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A SYSTEMATIC REVIEW OF THE DEVELOPMENT SUBCLASS PROPERTIES OF STARLIKE FUNCTIONS

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ABSTRACT

Starlike function is a class of analytic functions in complex analysis that have gained a lot of research attention due to their rich mathematical history and wide-ranging applications. This Systematic Literature Review (SLR) applying the Preferred Reporting Items for Systematic Reviews and Meta-Analyses (PRISMA) model requires to provide a comprehensive synthesis of existing research on the development of subclass properties of starlike functions. The data search was conducted using criteria determined by researchers from various search engines such as ResearchGate, Scopus, Google Scholar, and others. Only 12 out of 52 articles pertinent to this research. In addition, qualitative analysis was applied to the data to describe the research findings. Three main themes related to the study have been found, including type of class, the type of subclasses and the equation of the subclasses. Subclasses of starlike functions with unique geometric and analytic properties have been extensively studied throughout the years. The purpose of this review is to present an up-to-date, comprehensive summary of the development of subclass properties of starlike functions.

Keywords*: Includes three to six keywords, separated by commas, capitalise each word, use Italic and sorted alphabetically.*

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1. Introduction

The purpose of this study is to review, analyse, and classify the previous research that had been conducted on the topic related to starlike function and its connection with the exponential function. This systematic literature review (SLR) investigates current trends in research to determine which studies have comparable aims, targeted aims, theoretical and conceptual framework, and research designs. Table 1 below listed the main points of this analysis.

Table 1: Themes of this study

No.	Themes of the study
1	Type of class
2	Type of subclasses
3	Equation of the subclasses

In geometric function theory, researchers have explored the possibility of discovering common geometric properties among many types of analytic univalent functions. This field is a part of complex analysis. Carathéodory class denotes the set of all functions that have a positive real component. Functions with a variety of analytic and univalent functions can all be defined using this class.

The definition and basic result concerning to the subclasses of S. Let S represent the subclass of A containing normalised univalent functions. There are three main subclasses of normalised univalent functions, S which are the class of convex functions K, the class of starlike functions with respect to the origin S^* and the class of close-to-convex functions C but we will focus on the class of starlike functions.

1.1 The class of starlike functions

A set I in the complex plane is starlike with respect to w_0 if every line segment or ray with an initial point w_0 intersects the boundary of the domain only once. A function $f(z)$ is starlike with respect to w_0 if and only if the function maps E onto a starlike domain with respect to w_0. S^* denotes the class of starlike functions.

If a function $f(z) \in S$, then $f(z)$ is a starlike function if and only if it satisfies the condition

$$\operatorname{Re}\left(\frac{zf''(z)}{f'(z)}\right) > 0, \qquad z \in E.$$

2. Literature Review

Geometric function theory is a complex analysis field. It delves into the geometry behind numerous different types of analytic univalent functions (Oros, 2023). This subject of study first came to light in the 18th century and has since gained widespread attention. The purpose of geometric function theory is to investigate the analytical features of uniformization maps on the unit disc and their connection to the geometry of different domains in the complex plane.

Univalent functions theory had begun in the year 1907 by Koebe. Hussain (2009) pointed out the most of this study involves the class E of analytical and univalent functions in the open unit disc and normal uniform. Univalent functions have many classes: starlike, close-to-convex, and convex.

Koebe established the field of univalent functions theory in 1907. Most of this research is focused on Hussain's (2009) class E of analytical and univalent functions in the open unit disc and normal uniform. Among the various types of univalent functions are the starlike, close-to-convex, and convex.

The basic class of starlike functions has been improved and expanded by earlier researchers by including new parameters and in terms of subordination. In relation to a certain point, a symmetric point, a symmetric conjugate point, and a conjugate point, this resulted in the creation of a new subclass of starlike functions. Different results on the properties were found using the new subclass of the univalent functions.

