t}{t!}, \quad \theta_j = \frac{\lambda_j \beta_j^2}{\sigma^2}, \quad t = 0, 1, 2, \ldots,$$

$$h_q(\mu_j) = \frac{\exp(-\frac{\mu_j^2}{2})\left(\frac{\mu_j^2}{2}\right)^q}{q!}, \quad \mu_j = \frac{\sqrt{\lambda_j}\beta_j}{\sigma}, \quad q = 0, 1, 2, \ldots,$$

$$K_{t,q}(m,d) = \frac{\Gamma(\frac{1+\nu}{2}+t)\Gamma(d+q+\frac{1}{2})\nu^{\frac{\nu}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{\nu}{2})\Gamma(q+\frac{1}{2})}$$
$$\int_0^1 \left[(\vartheta_j)^{m+\frac{\nu}{2}-1}\left(1-\vartheta_j\right)^{t-\frac{1}{2}} (\nu\vartheta_j - \vartheta_j + 1)^{-(\frac{1+\nu}{2}+t)}\right] d\vartheta_j ,$$

$$\nu = n - p, \quad \vartheta_j = \frac{\hat{c}_j - 1}{\lambda_j + \hat{c}_j} \quad \text{and} \quad \hat{c}_j = \frac{\hat{\sigma}^2(\lambda_j + 1) + \lambda_j \hat{\beta}_j^2}{\lambda_j \hat{\beta}_j^2}.$$

Proof. For $r = 2d$, where $d = 1, 2, 3, \ldots$ and $r = 2, 4, 6, \ldots$, the r-th moment of $(\hat{\beta}_{\hat{g}})_j$, $j = 1, 2, \ldots, p$, can be written as

$$\mathrm{E}\left[((\hat{\beta}_{\hat{g}})_j)^r\right] = \sum_{m=0}^{r} {}^rC_m (-1)^m \, \mathrm{E}\left[(\vartheta_j)^m \left(\hat{\beta}_j\right)^r\right],$$

where, $G(m,r) = \mathrm{E}\left[(\vartheta_j)^m \left(\hat{\beta}_j\right)^r\right]$ for even numbers of r, $m = 0, 1, \ldots, r$, $\vartheta_j = \dfrac{\hat{c}_j - 1}{\lambda_j + \hat{c}_j}$ and $\hat{c}_j = \dfrac{\hat{\sigma}^2(\lambda_j + 1) + \lambda_j \hat{\beta}_j^2}{\lambda_j \hat{\beta}_j^2}$.

In order to obtain a formula for $\mathrm{E}\left[((\hat{\beta}_{\hat{g}})_j)^r\right] = \sum_{m=0}^{r} {}^rC_m (-1)^m \,\mathrm{E}\left[(\vartheta_j)^m \left(\hat{\beta}_j\right)^r\right]$, we should first obtain a formula for $G(m,r) = \mathrm{E}\left[(\vartheta_j)^m \left(\hat{\beta}_j\right)^r\right]$, where r is an even number. Note that the probability density function of $\tau_j = \dfrac{\sqrt{\lambda_j}\hat{\beta}_j}{\sigma}$, $f_{\tau_j}(\tau_j) = \dfrac{\exp(-\frac{\tau_j^2}{2})\exp(-\frac{\mu_j^2}{2})\exp(\tau_j\mu_j)}{\sqrt{2\pi}}$ (Equation 7), and the probability density function of ϑ_j, $f_{\vartheta_j}(\vartheta_j) = \sum_{t=0}^{\infty} \dfrac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t \left(1-\vartheta_j\right)^{t-\frac{1}{2}} \Gamma(\frac{1+v}{2}+t) v^{\frac{v}{2}} \vartheta_j^{\frac{v}{2}-1}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{v}{2})(v\vartheta_j + 1 - \vartheta_j)^{\frac{1+v}{2}+t}(t!)}$ (Equation 10).

