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MAKING CONTINUITY VISIBLE: PIECEWISE CUBIC BÉZIER RECONSTRUCTION OF THE APPLE LOGO IN WOLFRAM MATHEMATICA

Amirah Adlina Samsuri[1], Nur Aina Najihah Mohd Romzi[2], Syed Ahmad Aidil Adha Said Mad Zain[3] and *Azhar Ahmad[4]

*2023699632@student.uitm.edu.my[1], 2023690318@student.uitm.edu.my[2], syedaidil@uitm.edu.my[3], *azhar251@uitm.edu.my[4]*

[1,2,3]Fakulti Sains Komputer dan Matematik, Universiti Teknologi MARA Cawangan Negeri Sembilan Kampus Seremban 70300 Seremban, Negeri Sembilan, Malaysia

[4]Jabatan Sains Komputer dan Matematik,
Universiti Teknologi MARA Cawangan Pulau Pinang Kampus Permatang Pauh 13500 Permatang Pauh, Pulau Pinang, Malaysia

**Corresponding author*

ABSTRACT

Continuity conditions between adjacent Bézier segments are simple to state and difficult to picture. A student can write $P_3 = Q_0$ for positional continuity and $Q_1 = 2P_3 - P_2$ for tangent continuity long before recognising which of two drawings satisfies the second. This article reports a curve fitting activity built around that gap. The outline of the Apple logo was reconstructed in Wolfram Mathematica from fourteen piecewise cubic Bézier segments, twelve for the body and two for the leaf, using control points read by hand from the boundary rather than produced by automatic vectorisation. The same set of boundary points was then used twice, once with positional continuity alone and once with tangent continuity imposed at every joint that should be smooth. The two outlines differ by a single algebraic constraint, yet the first shows a visible corner on the left shoulder of the apple while the second does not. Joints carrying sharp features, the leaf tips and the bite, were deliberately left outside the tangent condition, which turns smoothness into a design decision rather than a default. The activity needs no CAD system and makes the geometric meaning of each Bernstein property directly observable.

Keywords: Bézier curve, curve fitting, parametric continuity, Wolfram Mathematica, geometric modelling

Introduction

Computer-Aided Geometric Design (CAGD) gives design software its mathematical vocabulary, and the Bézier curve is the first word most students learn (Farin, 2002). A cubic segment needs four control points. It starts at the first and ends at the last, it never leaves the convex hull of the polygon those points form, and it takes its end tangent directions from the polygon legs. Guarantees of this kind are what make the curve safe to place in the hands of a beginner, and they are the reason the same construction reappears in font design, animation, industrial modelling and the computer-aided design systems that support them (Sabancigil et al., 2020; Knight, 2024).

The difficulty appears once a single segment is no longer enough. A real outline is assembled from many segments, and the quality of the result depends less on any one curve than on what happens where two curves meet (Sule et al., 2025). Positional continuity, written C^0, asks only that consecutive segments share an endpoint. Tangent continuity, written C^1, asks in addition that their derivatives agree

there. Each condition is one line of algebra, and students reproduce both accurately under examination conditions. Shown two drawings, many of the same students cannot say which one satisfies the second. The gap is not mathematical. It is that the condition has never been attached to anything they can see (Piegl, 1989).

Concrete targets close that gap. Bézier curves have been used to trace Arabic calligraphy (Razali et al., 2019), to approximate the characteristic curves of photovoltaic modules (Louzazni & Al-Dahidi, 2021) and to reconstruct shapes from scattered boundary data (Cui et al., 2023), and in each case the mathematics is checked against something the eye can judge. Curve fitting itself is nothing more than finding a function that follows a set of points closely enough to be useful (Molugaram & Rao, 2017), so a target whose correct answer is already familiar removes the need for a separate error measure. A brand logo suits the purpose: its boundary is smooth, its proportions are recognisable, and a poor reconstruction announces itself.

What follows describes an activity of that kind. The outline of the Apple logo is rebuilt from fourteen piecewise cubic Bézier segments whose control points are read by hand from the boundary, and the same boundary is then reconstructed twice, once with positional continuity alone and once with tangent continuity imposed at the joints that should be smooth. The effect of a single algebraic constraint becomes something students can point at.