Not only are systematic reviews important for academics, policy makers, and other decision-makers, but they are also crucial for students who without them, would be faced with an overwhelming volume of literature on which to construct the "state of the art" section of their research (Trifu et al., 2022). Users should be able to evaluate the reliability and relevance of the review findings by examining the procedures and results of the review in sufficient detail.

3. Research Methodology

A thorough understanding of the development of star-like functions requires synthesize of existing knowledge, identification of research gaps, and development of a comprehensive perspective, all of which can be accomplished through a systematic literature review. We use the PRISMA technique, which provides a systematic and transparent approach to perform systematic reviews in a way that can be replicated, to accomplish this. According to the research of Mengist et al. (2020), literature review (LR) is a method for collecting data on a topic that meets certain criteria. To ensure the quality and reproducibility of the revision process, PRISMA develops a standard, peer-reviewed methodology based on checklists of optimal practises (Conde et al., 2020). The key components of PRISMA are the processes of identification, screening, eligibility, and inclusion. Figure 1 below illustrates the methodical process that was followed to resolve this study inquiry. In addition, 12 articles were examined in further detail after the data search obtained the findings shown in the following chart.

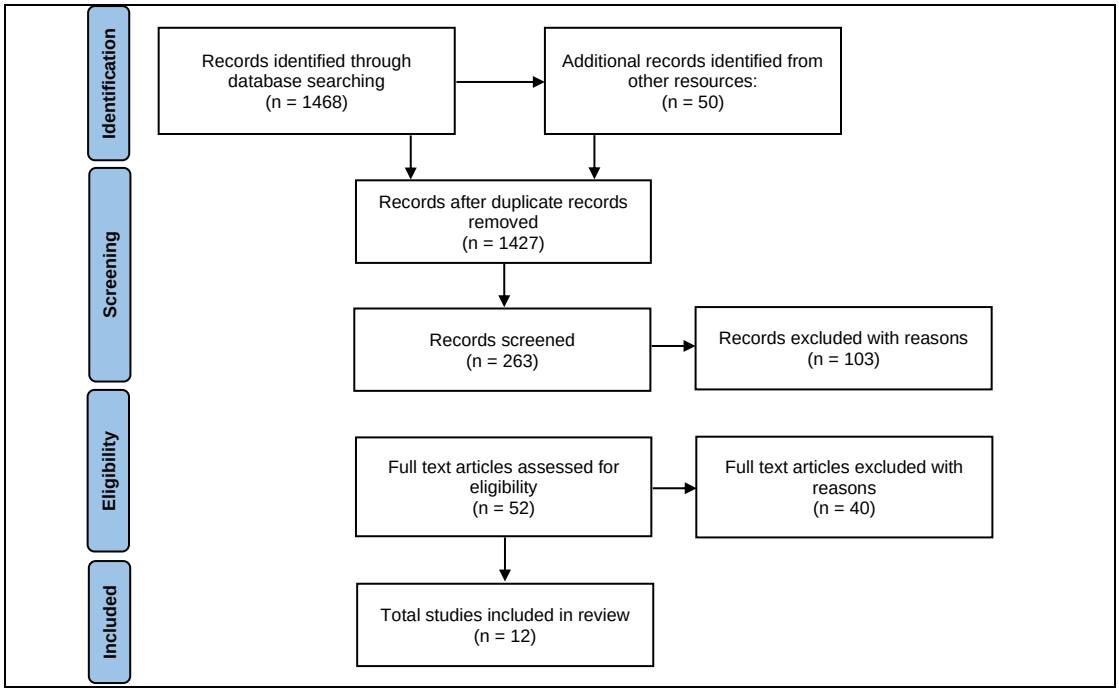

Figure 1: Identification of the research data search and selection flow

Table 2 provides a brief description of the selected articles including article names, year of publication, author's name, and journal's name.

Table 2: Identify of the articles to be analyzed further.

