Mathematical Sciences and Informatics Journal
Vol. 1, No. 2, Nov. 2020, pp. 1-11
http://www.mijuitmjournal.com
DOI : 10.24191/mij.v1i2.14179

Lyapunov Function for Two-species Mutualism Model with Constant Harvesting

Rusliza Ahmad
Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, Perak Branch Tapah Campus, Perak, Malaysia
rusliza259@uitm.edu.my

Article Info

Article history:

Received Apr. 05, 2020
Revised Aug. 15, 2020
Accepted Oct. 10, 2020

Keywords:

Mutualism Model
Constant Harvesting
Global Stability
Lyapunov Function

ABSTRACT

In this paper, the researcher proposes a simple mathematical model consisting of mutualistic interactions among two-species with constant harvesting. Mutualism is one kind of interaction that ends up being a win-win situation for both species involved. The interacting species benefit from this interaction and ultimately are better adapted for continuous existence. The harvesting function is implemented to describe the rate of removal of the species. This paper aims to investigate the global stability of the unique positive equilibrium point of the model. The global stability of the model is studied by using Lyapunov function method. By constructing a suitable Lyapunov function, it has been proven that the unique positive equilibrium point is globally asymptotically stable in a nonlinear system. Finally, numerical simulation is shown to illustrate theoretical results and to simulate the trajectories around the stable equilibrium point. From the numerical analysis, it is observed that both the species persist.

Corresponding Author:

Rusliza Ahmad
Faculty of Computer and Mathematical Sciences,
Universiti Teknologi MARA, Perak Branch, Tapah Campus
Perak, Malaysia
Email: rusliza259@uitm.edu.my

1. Introduction

Mutualism is an interaction between two species that benefits both species [1],[2]. Some examples of two-species mutualism include zebra and wildebeest [3], fungi and algae [4], yucca moths and yucca plants [5] and damselfish and sea anemone [6]. Mutualism can be categorised into four types; seed dispersal, pollination, digestive and protection [7]. In general, mutualism may be facultative or obligate [8]. In a facultative mutualism, each species can survive independently, but both benefit when they are found together. Zebra and wildebeest, damselfish and sea anemone are some examples of facultative mutualism. In an obligate mutualism, both species require each other to survive. Fungi and algae, yucca moths and yucca plants are some examples of obligate mutualism.

Global stability is one of the important issues in a mutualism system. Some biologists believe that local asymptotically stable equilibrium point is globally asymptotically stable in an ecological system [9]. A general method to prove global stability in a mutualism system is by constructing a Lyapunov function. The other methods that previous researchers usually employ to prove global stability are Dulac Criterion and limit cycle stability analysis.

The authors in [10] and [11] constructed two different forms of Lyapunov functions to prove global stability in the same basic model of mutualism. Georgescu et al. in [12] proved the global stability of three systems of mutualism via the method of Lyapunov function. Yang et al. [13] studied global stability in a discrete mutualism model using the iterative method. There are adequate conditions to verify the model's global stability. Lei [14] studied the global stability of the stage-

PENERBIT PRESS
UNIVERSITI TEKNOLOGI MARA

structured commensalism system by using the Lyapunov method. The author gives an example to illustrate the theoretical discussion.

The removal of some species of the population from their habitat is known as harvesting. Harvest management is used to control the increasing population and to meet the demands of the community for animal damage control, recreation or commercial harvesting [15]. Harvesting often decreases the equilibrium point of the population level since it increases the mortality rate [16].

León in [17] studied the global stability of mutualism system with proportional harvesting via the method of Lyapunov function. More than one Lyapunov function was constructed by the author to prove the global stability of the same system. The author in [3] developed a Lyapunov function to prove that the unique positive equilibrium point is globally asymptotically stable in mutualism model with proportional harvesting. The model is different from a model appearing in [17]. The author illustrates the results by an example. A fractional mutualism model with harvesting has been investigated by Supajaidee and Moonchai [18]. The authors use the Lyapunov function approach to achieve sufficient conditions for global stability of the coexistence equilibrium point.

In this paper, we construct an appropriate Lyapunov function to prove the global stability of the unique positive equilibrium point of a mutualism model with constant harvesting and illustrating our results with a numerical example.