The cubic Bernstein basis and what its properties do for the drawing

For a non-negative integer n, the Bernstein basis polynomial of degree n is

$$B^{n}_{i}(t) = [\, n! \,/\, (\, i!\,(n - i)!\,)\,]\; t^{i}(1 - t)^{n-i}, \quad i = 0, 1, \ldots, n, \quad t \in [0, 1]. \tag{1}$$

Setting $n = 3$ gives the four functions that carry the whole construction (Raseli et al., 2022):

$$B^{3}_{0}(t) = (1 - t)^{3}, \quad B^{3}_{1}(t) = 3t(1 - t)^{2},$$

$$B^{3}_{2}(t) = 3t^{2}(1 - t), \quad B^{3}_{3}(t) = t^{3}. \tag{2}$$

Figure 1 shows the four functions over the unit interval. Each one is non-negative there, and the four sum to one at every value of t, which together mean that a point on the curve is a weighted average of the four control points with weights that never go negative. The curve therefore cannot escape the quadrilateral formed by its control points. During tracing this becomes a diagnostic rather than a theorem: a curve that appears to run outside its dashed polygon indicates a mistyped coordinate, not an unusual curve.

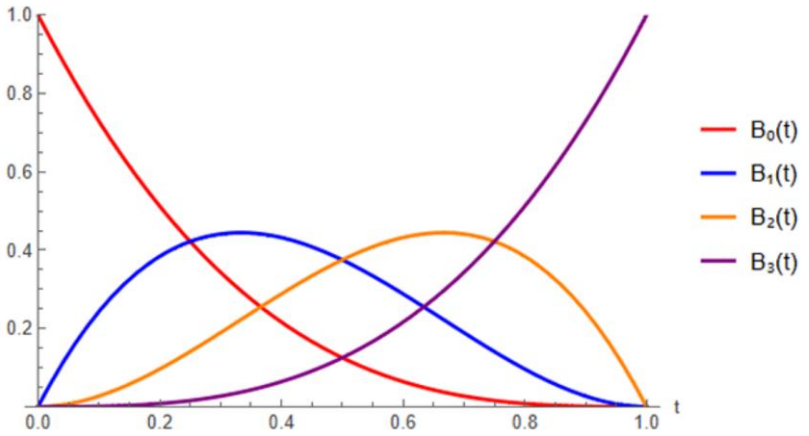


Figure 1: The cubic Bernstein basis functions over the unit interval

A second property has a practical consequence that is easy to overlook. The basis is symmetric, in the sense that $B^3_i(t) = B^3_{3-i}(1 - t)$, so reversing the order of the control points produces the same curve traversed backwards. A boundary may be read clockwise or anticlockwise, but it must be read one way throughout, because the continuity conditions below refer to the last control point of one segment and the first of the next.

Given control points P_0, P_1, P_2 and P_3, the cubic Bézier curve is the weighted sum

$$B(t) = (1 - t)^3P_0 + 3t(1 - t)^2P_1 + 3t^2(1 - t)P_2 + t^3P_3, \quad t \in [0, 1]. \tag{3}$$

Substituting $t = 0$ and $t = 1$ leaves $B(0) = P_0$ and $B(1) = P_3$, because three of the four basis functions vanish at each end. Endpoint interpolation is what allows a boundary to be split at chosen points: the first and last control points of every segment are pinned to coordinates read from the image, and only the two interior points remain free. Differentiating (3) gives

$$B'(t) = 3(1 - t)^2(P_1 - P_0) + 6t(1 - t)(P_2 - P_1) + 3t^2(P_3 - P_2), \tag{4}$$

and evaluating the derivative at the ends of the interval gives

$$B'(0) = 3(P_1 - P_0), \quad B'(1) = 3(P_3 - P_2). \tag{5}$$

The curve leaves P_0 along the first leg of the polygon and arrives at P_3 along the last. Students can predict the direction of departure before plotting anything, and the length of the leg tells them how strongly the curve is pulled that way. Equation (5) is also the whole of what is needed for the continuity analysis, since it is the only place where the derivative and the control polygon meet.

Tracing the Apple outline in Wolfram Mathematica

The reference image is imported, and its opacity reduced, in this case to 0.15 with SetAlphaChannel, so that fitted curves remain clearly visible on top of the original shape. Coordinates are then read along the

outer boundary in one consistent direction and grouped into sets of four, so that the first polygon reads Points1 = {{132, 313}, {146, 343}, {169, 365}, {211, 370}} and the remainder follow the contour from there.