No.	Article Title	Year	Author's Name	Journal Name
1	On a certain univalent mapping	1959	Kôichi Sakaguchi	Journal of the Mathematical Society of Japan
2	Functions Starlike with Respect to Other Points	1991	Suzeini Abdul Halim	International Journal of Mathematics and Mathematical Sciences
3	Coefficient bound of a generalized close-to-convex function	2013	Abdullah Yahya, Shaharuddin C. Soh, Daud Mohamad	International Journal of Pure and Applied Mathematics
4	A subclass of starlike functions	1992	R.M. Goel, Beant Singh	Tamkang Journal of

	with respect to symmetric points		Mehrok	Mathematics
5	On Coefficient Problems for Functions Connected with the Sine Function	2021	Katarzyna Tra˛bka-Więcław	Symmetry
6	Third Hankel Determinant for a Class of Functions with Respect to Symmetric Points Associated with Exponential Function	2020	Ganesh Koride, Bharavi Sharma Rayaprolu, Rajya Laxmi	WSEAS Transactions on Mathematics
7	Some subclasses of close-to-convex functions	1987	Rabha Md El-Ashwah, D. K. Thomas	J. Ramanujan Math. Soc.
8	Some properties of a new subclass of tilted star-like functions with respect to symmetric conjugate points	2023	Daud Mohamad, Nur Hazwani Aqilah Abdul Wahid, Nurfatin Nabilah Md Fauzi	American Institute of Mathematical Sciences (AIMS)
9	Subclass of Starlike Functions with Respect to Symmetric Conjugate Points	2011	Loo Chien Ping, Aini Janteng	International Journal of Algebra
10	Coefficient Problems for Star-like Functions with Respect to Symmetric Conjugate Points Connected to the Sine Function	2023	Daud Mohamad, Nur Hazwani Aqilah Abdul Wahid, Nurul Natasya Hasni	European Journal of Pure and Applied Mathematics (EJPAM)
11	A Subclass of Starlike Functions with Respect to Conjugate Points	2009	Suci Aida Fitri Mad Dahhar, Aini Janteng	International Mathematical Forum
12	Second Hankel Determinant for a Subclass of Tilted Starlike Functions with Respect to Conjugate Points	2015	Nur Hazwani Aqilah Abdul Wahid, Daud Mohamad, Shaharuddin Cik Soh	MATEMATIKA

The table above presents the observed fluctuations in the total number of articles selected for this systematic literature review (SLR) from 1959 to 2023. The present systematic literature review focuses on the investigation of the development of subclass properties of starlike functions. It is worth noting that there exists a considerable temporal gap in the literature, which may be attributed to the involvement of the researchers who were the pioneers in establishing this class.

4. Data Collection and Analysis

This study used several digital databases, including Google Scholar, Research Gate, Scopus, and others, to search for relevant articles and other sources of information. The terms “starlike functions,” “symmetric points,” “symmetric conjugate points,” and “conjugate points” are all utilised in data searches. These themes were selected using the Preferred Reporting Items for Systematic Reviews and Meta-analysis (PRISMA) SLR model equation. The articles come from prestigious scientific journals. The results of this inquiry are displayed in Table 3, which include data collected from the studies considered using PRISMA model.

Table 3: Analysis of selected article based on themes.

No.	Article Title	Type of Class	Type of Subclasses	Equation of the Subclasses
1	On a certain univalent mapping	Starlike function	Symmetric	$\mathrm{Re}\left(\frac{zf'(z)}{f(z)-f(-z)}\right)>0$