2. Lyapunov's Second Method

Lyapunov's second method or known as direct method applies an energy-like function called the Lyapunov function to analyse the behavior of dynamical systems analytically [19]. This method is called a direct method since there is no need for understanding of the solution of the differential equation system. Lyapunov's second method enables the analysis to extend beyond a specific area close to the equilibrium point (global analysis). The basic idea of this stability verification technique is to look for an aggregated summarizing function that continues to decrease to a minimum as the system changes. The following Definition 1 and Theorem 1 can be obtained from [20].

Definition 1: Suppose that (x_0, y_0) is an equilibrium point of a given nonlinear system

$$\frac{dx}{dt} = f(x,y),$$
$$\frac{dy}{dt} = g(x,y).$$

A function *V(x,y)* defined on a region Ω of the state space and containing (x_0, y_0) is a Lyapunov function if it satisfies the following three requirements:

1. $V(x,y)$ is continuous and has continuous first partial derivatives.
2. $V(x,y)$ is positive definite.
3. $\dot{V}(x,y) = \frac{\partial V}{\partial x} \bullet \frac{dx}{dt} + \frac{\partial V}{\partial y} \bullet \frac{dy}{dt}$ is negative semidefinite.

Theorem 1. If there exists a Lyapunov function $V(x,y)$, then the equilibrium point (x_0, y_0) is stable. If, furthermore, the function $\dot{V}(x,y)$ is strictly negative for every point then the stability is asymptotic.

3. Mathematical Model

We consider the constant harvesting model of mutualism as follows

$$\begin{aligned} \frac{dx}{dt} &= rx - bx^2 + \alpha xy - h_x, \\ \frac{dy}{dt} &= sy - ey^2 + \beta xy - h_y, \end{aligned} \qquad (1)$$

where $b=\frac{r}{K_x}$ and $e=\frac{s}{K_y}$. The symbols x and y denote the population size of the first species and the second species respectively, r and s are the intrinsic growth rates, K_x and K_y are the carrying capacities, α and β measure the mutualism effect of y on x and x on y respectively. The terms h_x and h_y in model (1) are the harvesting rates of the first and second species respectively and assumed to be positive. All the parameters are positive constants.

The equilibrium point of model (1) is obtained by setting the equations equal to zero to get $y=\frac{bx^2-rx+h_x}{\alpha x}$ and $x=\frac{ey^2-sy+h_y}{\beta y}$. Let $F(x)=\frac{bx^2-rx+h_x}{\alpha x}$ and $G(y)=\frac{ey^2-sy+h_y}{\beta y}$. To determine the critical point of $F(x)$ and $G(y)$, we set first derivatives $F'(x)=\frac{b}{\alpha}-\frac{h_x}{\alpha x^2}$ and $G'(y)=\frac{e}{\beta}-\frac{h_y}{\beta y^2}$ equal to zero and obtain the critical point $x_c=\sqrt{\frac{h_x}{b}}$ and $y_c=\sqrt{\frac{h_y}{e}}$. Subsequently, we have $F(x_c)=\frac{1}{\alpha}(2\sqrt{h_x b}-r)$ and $G(y_c)=\frac{1}{\beta}(2\sqrt{h_y e}-s)$. Since $F(x)$ is concave upward and $G(y)$ is concave to the right, we know that $F(x_c)$ is the minimum value as well as $G(y_c)$. If $F(x_c)<y_c$ and $G(y_c)<x_c$, this implies that there are either two or three or four intersections between the curves $F(x)$ and $G(y)$ in the positive quadrant. This means that there exist either two or three or four positive equilibrium points in the positive quadrant for model (1). Under conditions $F(x_c)<y_c$ and $G(y_c)<x_c$, one of the equilibrium points in the positive quadrant is qualitatively stable. These conditions are only sufficient conditions and not necessary conditions to get one stable equilibrium point in positive quadrant.