Thus, $G(m,r)$ can be expressed as

$$
\begin{aligned}
& G(m,r) \\
&= G(m,2d) \\
&= \mathrm{E}\left[(\vartheta_j)^m \left(\hat{\beta}_j\right)^r\right] \\
&= \mathrm{E}\left[(\vartheta_j)^m \left(\frac{\tau_j\sigma}{\sqrt{\lambda_j}}\right)^{2d}\right] \\
&= \left(\frac{\sigma}{\sqrt{\lambda_j}}\right)^{2d} \mathrm{E}\left[(\vartheta_j)^m \left(\tau_j\right)^{2d}\right] \\
&= \left(\tfrac{\sigma}{\sqrt{\lambda_j}}\right)^{2d} \int_0^1 \int_{-\infty}^{\infty} (\vartheta_j)^m \left(\tau_j\right)^{2d} \\
&\quad \left[\sum_{t=0}^{\infty} \frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t \left(1-\vartheta_j\right)^{t-\frac{1}{2}} \Gamma(\frac{1+v}{2}+t) v^{\frac{v}{2}} \vartheta_j^{\frac{v}{2}-1}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{v}{2})(v\vartheta_j + 1 - \vartheta_j)^{\frac{1+v}{2}+t}(t!)}\right] \\
&\quad \times \left[\frac{\exp(-\frac{\tau_j^2}{2})\exp(-\frac{\mu_j^2}{2})\exp(\tau_j\mu_j)}{\sqrt{2\pi}}\right] d\tau_j d\vartheta_j
\end{aligned}
$$

$$=\sum_{t=0}^{\infty}\left[\left(\frac{\sigma}{\sqrt{\lambda_j}}\right)^{2d}\frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t}{t!}\frac{\exp(-\frac{\mu_j^2}{2})}{\sqrt{2\pi}}\frac{\Gamma(\frac{1+v}{2}+t)v^{\frac{v}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{v}{2})}\right]$$

$$\times\int_0^1\int_{-\infty}^{\infty}\Big[(\vartheta_j)^{m+\frac{v}{2}-1}\left(\tau_j\right)^{2d}\exp(-\tfrac{\tau_j^2}{2})\exp(\tau_j\mu_j)\left(1-\vartheta_j\right)^{t-\frac{1}{2}} \quad (11)$$

$$(v\vartheta_j-\vartheta_j+1)^{-(\frac{1+v}{2}+t)}\Big]d\tau_j d\vartheta_j$$

Using the expression, $\exp(\tau_j\mu_j)=\sum_{k=0}^{\infty}\frac{(\tau_j\mu_j)^k}{k!}$, $k=0, 1, 2, \ldots,$ the Equation 11 becomes

$$G(m,2d)=\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\left[\frac{\mu_j^k}{k!}\left(\frac{\sigma}{\sqrt{\lambda_j}}\right)^{2d}\frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t}{t!}\frac{\exp(-\frac{\mu_j^2}{2})}{\sqrt{2\pi}}\right.$$

$$\left.\frac{\Gamma(\frac{1+v}{2}+t)v^{\frac{v}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{v}{2})}\right]\times\int_0^1\int_{-\infty}^{\infty}\Big[\left(\tau_j\right)^{2d+k}\exp(-\tfrac{\tau_j^2}{2})(\vartheta_j)^{m+\frac{v}{2}-1}$$

$$\left(1-\vartheta_j\right)^{t-\frac{1}{2}}(v\vartheta_j-\vartheta_j+1)^{-(\frac{1+v}{2}+t)}\Big]d\tau_j d\vartheta_j$$

$$=\sum_{t=0}^{\infty}\sum_{k=0}^{\infty}\left[\frac{\mu_j^k}{k!}\left(\frac{\sigma}{\sqrt{\lambda_j}}\right)^{2d}\frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t}{t!}\frac{\exp(-\frac{\mu_j^2}{2})}{\sqrt{2\pi}}\frac{\Gamma(\frac{1+v}{2}+t)v^{\frac{v}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{v}{2})}\right]$$

$$\times\int_0^1\Big[(\vartheta_j)^{m+\frac{v}{2}-1}\left(1-\vartheta_j\right)^{t-\frac{1}{2}}(v\vartheta_j-\vartheta_j+1)^{-(\frac{1+v}{2}+t)}\Big]d\vartheta_j$$

$$\int_{-\infty}^{\infty}\Big[\left(\tau_j\right)^{2d+k}\exp(-\tfrac{\tau_j^2}{2})\Big]d\tau_j \quad (12)$$