No automatic vectorization is used, and that is deliberate. A tracing tool would produce a cleaner outline in a fraction of the time and teach almost nothing, because the decisions about where one segment should end, which way the boundary is being traversed and how far the interior points should be pushed are the content of the exercise. Automatic methods that place control points by minimizing an energy or a fitting residual are worth showing afterwards (Han & Zhu, 2020; Cui et al., 2023), once students know what the placement is for.

Fourteen polygons were needed for the complete outline, twelve for the body and two that close the leaf at its two tips. Each polygon becomes one parametric equation through (3), plotted with ParametricPlot over t in [0, 1], while the polygon itself is drawn with ListLinePlot as a dashed grey line. Plotting each segment in a different color is a small detail with a disproportionate effect. Students can see at once which stretch of boundary each of their fourteen polygons is responsible for, and a segment that has been asked to do too much appears as a single color spanning a region where the curvature reverses.

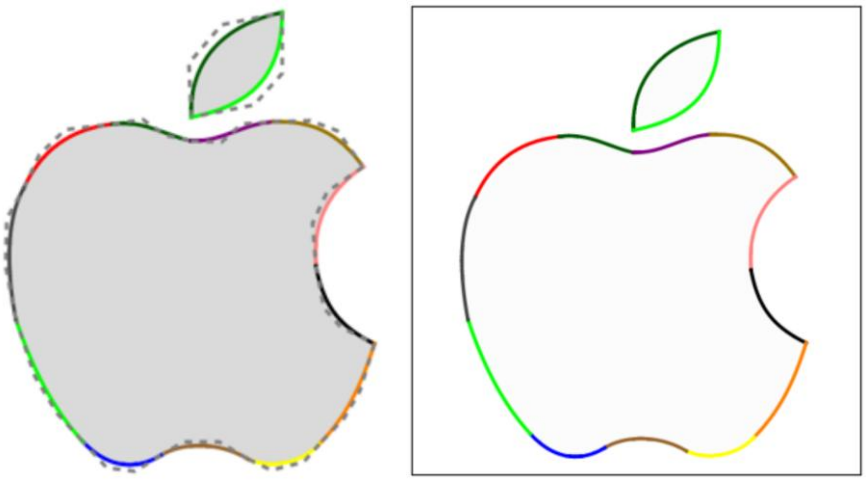


Figure 2: (a) the fourteen fitted segments with their dashed control polygons over the faded reference image, and (b) the completed outline with the reference image removed

The separation between the dashed polygon and the colored curve is widest at the shoulders of the apple, where the curvature is highest. That is the convex hull property doing visible work rather than evidence of a poor fit, and it is worth pausing on, because the instinct of most students is to drag the interior points back towards the boundary until polygon and curve almost coincide, which costs them the smoothness they were trying to obtain.

What changes when tangent continuity is imposed

Let $B_1(t)$ be the segment defined by P_0 to P_3 and $B_2(t)$ the segment that follows it, defined by Q_0 to Q_3. Requiring $B_1(1) = B_2(0)$ and using endpoint interpolation gives the positional condition

$$P_3 = Q_0. \tag{6}$$

Condition (6) closes the gap between two segments and says nothing at all about direction, so a joint that satisfies it may still turn a corner. Requiring the derivatives to agree as well, $B'_1(1) = B'_2(0)$, and substituting (5) gives $3(P_3 - P_2) = 3(Q_1 - Q_0)$, which together with (6) becomes

$$Q_1 = 2P_3 - P_2. \tag{7}$$

Read geometrically, (7) says that P_2, P_3 and Q_1 must be collinear with P_3 at the midpoint. The second control point of every following segment stops being a free choice and becomes a consequence of the segment before it, which is the first moment in the exercise when a piece of algebra takes something away from the student rather than giving them a new option.