4. Global Stability

This section aims to study the global stability of the unique positive equilibrium point $E^*=(x^*,y^*)$ in system (1) by using an appropriate Lyapunov function when the conditions $F(x_c)<y_c$ and $G(y_c)<x_c$ are satisfied. In order to prove the global stability of the equilibrium point $E^*=(x^*,y^*)$ of model (1), we analyze the associated linearization model with perturbation u and v. Let $u=x-x^*$ and $v=y-y^*$ and substitute into system (1) to get

$$\frac{du}{dt}=r(u+x^*)-b(u+x^*)^2+\alpha(u+x^*)(v+y^*)-h_x,$$
$$\frac{dv}{dt}=s(v+y^*)-e(v+y^*)^2+\beta(u+x^*)(v+y^*)-h_y.$$

or

$$\frac{du}{dt}=ru+rx^*-bu^2-2bx^*u-b(x^*)^2+\alpha uv+\alpha y^*u+\alpha x^*v+\alpha x^*y^*-h_x,$$
$$\frac{dv}{dt}=sv+sy^*-ev^2-2ey^*v-e(y^*)^2+\beta uv+\beta y^*u+\beta x^*v+\beta x^*y^*-h_y.$$

We get the linearized model after simplifying and neglecting the product terms

$$\frac{du}{dt} = (r - 2bx^* + \alpha y^*)u + \alpha x^* v \,, \qquad (2)$$
$$\frac{dv}{dt} = (s - 2ey^* + \beta x^*)v + \beta y^* u \,.$$

The corresponding characteristic equation is obtained from

$$\begin{vmatrix} r - 2bx^* + \alpha y^* - \lambda & \alpha x^* \\ \beta y^* & s - 2ey^* + \beta x^* - \lambda \end{vmatrix} = 0,$$

which leads to

$$\lambda^2 + a_{00} + a_{11}\lambda = 0, \qquad (3)$$

where

$$a_{00} = rs - 2ery^* + \beta rx^* - 2bsx^* + 4bex^*y^* - 2b\beta(x^*)^2 + \alpha sy^* - 2e\alpha(y^*)^2 > 0$$
$$a_{11} = (-r + 2bx^* - \alpha y^*) + (-s + 2ey^* - \beta x^*) > 0 \,.$$

We define a function

$$V(u,v) = Au^2 + Buv + Cv^2 \qquad (4)$$

where

$$A = \frac{(\beta y^*)^2 + (s - 2ey^* + \beta x^*)^2 + a_{00}}{2D} \,, \qquad (5)$$
$$B = \frac{(\alpha x^*)(-s + 2ey^* - \beta x^*) + (\beta y^*)(-r + 2bx^* - \alpha y^*)}{D} \,, \qquad (6)$$
$$C = \frac{(\alpha x^*)^2 + (r - 2bx^* + \alpha y^*)^2 + a_{00}}{2D} \,, \qquad (7)$$
$$D = a_{00}a_{11} \qquad (8)$$

This function is definitely continuous with continuous first partial derivatives. Next, we'd like to verify whether the function $V(u,v)$ is positive or negative definite. By completing the square we get

$$\begin{aligned} V(u,v) &= Au^2 + Buv + Cv^2 \\ &= A\left(u^2 + \frac{B}{A}uv + \frac{C}{A}v^2\right) \\ &= A\left[u^2 + \frac{B}{A}uv + \left(\frac{\frac{B}{A}v}{2}\right)^2 - \left(\frac{\frac{B}{A}v}{2}\right)^2 + \frac{C}{A}v^2\right] \\ &= A\left[\left(u + \frac{B}{2A}v\right)^2 + \frac{C}{A}v^2 - \frac{B^2}{4A^2}v^2\right] \\ &= A\left[\left(u + \frac{B}{2A}v\right)^2 + \left(\frac{C}{A} - \frac{B^2}{4A^2}\right)v^2\right] \end{aligned}$$