2	Functions Starlike with Respect to Other Points	Starlike function	Symmetric Symmetric conjugate Conjugate	$\mathrm{Re}\left(\frac{2zf'(z)}{f(z)-f(-z)}\right)>\beta$ $\mathrm{Re}\left(\frac{2zf'(z)}{f(z)-\overline{f(-\bar{z})}}\right)>\beta$ $\mathrm{Re}\left(\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}\right)>\beta$
3	Coefficient bound of a generalized close-to-convex function	Starlike function	Symmetric	$\mathrm{Re}\left(e^{i\alpha}\frac{2zf'(z)}{f(z)-f(-z)}\right)>\delta$
4	A subclass of starlike functions with respect to symmetric points	Starlike function	Symmetric	$\frac{2zf'(z)}{f(z)-f(-z)}\prec\frac{1+Az}{1+Bz}$
5	On Coefficient Problems for Functions Connected with the Sine Function	Starlike function	Symmetric	$\frac{2zf'(z)}{f(z)-f(-z)}\prec 1+\sin(z)$
6	Third Hankel Determinant for a Class of Functions with Respect to Symmetric Points Associated with Exponential Function	Starlike and convex function	Symmetric	$\frac{2[zf'(z)]}{f(z)-f(-z)}\prec e^{z}$ $\frac{2[zf'(z)]'}{\left(f(z)-f(-z)\right)'}\prec e^{z}$
7	Some subclasses of close-to-convex functions	Starlike function	Symmetric conjugate Conjugate	$\mathrm{Re}\left(\frac{zf'(z)}{f(z)-\overline{f(-\bar{z})}}\right)>0$ $\mathrm{Re}\left(\frac{zf'(z)}{f(z)+\overline{f(\bar{z})}}\right)>0$
8	Some properties of a new subclass of tilted star-like functions with respect to symmetric conjugate points	Starlike function	Symmetric conjugate	$\mathrm{Re}\left(e^{i\alpha}\frac{zf'(z)}{f(z)-\overline{f(-\bar{z})}}\right)>\delta$ $\left(e^{i\alpha}\frac{2zf'(z)}{f(z)-\overline{f(-\bar{z})}}-\delta-i\sin\alpha\right)\frac{1}{(\cos\alpha-\delta)}\prec\frac{1+Az}{1+Bz}$
9	Subclass of Starlike Functions with Respect to Symmetric Conjugate	Starlike function	Symmetric conjugate	$\frac{2zf'(z)}{f(z)-\overline{f(-\bar{z})}}\prec\frac{1+Az}{1+Bz}$

	Points			
10	Coefficient Problems for Star-like Functions with Respect to Symmetric Conjugate Points Connected to the Sine Function	Starlike function	Symmetric conjugate	$\frac{2zf'(z)}{f(z)-\overline{f(-\bar{z})}} \prec 1+\sin(z)$
11	A Subclass of Starlike Functions with Respect to Conjugate Points	Starlike function	Conjugate	$\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}} \prec \frac{1+Az}{1+Bz}$
12	Second Hankel Determinant for a Subclass of Tilted Starlike Functions with Respect to Conjugate Points	Starlike function	Conjugate	$\mathrm{Re}\left(e^{i\alpha}\frac{zf'(z)}{f(z)+\overline{f(\bar{z})}}\right)>\delta$ $\left(e^{i\alpha}\frac{2zf'(z)}{f(z)+\overline{f(\bar{z})}}-\delta-i\sin\alpha\right)\frac{1}{(\cos\alpha-\delta)} \prec \frac{1+Az}{1+Bz}$

Based on Table 3, the 12 articles have been classified into several categories, including classification based on class type, subclass type, and the equation of the subclasses as determined by the researchers in the selected articles. There exist two distinct classes of functions known as starlike functions and convex functions. In addition to the stated previously, the subclasses encompassed within these articles consist of symmetric points, symmetric conjugate points, and conjugate points.

5. Results and Discussion

The information gathered from this SLR was analysed. This section explains about the three themes chosen in analysing this data, namely type of class, type of subclass and equation of the subclasses.

5.1 Type of class

Table 4 summarises each class employed in the reviewed articles.

Table 4: The class used in the studies.

Starlike function $(\boldsymbol{S^*})$	**Starlike and convex function** $(\boldsymbol{S^*, C^*})$
n = 11	n = 1

where n represent the total number of articles.

From the table above, there are two different class used which are starlike functions and convex functions. Starlike and convex functions were used together in one of the

publications. Eleven of the studies relied only on starlike functions, while the other employed a combination of starlike and convex functions.