When $k=2q+1$, $q=0, 1, 2, \ldots$, the integral in Equation 12 is equal to 0 because the integral is an odd function of τ_j. When $k=2q$, $q=0, 1, 2, \ldots$, and considering the integration with respect to τ_j^2 instead of τ_j, the Equation 12 becomes

$$G(m,2d)$$

$$= \sum_{t=0}^{\infty} \sum_{2q=0}^{\infty} \left[\frac{\mu_j^{2q}}{(2q)!} \left(\frac{\sigma}{\sqrt{\lambda_j}} \right)^{2d} \frac{\exp(-\frac{\theta_j}{2}) \left(\frac{\theta_j}{2} \right)^t}{t!} \frac{\exp(-\frac{\mu_j^2}{2})}{\sqrt{2\pi}} \frac{\Gamma(\frac{1+\nu}{2}+t)\nu^{\frac{\nu}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{\nu}{2})} \right]$$

$$\times \int_0^1 \left[(\vartheta_j)^{m+\frac{\nu}{2}-1} \left(1-\vartheta_j\right)^{t-\frac{1}{2}} (\nu\vartheta_j - \vartheta_j + 1)^{-(\frac{1+\nu}{2}+t)} \right] d\vartheta_j$$

$$\int_0^{\infty} \left[2\left(\tau_j\right)^{2d+2q} \exp(-\tfrac{\tau_j^2}{2}) \left(\tfrac{1}{2\tau_j} \right) \right] d(\tau_j^2)$$

$$= \sum_{t=0}^{\infty} \sum_{q=0}^{\infty} \left[\frac{\mu_j^{2q}}{(2q)!} \left(\frac{\sigma}{\sqrt{\lambda_j}} \right)^{2d} \frac{\exp(-\frac{\theta_j}{2}) \left(\frac{\theta_j}{2} \right)^t}{t!} \frac{\exp(-\frac{\mu_j^2}{2})}{\sqrt{2\pi}} \frac{\Gamma(\frac{1+\nu}{2}+t)\nu^{\frac{\nu}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{\nu}{2})} \right]$$

$$\times \int_0^1 \left[(\vartheta_j)^{m+\frac{\nu}{2}-1} \left(1-\vartheta_j\right)^{t-\frac{1}{2}} (\nu\vartheta_j - \vartheta_j + 1)^{-(\frac{1+\nu}{2}+t)} \right] d\vartheta_j$$

$$\int_0^{\infty} \left[\left(\tau_j^2\right)^{d+q-\frac{1}{2}} \exp(-\tfrac{\tau_j^2}{2}) \right] d(\tau_j^2)$$

$$= \sum_{t=0}^{\infty} \sum_{q=0}^{\infty} \left[2^{(d+q+\frac{1}{2})-1} \left(\frac{\mu_j^{2q}}{(2q)!} \right) \left(\frac{\sigma}{\sqrt{\lambda_j}} \right)^{2d} \frac{\exp(-\frac{\theta_j}{2}) \left(\frac{\theta_j}{2} \right)^t}{t!} \right. \tag{13}$$

$$\left. \frac{\exp(-\frac{\mu_j^2}{2})}{\sqrt{2\pi}} \frac{\Gamma(\frac{1+\nu}{2}+t)\nu^{\frac{\nu}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{\nu}{2})} \right]$$

$$\times \int_0^1 \left[(\vartheta_j)^{m+\frac{\nu}{2}-1} \left(1-\vartheta_j\right)^{t-\frac{1}{2}} (\nu\vartheta_j - \vartheta_j + 1)^{-(\frac{1+\nu}{2}+t)} \right] d\vartheta_j$$

$$\int_0^{\infty} \left[\left(\frac{\tau_j^2}{2} \right)^{(d+q+\frac{1}{2})-1} \exp(-\tfrac{\tau_j^2}{2}) \right] d(\tau_j^2)$$