Function $V(u,v)$ is positive definite if, and only if $A>0$ and $4AC-B^2>0$, and is negative definite if, and only if $A<0$ and $4AC-B^2>0$. From (5) and (8), it is clear that $A>0$ and $D>0$. Then

$$4AC-B^2 = 4\left(\frac{(\beta y^*)^2+(s-2ey^*+\beta x^*)^2+a_{00}}{2D}\right)\left(\frac{(\alpha x^*)^2+(r-2bx^*+\alpha y^*)+a_{00}}{2D}\right)-$$
$$\left(\frac{(\alpha x^*)(-s+2ey^*-\beta x^*)+(\beta y^*)(-r+2bx^*-\alpha y^*)}{D}\right)^2$$
$$=\frac{2(rs-2ery^*+\beta rx^*-2bsx^*+4bex^*y^*-2b\beta(x^*)^2+\alpha sy^*-2e\alpha(y^*)^2)}{D^2}+$$
$$\frac{4bx^*(-r+bx^*-\alpha y^*)+4ey^*(-s+ey^*-\beta x^*)+\beta^2(x^*)^2+\alpha^2(x^*)^2}{D^2}+$$
$$\frac{\beta^2(y^*)^2+\alpha^2(y^*)^2+2\beta sx^*+2\alpha ry^*+s^2+r^2}{D^2}$$

We get

$rs-2ery^*+\beta rx^*-2bsx^*+4bex^*y^*-2b\beta(x^*)^2+\alpha sy^*-2e\alpha(y^*)^2=a_{00}>0$, $-r+bx^*-\alpha y^*<0$, $-s+ey^*-\beta x^*<0$ and $D^2>0$. Since,
$2a_{00}+\beta^2(x^*)^2+\alpha^2(x^*)^2+\beta^2(y^*)^2+\alpha^2(y^*)^2+2\beta sx^*+2\alpha ry^*+s^2+r^2>$
$-r+bx^*-\alpha y^*-s+ey^*-\beta x^*$

therefore $4AC-B^2>0$. Hence, the function $V(u,v)$ is positive definite. The chosen function $V(u,v)$ meets the first two criteria of a Lyapunov function.

For the final requirement, referring to the linearized system (2), we have

$$\frac{\partial V}{\partial u}\frac{du}{dt}+\frac{\partial V}{\partial v}\frac{dv}{dt}=(2Au+Bv)((r-2bx^*+\alpha y^*)u+\alpha x^*v)+ (Bu+2Cv)((s-2ey^*+\beta x^*)v+\beta y^*u) \quad (9)$$

Substituting the values of A,B and C from (5), (6) and (7) in (9) we get

$$\frac{\partial V}{\partial u}\frac{du}{dt}+\frac{\partial V}{\partial v}\frac{dv}{dt}=\Bigg(\bigg(\frac{(\beta^2(y^*)^2+(s-2ey^*+\beta x^*)^2}{D}+$$
$$\frac{(r-2bx^*+\alpha y^*)(s-2ey^*+\beta x^*)-\alpha\beta x^*y^*}{D}\bigg)u+\bigg(\frac{\beta y^*(-r+2bx^*-\alpha y^*)}{D}+$$
$$\frac{\alpha x^*(-s+2ey^*-\beta x^*)}{D}\bigg)v\Bigg)((r-2bx^*+\alpha y^*)u+\alpha x^*v)+$$
$$\Bigg(\bigg(\frac{\beta y^*(-r+2bx^*-\alpha y^*)+\alpha x^*(-s+2ey^*-\beta x^*)}{D}\bigg)u+\bigg(\frac{\alpha^2(x^*)^2}{D}+$$
$$\frac{(r-2bx^*+\alpha y^*)^2+(r-2bx^*+\alpha y^*)(s-2ey^*+\beta x^*)-\alpha\beta x^*y^*}{D}\bigg)v\Bigg)\times$$
$$((s-2ey^*+\beta x^*)v+\beta y^*u)$$

$$= \frac{-(u^2+v^2)(-r+2bx^*-\alpha y^*-s+2ey^*-\beta x^*)}{D} \times$$
$$((r-2bx^*+\alpha y^*)(s-2ey^*+\beta x^*)-\alpha\beta x^* y^*)$$
$$= \frac{-(u^2+v^2)D}{D}$$
$$= -(u^2+v^2) \tag{10}$$

which is clearly negative definite. So $V(u,v)$ is a Lyapunov function for the linear system (2).