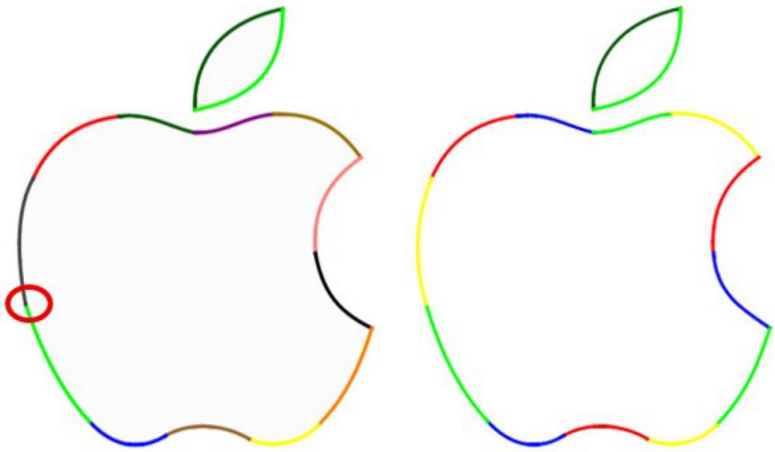


Figure 3: the same boundary points reconstructed under (a) positional continuity only and (b) tangent continuity, with the affected joint circled

Figure 3 places the two reconstructions side by side. The circled joint on the left shoulder is where the difference is easiest to see: in the left-hand outline the two segments meet at a common point but arrive from different directions, and the corner is unmistakable once it has been pointed out. In the right-hand outline the same joint passes without incident. Nothing else was changed. Table 1 sets out what each condition requires and what it produces.

Table 1: Continuity conditions at the segment joints and their effect on the reconstructed outline

Condition	**Requirement on the control points**	**Effect on the drawing**
C^0 (position)	$P_3 = Q_0$	The segments meet without a gap, but their tangent directions may differ, so a corner can survive at the joint.
C^1 (tangent)	$Q_1 = 2P_3 - P_2$, so that P_2, P_3 and Q_1 are collinear	The segments share a tangent direction and the joint is no longer visible in the outline.
None imposed	The control points of the two segments are chosen independently	A sharp feature is preserved, which is what the leaf tips and the bite require.

The instructive part is what has been left out. Applying (7) at the leaf tips would round them into an eye shape and applying it at the bite on the right of the logo would smooth away the feature the logo is known for. Those joints are held at C^0 on purpose. Smoothness turns out to be a decision taken joint by joint rather than a condition applied everywhere, which is a point student rarely take from a lecture and almost always take from watching their own leaf go wrong.

Using the activity in a computer mathematics class

Three features make the exercise work in a laboratory session. The feedback loop is short, because a mistyped coordinate or a badly placed interior point shows up in the plot within seconds. The target is recognizable, so students assess their own work without waiting for a marking scheme. And the entire activity runs inside one computer algebra system on a standard campus license, with no CAD package, no plug-in and no file conversion between tools.

Some practical notes are worth passing on. Reading coordinates by hand is the slow part, so a two-hour session is realistic for the full logo and a shorter one should be restricted to the leaf, which needs only two polygons and eight points. Distributing the reference image at a fixed pixel size keeps coordinates comparable between groups and makes it possible to discuss one group’s control points in front of another. Wrapping the plot in Manipulate, with sliders bound to one interior control point, turns the convex hull property and the endpoint tangent property into something students explore rather than accept.

For assessment, the notebook alone is a weak piece of evidence. A more useful submission is the notebook together with a table of the fourteen control polygons and a short written justification of which joints carry (7) and which do not. The marks belong to the justification. Rewarding pixel accuracy pushes students towards nudging coordinates for an hour, which is neither mathematics nor design.

The activity transfers without modification to any logo bounded by smooth curves, and it scales in either direction by changing the number and position of the control points. A natural follow-up asks

students to reconstruct the same outline with fewer segments and to report where the fit first degrades, which introduces the trade-off between segment count and approximation quality in a form they have already felt. For stronger groups, the same fourteen polygons can be fed to a different basis, whether a shape-parameter family (Chen, 2013), an interpolating spline (Tayebi Arasteh & Kalisz, 2021) or one of the fractal constructions now used in graphics (Vijay et al., 2025), and the reconstructions compared on a boundary they know well.

Conclusion

Fourteen piecewise cubic Bézier segments reproduce the Apple outline, and the difference between a reconstruction with positional continuity and one with tangent continuity comes down to the single constraint $Q_1 = 2P_3 - P_2$. Placing the two outlines side by side makes that constraint visible in a way no amount of algebraic verification achieves and leaving it off at the leaf tips and the bite makes the complementary point that smoothness is chosen rather than assumed.

The requirements are modest: an image, a computer algebra system and a session long enough to read coordinates carefully. What students take away is not the outline but the habit of asking, at every joint, what the mathematics is guaranteeing and what it is not.

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