By noting that $\int_0^\infty \left[\left(\frac{\tau_j^2}{2}\right)^{(d+q+\frac{1}{2})-1} \exp(-\frac{\tau_j^2}{2}) \right] d(\tau_j^2) = 2\Gamma(d+q+\frac{1}{2})$,

which is the Gamma function, Equation 13 reduces to

$$G(m,2d)$$

$$= \sum_{t=0}^{\infty} \sum_{q=0}^{\infty} \Bigg[2^{(d+q+\frac{1}{2})-1} \left(\frac{\mu_j^{2q}}{(2q)!} \right) \left(\frac{\sigma}{\sqrt{\lambda_j}} \right)^{2d} \frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t}{t!} \frac{\exp(-\frac{\mu_j^2}{2})}{\sqrt{2\pi}}$$

$$\frac{\Gamma(\frac{1+\nu}{2}+t)\nu^{\frac{\nu}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{\nu}{2})} \Bigg] \times \left[2\Gamma(d+q+\tfrac{1}{2}) \right]$$

$$\int_0^1 \left[(\vartheta_j)^{m+\frac{\nu}{2}-1} \left(1-\vartheta_j\right)^{t-\frac{1}{2}} (\nu\vartheta_j - \vartheta_j + 1)^{-(\frac{1+\nu}{2}+t)} \right] d\vartheta_j$$

$$= \sum_{t=0}^{\infty} \sum_{q=0}^{\infty} \Bigg[2^{d+q} \left(\frac{\sigma}{\sqrt{\lambda_j}} \right)^{2d} \frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t}{t!} \frac{\exp(-\frac{\mu_j^2}{2})\left(\mu_j^2\right)^q}{1}$$

$$\frac{\Gamma(\frac{1+\nu}{2}+t)\Gamma(d+q+\frac{1}{2})\nu^{\frac{\nu}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{\nu}{2})\left[\sqrt{\pi}(2q)!\right]} \Bigg]$$

$$\times \int_0^1 \left[(\vartheta_j)^{m+\frac{\nu}{2}-1} \left(1-\vartheta_j\right)^{t-\frac{1}{2}} (\nu\vartheta_j - \vartheta_j + 1)^{-(\frac{1+\nu}{2}+t)} \right] d\vartheta_j$$

$$= \sum_{t=0}^{\infty} \sum_{q=0}^{\infty} \Bigg[2^{d+2q} \left(\frac{\sigma^2}{\lambda_j} \right)^{d} \frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t}{t!} \frac{\exp(-\frac{\mu_j^2}{2})\left(\frac{\mu_j^2}{2}\right)^q}{q!}$$

$$\frac{\Gamma(\frac{1+\nu}{2}+t)\Gamma(d+q+\frac{1}{2})\nu^{\frac{\nu}{2}}(q!)}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{\nu}{2})\left[\sqrt{\pi}(2q)!\right]} \Bigg]$$

$$\times \int_0^1 \left[(\vartheta_j)^{m+\frac{\nu}{2}-1} \left(1-\vartheta_j\right)^{t-\frac{1}{2}} (\nu\vartheta_j - \vartheta_j + 1)^{-(\frac{1+\nu}{2}+t)} \right] d\vartheta_j \qquad (14)$$