Next we show that $V(u,v)$ is also a Lyapunov function for the nonlinear system (1). Let F_1 and F_2 be two functions of x and y defined by

$$F_1(x,y) = rx - bx^2 + \alpha xy - h_x,$$
$$F_2(x,y) = sy - ey^2 + \beta xy - h_y\ .$$

We have to prove that $\frac{\partial V}{\partial u}F_1 + \frac{\partial V}{\partial v}F_2$ is negative definite. By letting $u = x - x^*$ and $v = y - y^*$ in (2), we get

$$\frac{du}{dt} = ru + rx^* - bu^2 - 2bx^*u - b(x^*)^2 + \alpha uv + \alpha y^*u + \alpha x^*v + \alpha x^*y^* - h_x$$

After simplifying, we have

$$\frac{du}{dt} = (rx^* - b(x^*)^2 + \alpha x^*y^* - h_x) + (r - 2bx^* + \alpha y^*)u + \alpha x^*v - bu^2 + \alpha uv$$

The first terms of $\frac{du}{dt}$ is equal to zero at equilibrium point (x^*, y^*) . This simplifies to

$$\begin{aligned} \frac{du}{dt} &= (r-2bx^*+\alpha y^*)u + \alpha x^*v - bu^2 + \alpha uv \\ &= (r-2bx^*+\alpha y^*)u + \alpha x^*v + f_1(u,v) = F_1(u,v), \end{aligned} \tag{11}$$

where

$$f_1(u,v) = -bu^2 + \alpha uv$$

Similiarly, we obtain

$$\frac{dv}{dt} = sv + sy^* - ev^2 - 2ey^*v - e(y^*)^2 + \beta uv + \beta y^*u + \beta x^*v + \beta x^*y^* - h_y$$

After simplifying, we have

$$\frac{dv}{dt} = (sy^* - e(y^*)^2 + \beta x^*y^* - h_y) + (s - 2ey^* + \beta x^*)v + \beta y^*u - ev^2 + \beta uv$$

The first term of $\frac{dv}{dt}$ is equal to zero at equilibrium point (x^*, y^*). This simplifies to

$$\frac{dv}{dt} = (s - 2ey^* + \beta x^*)v + \beta y^* u - ev^2 + \beta uv$$
$$= (s - 2ey^* + \beta x^*)v + \beta y^* u + f_2(u,v) = F_2(u,v), \tag{12}$$

where

$$f_2(u,v) = -ev^2 + \beta uv.$$

From (4) we have

$$\frac{\partial V}{\partial u} = 2Au + Bv \text{ and } \frac{\partial V}{\partial v} = Bu + 2Cv.$$

Hence,

$$\frac{\partial V}{\partial u}F_1 + \frac{\partial V}{\partial v}F_2 = (2Au + Bv)((r - 2bx^* + \alpha y^*)u + \alpha x^* v + f_1(u,v)) +$$
$$(Bu + 2Cv)((s - 2ey^* + \beta x^*)v + \beta y^* u + f_2(u,v))$$
$$= (2Au + Bv)((r - 2bx^* + \alpha y^*)u + \alpha x^* v) +$$
$$(Bu + 2Cv)(s - 2ey^* + \beta x^*)v + \beta yu) +$$
$$(2Au + Bv)f_1(u,v) + (Bu + 2Cv)f_2(u,v)$$

From (9) and (10), we have

$$\frac{\partial V}{\partial u}F_1 + \frac{\partial V}{\partial v}F_2 = -(u^2 + v^2) + (2Au + Bv)f_1(u,v) + (Bu + 2Cv)f_2(u,v) \tag{13}$$

Introducing polar co-ordinates $u = R\cos\theta$, $v = R\sin\theta$, (13) can be written as

$$\frac{\partial V}{\partial u}F_1 + \frac{\partial V}{\partial v}F_2 = -R^2 + R\left[(2A\cos\theta + B\sin\theta)f_1(u,v) + (B\cos\theta + 2C\sin\theta)f_2(u,v)\right]$$