5.2 Type of the subclasses

The Table 5 below shows the different type of subclasses of the studies. The subclass of each study differs according to the purpose of the study.

Table 5: Type of subclasses used in the studies.

Symmetric	**Symmetric Conjugate**	**Conjugate**	**Both Symmetric Conjugate and Conjugate**	**All Subclass (Symmetric, Symmetric Conjugate and Conjugate)**
n = 5	n = 3	n = 2	n = 1	n = 1

From the 12 chosen articles, five articles represented symmetric points while three articles used symmetric conjugate points. There were two articles for the subclasses of starlike functions with respect to conjugate points. One article used both symmetric conjugate and conjugate points while another one article use all subclasses in their study. From this data, we can develop the subclasses of starlike functions with respect to symmetric, symmetric conjugate and conjugate points. As a result, researchers of the future can concentrate on previously unexplored parameters.

5.3 Equation of the subclasses

From Table 3, we conclude the development of the subclasses of starlike functions with respect to other points. Some of the articles introduced two or three subclasses while most of the articles propose only one subclasses. Figure 2 shows a clear vision of the study by the past researchers on the class of starlike functions with respect to other points. They expand and study the class by adding a new parameter to the subclass introduced by the other researchers.

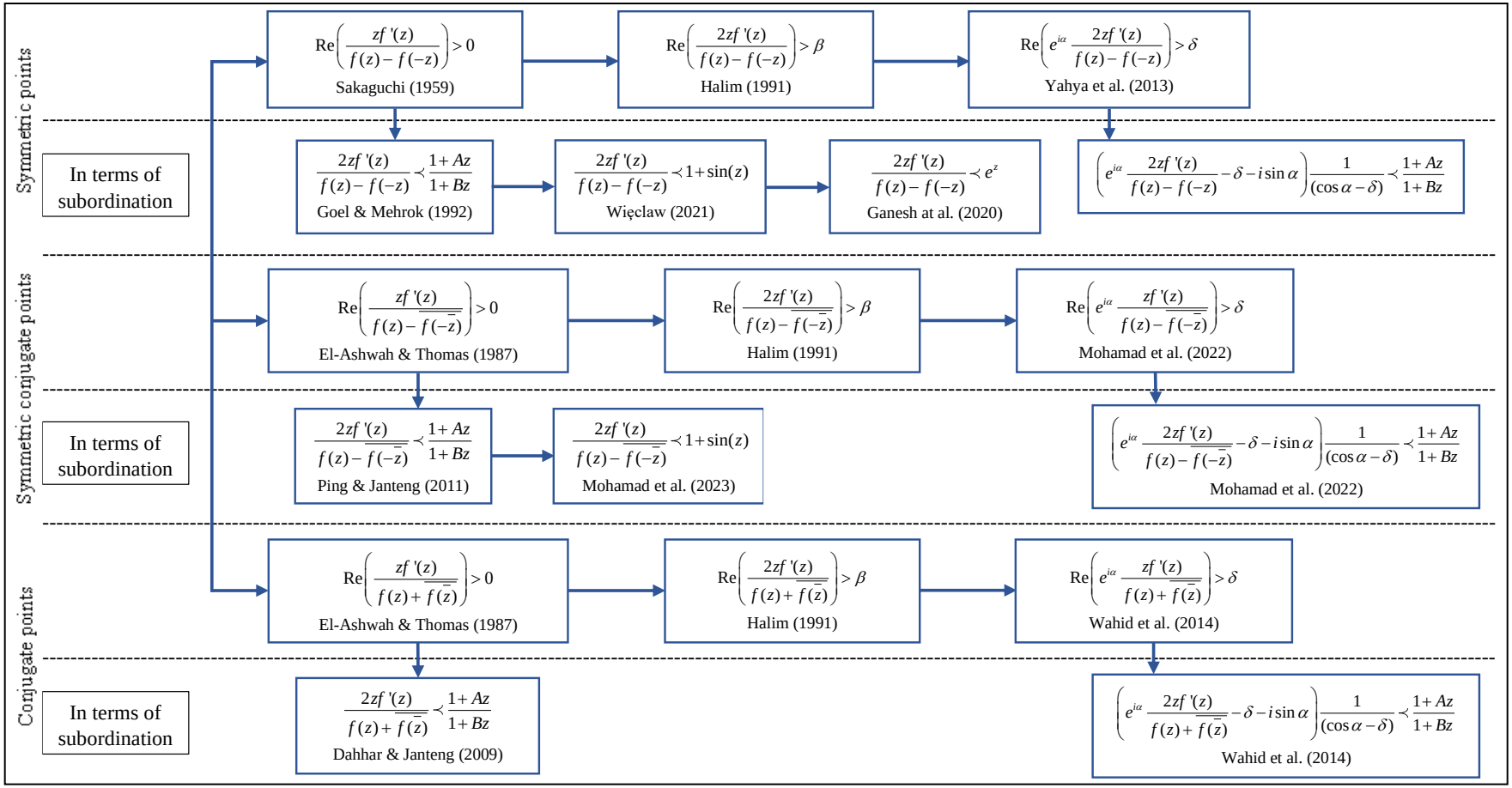

Figure 2: A Development of the Subclasses of Starlike Functions with Respect to Other Point.

Figure 2 provides a useful summary of the previous study done on the class of starlike functions with respect to certain points. By introducing a new parameter to the subclass introduced by the other researchers, they are able to enlarge and further investigate the class. The first to define the category of starlike functions with respect to certain points was Sakaguchi in 1959, then followed by El-Ashwah and Thomas (1987). Sakaguchi (1959) introduced the class of starlike functions with respect to symmetric points denoted by S_S^*. Then, Halim (1991) extends the class introduced by Sakaguchi (1959) in order β while Yahya et al. (2013) does the same in order δ. In term of subordination, Goel and Mehrok (1992) introduced a subclass of starlike functions with respect to symmetric points with parameters A and B denoted by $S_S^*(A,B)$. Next, Wieclaw (2021) continues the class by adding the sine function. Ganesh at al. (2020) further the class by adding the exponential function.