Applying the *duplication formula* for the Gamma function (Stroud, 2003), $\sqrt{\pi}(2q)! = 2^{2q}\Gamma(q+\frac{1}{2})q!$, Equation 14 reduces to

$$G(m,2d) = \sum_{t=0}^{\infty}\sum_{q=0}^{\infty}\Bigg[2^{d+2q}\left(\tfrac{\sigma^2}{\lambda_j}\right)^d \frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t}{t!}\frac{\exp(-\frac{\mu_j^2}{2})\left(\frac{\mu_j^2}{2}\right)^q}{q!}$$

$$\frac{\Gamma(\frac{1+v}{2}+t)\Gamma(d+q+\frac{1}{2})v^{\frac{v}{2}}(q!)}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{v}{2})\left[2^{2q}\Gamma(q+\frac{1}{2})q!\right]}\Bigg]$$

$$\times\int_0^1\left[(\vartheta_j)^{m+\frac{v}{2}-1}\left(1-\vartheta_j\right)^{t-\frac{1}{2}}(v\vartheta_j-\vartheta_j+1)^{-(\frac{1+v}{2}+t)}\right]d\vartheta_j$$

$$= 2^d\left(\tfrac{\sigma^2}{\lambda_j}\right)^d\sum_{t=0}^{\infty}\sum_{q=0}^{\infty}\Bigg[\frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t}{t!}\frac{\exp(-\frac{\mu_j^2}{2})\left(\frac{\mu_j^2}{2}\right)^q}{q!}$$

$$\frac{\Gamma(\frac{1+v}{2}+t)\Gamma(d+q+\frac{1}{2})v^{\frac{v}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{v}{2})\Gamma(q+\frac{1}{2})}\Bigg]$$

$$\times\int_0^1\left[(\vartheta_j)^{m+\frac{v}{2}-1}\left(1-\vartheta_j\right)^{t-\frac{1}{2}}(v\vartheta_j-\vartheta_j+1)^{-(\frac{1+v}{2}+t)}\right]d\vartheta_j$$

$$= 2^d\left(\tfrac{\sigma^2}{\lambda_j}\right)^d\sum_{t=0}^{\infty}\sum_{q=0}^{\infty}h_t(\theta_j)h_q(\mu_j)\frac{\Gamma(\frac{1+v}{2}+t)\Gamma(d+q+\frac{1}{2})v^{\frac{v}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{v}{2})\Gamma(q+\frac{1}{2})}$$

$$\times\int_0^1\left[(\vartheta_j)^{m+\frac{v}{2}-1}\left(1-\vartheta_j\right)^{t-\frac{1}{2}}(v\vartheta_j-\vartheta_j+1)^{-(\frac{1+v}{2}+t)}\right]d\vartheta_j$$

$$= 2^d\left(\tfrac{\sigma^2}{\lambda_j}\right)^d\sum_{t=0}^{\infty}\sum_{q=0}^{\infty}h_t(\theta_j)\,h_q(\mu_j)\,K_{t,q}(m,d), \qquad (15)$$

where $h_t(\theta_j) = \dfrac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t}{t!}$, $\theta_j = \dfrac{\lambda_j\beta_j^2}{\sigma^2}$, $t = 0, 1, 2, \ldots,$

$$h_q(\mu_j) = \frac{\exp(-\frac{\mu_j^2}{2})\left(\frac{\mu_j^2}{2}\right)^q}{q!}, \quad \mu_j = \frac{\sqrt{\lambda_j}\beta_j}{\sigma}, \quad q = 0, 1, 2, \ldots,$$

$$K_{t,q}(m,d) = \frac{\Gamma(\frac{1+v}{2}+t)\Gamma(d+q+\frac{1}{2})v^{\frac{v}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{v}{2})\Gamma(q+\frac{1}{2})}$$

$$\int_0^1 \left[(\vartheta_j)^{m+\frac{v}{2}-1} \left(1-\vartheta_j\right)^{t-\frac{1}{2}} (v\vartheta_j - \vartheta_j + 1)^{-(\frac{1+v}{2}+t)} \right] d\vartheta_j \ ,$$

$$\vartheta_j = \frac{\hat{c}_j - 1}{\lambda_j + \hat{c}_j}\ ,\ \hat{c}_j = \frac{\hat{\sigma}^2(\lambda_j+1) + \lambda_j \hat{\beta}_j^2}{\lambda_j \hat{\beta}_j^2}\ ,$$

$v = n - p$,
$r = 2d = 2, 4, 6, \ldots,\ d = 1, 2, 3, \ldots$ and $m = 0, 1, \ldots, r$.