Let us denote the largest of the numbers $|2A|$, $|B|$ and $|2C|$ by K. Our assumptions imply that $|f_1(u,v)| < \frac{R}{12K}$ and $|f_2(u,v)| < \frac{R}{12K}$ for all sufficiently small $R > 0$, so that

$$\frac{\partial V}{\partial u}F_1 + \frac{\partial V}{\partial v}F_2 < -R^2 + R\left[(2A\cos\theta + B\sin\theta)\frac{R}{12K} + (B\cos\theta + 2C\sin\theta)\frac{R}{12K}\right]$$
$$< -R^2 + \frac{R^2}{12K}((|2A| + |B|)\cos\theta + (|B| + |2C|)\sin\theta)$$
$$< -R^2 + \frac{R^2}{12K}(2K\cos\theta + 2K\sin\theta)$$
$$< -R^2 + \frac{R^2}{12K}(4K)$$
$$< \frac{-2R^2}{3} < 0 \tag{14}$$

Thus, $V(u,v)$ is a positive definite function with the property that $\frac{\partial V}{\partial u}F_1 + \frac{\partial V}{\partial v}F_2$ is negative definite. So $V(u,v)$ is a Lyapunov function for the nonlinear system (1). Therefore, the unique positive equilibrium point $E^* = (x^*, y^*)$ of model (1) is globally asymptotically stable.

5. Numerical Simulation

We present the following system of wildebeest and zebra without the involvement of lion as suggested in [21] with the addition of the constant harvesting terms:

$$\frac{dx}{dt} = 0.405x - 0.03375x^2 + 0.015xy - 0.8,$$
$$\frac{dy}{dt} = 0.34y - 0.02833y^2 + 0.020xy - 0.9, \tag{15}$$

where x and y are population size of wildebeest and zebra respectively (both measured in thousands). In this model, harvesting refers to the process of cropping the species, i.e. removing the species to a new location to avoid overcrowding. The harvesting rates $h_x = 0.8$ and $h_y = 0.9$ are chosen based on the feasible region as in Figure 1.

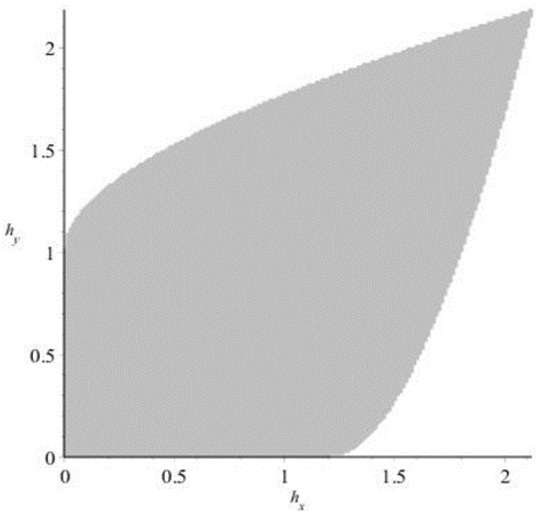

Figure 1. Feasible region for h_x and h_y satisfying $F(x_c) < y_c$ and $G(y_c) < x_c$

Model (15) compared with model (1), we have $r = 0.405$, $b = 0.03375$, $\alpha = 0.015$, $s = 0.34$, $e = 0.02833$, $\beta = 0.020$, $h_x = 0.8$ and $h_y = 0.9$.

With these parameters, the conditions $F(x_c) = -5.0911 < y_c = 5.6360$ and $G(y_c) = -1.0313 < x_c = 4.8686$ are satisfied. Model (15) has four equilibrium points in the positive quadrant; $E_1 = (10.5539, 1.7997)$, $E_2 = (2.1083, 3.0403)$, $E_3 = (1.6001, 9.9309)$ and $E_4 = (22.9948, 27.0577)$. The stability of each equilibrium points is given in Table 1.

Table 1. Equilibrium Points of Model (15) and Their Stabilities

Equilibrium Points	**Eigenvalues**	**Stability**
$E_1 = (10.5539, 1.7997)$	-0.28812 , 0.45682	Unstable Saddle Point
$E_2 = (2.1083, 3.0403)$	0.32499 , 0.19318	Unstable Source
$E_3 = (1.6001, 9.9309)$	0.45336 , -0.19815	Unstable Saddle Point
$E_4 = (22.9948, 27.0577)$	-1.16938 , -0.30527	Asymptotically Stable Sink

We consider the equilibrium point $E_4 = E^* = (22.9948, 27.0577)$ where this equilibrium point is asymptotically stable. In Figure 2, the equilibrium point $(22.9948, 27.0577)$ is a stable sink. This implies that close solution trajectories tend without oscillation to the equilibrium point. The population dynamics of zebra and wildebeest can be independently analyzed. From Figure 3 and Figure 4, both the zebra and the wildebeest populations converge in finite time to their equilibrium point $x^* = 22.9948$ and $y^* = 27.0577$ respectively.