For symmetric conjugate, the researchers who introduced this class were El-Ashwah and Thomas in 1987, where the class is denoted as S_{SC}^*. Halim (1991) then followed by adding β while Mohamad (2022) expand the class with the parameter δ. In term of subordination, Ping and Janteng (2011) add A and B as a new parameter. Mohamad et al. (2023) further the class by adding the sine function. The, Wahid et al. (2015) expand the class with the parameters $\alpha,\ \delta, A$ and B denoted by $S_{SC}^*(\alpha,\delta,A,B)$.

Conjugate points were also first introduced by El-Ashwah and Thomas (1987) who were also responsible for the introduction of symmetric conjugate points. Next, Halim (1991) is employing this class to β. In order to expand the class even more, Wahid et al. (2013) include an exponential function and in order δ. In term of subordination, Dahhar and Janteng (2009) referred to the class of starlike functions with respect to conjugate points, which they denoted as $S_C^*(A,B)$.

6. Conclusion

In conclusion, this comprehensive literature review concludes with an overview of the substantial studies into the development of subclass properties for starlike function. As the results show, these properties have important implications in many areas of mathematics and practical science, such as geometric function theory and univalent mappings. Complex geometric and analytic properties of starlike functions have been better understood due to the classification of numerous subclasses based on unique criteria, providing important knowledge for researchers in complex analysis and related fields.

The array of methodologies used by researchers to explore subclass properties, including as coefficient estimations, integral operators, and boundary behaviour analysis, is also highlighted in the paper. The study of starlike functions gains from this wide variety of methods, and it also opens up exciting possibilities for cross-disciplinary work. With the PRISMA model as an outline, this study combines a thorough and credible synthesis of available research on the topic, ensuring the trustworthiness and transparency of the selection process.

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9. Conflict of Interest

The authors have no conflicts of interest to declare.

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