The proof for Theorem 1 is completed.

Theorem 2 is given here with the proof omitted. The procedure to obtain the proof for Theorem 2 is similar to the procedure to obtain the proof for Theorem 1.

Theorem 2. For an odd number of *r*, that is $r = 2d+1,\ d = 0, 1, 2, \ldots,$ the *r*-th moment of $(\hat{\beta}_{\hat{g}})_j,\ j = 1, 2, \ldots, p$, is defined by

$$\mathrm{E}\left[((\hat{\beta}_{\hat{g}})_j)^r \right] = \sum\nolimits_{m=0}^{r} {}^rC_m (-1)^m J(m,r)\ ,$$

where ${}^rC_m = \dfrac{r!}{m!(r-m)!}$,

$r = 2d+1 = 1, 3, 5, \ldots,\ d = 0, 1, 2, \ldots,\ m = 0, 1, \ldots, r$,

$$J(m,r) = J(m, 2d+1)$$
$$= 2^d \left(\tfrac{\sigma^2}{\lambda_j}\right)^d \beta_j \sum\nolimits_{t=0}^{\infty} \sum\nolimits_{q=0}^{\infty}\ h_t(\theta_j) h_q(\mu_j) K_{t,q+1}(m,d),$$

$$h_t(\theta_j) = \frac{\exp(-\frac{\theta_j}{2})\left(\frac{\theta_j}{2}\right)^t}{t!}\ ,\ \theta_j = \frac{\lambda_j \beta_j^2}{\sigma^2}\ ,\ t = 0, 1, 2, \ldots,$$

$$h_q(\mu_j) = \frac{\exp(-\frac{\mu_j^2}{2})\left(\frac{\mu_j^2}{2}\right)^q}{q!}\ ,\ \mu_j = \frac{\sqrt{\lambda_j}\beta_j}{\sigma}\ ,\ q = 0, 1, 2, \ldots,$$

$$K_{t,q+1}(m,d) = \frac{\Gamma(\frac{1+v}{2}+t)\Gamma(d+q+\frac{3}{2})v^{\frac{v}{2}}}{\Gamma(\frac{1}{2}+t)\Gamma(\frac{v}{2})\Gamma(q+\frac{3}{2})}$$

$$\int_0^1 \left[(\vartheta_j)^{m+\frac{v}{2}-1} \left(1-\vartheta_j\right)^{t-\frac{1}{2}} (v\vartheta_j - \vartheta_j + 1)^{-(\frac{1+v}{2}+t)} \right] d\vartheta_j,$$

$$\nu = n - p,$$

$$\vartheta_j = \frac{\hat{c}_j - 1}{\lambda_j + \hat{c}_j} \text{ and } \hat{c}_j = \frac{\hat{\sigma}^2(\lambda_j + 1) + \lambda_j \hat{\beta}_j^2}{\lambda_j \hat{\beta}_j^2}.$$

The Bias, Variance, Mean Squared Error, Skewness, and Kurtosis

The first, second, third, and fourth moments of $(\hat{\beta}_{\hat{g}})_j$ are given by Equations 16 to 19 respectively.

$$\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j\right) = J(0,1) - J(1,1) \tag{16}$$

$$\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j^2\right) = G(0,2) - 2G(1,2) + G(2,2) \tag{17}$$

$$\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j^3\right) = J(0,3) - 3J(1,3) + 3J(2,3) - J(3,3) \tag{18}$$

$$\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j^4\right) = G(0,4) - 4G(1,4) + 6G(2,4) - 4G(3,4) + G(4,4) \tag{19}$$