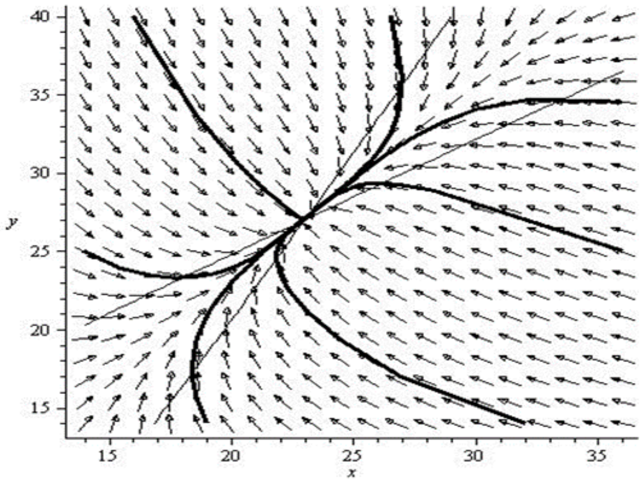

Figure 2. Phase portrait of $(x(t), y(t))$

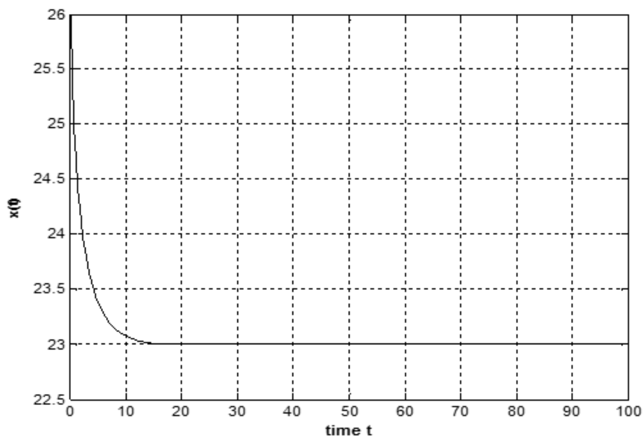

Figure 3. Trajectory of $x(t)$ at $h_x = 0.8$ and $x(0) = 26$

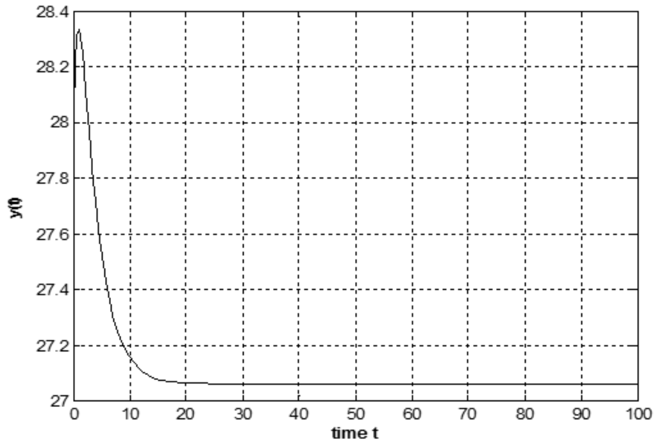

Figure 4. Trajectory of $y(t)$ at $h_y = 0.9$ and $y(0) = 28$

6. Conclusion

This study focuses on the global stability of the two-species mutualism model with a constant rate of harvesting function. The global stability of the unique positive equilibrium point of the model is proven by constructing an appropriate Lyapunov function. From the numerical analysis, it is observed that both the species persist. As seen from the Figure 2 – Figure 4, both the wildebeest and zebra populations tend toward the equilibrium point $(22.9948, 27.0577)$.