Thus, the bias, variance, mean squared error, skewness and kurtosis for $(\hat{\beta}_{\hat{g}})_j$ can be computed using the following equations:

$$\begin{aligned}\mathrm{bias}((\hat{\beta}_{\hat{g}})_j) &= \mathrm{E}\left((\hat{\beta}_{\hat{g}})_j\right) - \beta_j \\ &= J(0,1) - J(1,1) - \beta_j\end{aligned} \tag{20}$$

$$\mathrm{var}((\hat{\beta}_{\hat{g}})_j) = \mathrm{E}\left((\hat{\beta}_{\hat{g}})_j^2\right) - \left[\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j\right)\right]^2$$

$$= G(0,2) - 2G(1,2) + G(2,2) - [J(0,1)]^2 + 2J(0,1)J(1,1) - [J(1,1)]^2 \tag{21}$$

$$\mathrm{mse}((\hat{\beta}_{\hat{g}})_j) = \mathrm{var}((\hat{\beta}_{\hat{g}})_j) + \left[\mathrm{bias}((\hat{\beta}_{\hat{g}})_j)\right]^2$$

$$= G(0,2) - 2G(1,2) + G(2,2) + \beta_j^2 - 2\beta_j J(0,1) + 2\beta_j J(1,1) \tag{22}$$

$$\text{skewness} = \frac{\mathrm{E}\left[[(\hat{\beta}_{\hat{g}})_j - \mathrm{E}((\hat{\beta}_{\hat{g}})_j)]^3 \right]}{\left[\mathrm{var}((\hat{\beta}_{\hat{g}})_j) \right]^{3/2}}$$

$$= \frac{\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j^3\right) - 3\,\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j^2\right)\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j\right) + 2[\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j\right)]^3}{\left[\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j^2\right) - \left[\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j\right)\right]^2 \right]^{\frac{3}{2}}} \quad (23)$$

$$\text{kurtosis} = \frac{\mathrm{E}\left[[(\hat{\beta}_{\hat{g}})_j - \mathrm{E}((\hat{\beta}_{\hat{g}})_j)]^4 \right]}{\left[\mathrm{var}((\hat{\beta}_{\hat{g}})_j) \right]^{2}}$$

$$= \Big\{ \mathrm{E}\left((\hat{\beta}_{\hat{g}})_j^4\right) - 4\,\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j^3\right)\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j\right) + 6\,\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j^2\right)[\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j\right)]^2 - 3[\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j\right)]^4 \Big\} / \left\{ \left[\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j^2\right) - \left[\mathrm{E}\left((\hat{\beta}_{\hat{g}})_j\right)\right]^2 \right]^2 \right\} \quad (24)$$

Conclusion

In this paper, a further study is done on the biased estimator that is derived in Ng et al. (2007). The general expression for the moments of the biased estimator is developed in this paper. The bias, variance, mean squared error, skewness, and kurtosis of the estimator are further derived from the higher moments of the estimator. The results enable us to further study the sampling properties of the biased estimator.

References

Hoerl, A. E., & Kennard, R. W. (1970). Ridge regression: Biased estimation for non-orthogonal problems. *Technometrics*, *12*, 55–67.

Liu, K. (1993). A new class of biased estimate in linear regression. *Communications in Statistics-Theory and Methods*, *22*(2), 393–402.

Ng, S. F., Low, H. C., & Quah, S. H. (2007). A study on the Liu-type estimator. *Proceedings of the Third IMT-GT Regional Conference on Mathematics, Statistics and Applications: Strengthening Regional Cooperation through the Mathematical Sciences.* Universiti Sains Malaysia, Penang.

Singh, B., Chaubey, Y. P., & Dwivedi, T. D. (1986). An almost unbiased ridge estimator. *Sankhya: The Indian Journal of Statistics, 48*(B), 342–346.

Stroud, K. A. (2003). *Advanced engineering mathematics.* New York: Industrial Press.

Trenkler, G. (1978). An iteration estimator for the linear model. *Compstat,* 125–131.

NG SET FOONG, Department of Information Technology and Quantitative Sciences, Universiti Teknologi MARA Pulau Pinang, 13500 Permatang Pauh, Pulau Pinang, MALAYSIA. E-mail: setfoong@yahoo.com

LOW HENG CHIN & QUAH SOON HOE, School of Mathematical Sciences, Universiti Sains Malaysia, 11800 USM Pulau Pinang, MALAYSIA.