Acknowledgements

The author acknowledges Universiti Teknologi MARA (UiTM) for giving an opportunity and support to accomplish this research.

References

[1] L. L. Rockwood, Introduction to Population Ecology. Hoboken, USA: John Wiley and Sons, 2015.

[2] D. H. Boucher, S. James and K. H. Keeler, "The Ecology of Mutualism," *Annual Review of Ecology and Systematic*, vol .13, pp. 315 – 347, 1982.

[3] R. Ahmad, "Global Stability of Two-Species Mutualism Model with Proportional Harvesting," *International Journal of Advanced and Applied Sciences*, vol. 4, no. 7, pp. 74 – 79, 2017.

[4] S. Raghukumar, Fungi in Coastal and Oceanic Marine Ecosystems: Marine Fungi. India: Springer, 2017, pp.120 – 121.

[5] C. Starr, R. Taggart, C. Evers and L. Starr, Biology: The Unity and Diversity of Life, 13th ed. Boston: Cengage Learning, 2012.

[6] R. Benz, Ecology and Evolution: Islands of Change. Virginia, USA: NSTA Press, 2000.

[7] D. H. Janzen, The Natural History of Mutualism. In: D. L. Boucher. *Editors*. The Biology of Mutualism: Ecology and Evolution. Oxford University Press, New York, USA, 1985, pp.40 – 99.

[8] P. J. Morin, Community Ecology. Hoboken, USA: John Wiley and Sons, 2011.

[9] K. S. Cheng, S. B. Hsu and S. S. Lin, "Some Results on Global Stability of a Predator-Prey System," *Journal of Mathematical Biology*, vol. 12, no. 1, pp. 115 – 126, 1981.

[10] B. S. Goh, "Stability in Models of Mutualism," *The American Naturist*, vol. 113, no. 2, pp. 261 – 275, 1979.

[11] B. R. Reddy, K. L. Narayan, and N. C. Pattabhiramacharyulu, "On Global Stability of Two Mutually Interacting Species with Limited Resources for Both the Species," *International Journal of Contemporary Mathematical Sciences*, vol. 6, no. 9, pp. 401 – 407, 2011.

[12] P. Georgescu, H. Zhang and D. Maxin, "The Global Stability of Coexisting Equilibria for Three Models of Mutualism," *Mathematical Biosciences and Engineering*, vol. 13, no. 1, pp. 101 – 118, 2016.

[13] K. Yang, X. Xie and F. Chen, "Global Stability of a Discrete Mutualism Model," *Abstract and Applied Analysis*, vol. 2014, Article ID 928726, 6 pages, 2014.

[14] C. Lei, "Dynamic Behaviors of a Stage-Structured Commensalism System," *Advances in Difference Equations*, vol.301, no.2018, 2018.

[15] R. Ouncharoen, S. Pinjai, T. Dumrongpokaphan and Y. Lenbury, "Global Stability Analysis of Predator-Prey Model with Harvesting and Delay," *Thai Journal of Mathematics*, vol. 8, no. 3, pp. 589 – 605, 2012.

[16] A. Martin, Predator-Prey Models with Delays and Prey Harvesting, Unpublished Master's Thesis. Dalhousie University Halifix, Nova Scotia, 1999.

[17] C. V. D. León, "Lyapunov Function for Two-Species Cooperative Systems," *Applied Mathematics and Computation*, vol. 219, no. 5, pp. 2493 – 2497, 2012.

[18] N. Supajaidee and S. Moonchai, "Stability Analysis of a Fractional-Order-Two-Species Facultative Mutualism Model with Harvesting," *Advances in Difference Equations*, vol. 372, no. 2017, 2017.

[19] K. D. Do and J. Pan, Control of Ships and Underwater Vehicles. Berlin, Germany: Springer Science and Business Media, 2009.

[20] W. E. Boyce and R. C. DiPrima, Elementary Differential Equations and Boundary Value Problems. Hoboken, USA: John Wiley and Sons, 1992.

[21] T. H. Fay and J. C. Greeff, "Lion, Wildebeest and Zebra: A Predator-Prey Model," *Ecological Modeling,* vol. 196, no. 1, pp. 237 – 244, 2006.