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ENVELOPING LIE ALGEBRAS OF LEIBNIZ ALGEBRAS AND ASSOCIATED LIE GROUP-LIKE OBJECTS

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ABSTRACT

Leibniz algebras are generalization of well-known Lie algebras. Lie algebra has its Lie group, most of the applications of Lie algebras go through the Lie groups. For Leibniz algebras Lie group-like object was not found and it is of great interest. In this paper we turn in two constructions of Lie algebras (the enveloping Lie algebras of Leibniz algebras) closely related to Leibniz algebras. One hopes the Lie group-like objects for Leibniz algebras can be constructed as some combinations of Lie groups of the Lie algebras raised. We describe the enveloping Lie algebras of Leibniz algebras of dimensions two and three. These results can be used to associate geometric objects (loop spaces) to low dimensional Leibniz algebras.

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1. Introduction

The concept of Leibniz algebra was introduced by Loday (see [7]) in the study of Leibniz (co)homology as a noncommutative analogue of Lie algebras (co)homology.

A Leibniz algebra is a vector space over a field F equipped with a F-bilinear map

$[\cdot,\cdot]:L \times L \to L$

satisfying the Leibniz identity

$$[x,[z,y]]=[[x,z],y]+[z,[x,y]], \qquad (1)$$

for all $x, y, z \in L$.

In fact, the definition above is a definition of the left Leibniz algebra, whereas the identity for the right Leibniz algebra is as follows

$$[[x,y],z]= [[x,z],y]+[x,[y,z]], \qquad (2)$$

for all $x, y, z \in L$.

The passage from the left to the right Leibniz algebra can be easily done by considering a new product "$[\cdot,\cdot]^{opp}$" on the same vector space defined by "$[x,y]^{opp} =[y ,x]$". Therefore, the results proved for the right Leibniz algebras $(L,[\cdot,\cdot])$ can be easily reformulated for the left Leibniz algebras $(L, [x,y]^{opp})$ and vice versa.

Clearly, a Lie algebra is a Leibniz algebra, and conversely, a Leibniz algebra L with property $[x,y]=-[y,x]$, for all $x,y \in L$, is a Lie algebra. The inherent properties of non-Lie Leibniz algebras imply that the subspace spanned by squares of elements of the algebra L is a non-trivial ideal.

During the last 25 years the theory of Leibniz algebras has been a topic of active research. Some (co)homology and deformations properties; relations with R-matrices and Yang-Baxter equations; result on various types of decompositions; structure of semisimple, solvable and nilpotent Leibniz algebras; classifications of some classes of graded nilpotent Leibniz algebras were obtained in numerous papers devoted to Leibniz algebras. In fact, many results of theory of Lie algebras have been extended to the Leibniz algebras case. For instance, the classical results on Cartan subalgebras, Levi's decomposition, and properties of solvable algebras with a given nilradical and others from the theory of Lie algebras are also true for Leibniz algebras.

From classical theory of finite-dimensional Lie algebras it is known that an arbitrary Lie algebra is decomposed into a semidirect sum of the solvable radical and its semisimple subalgebra (Levi's theorem). According to the Cartan-Killing theory a semisimple Lie algebra can be represented as a direct sum of simple ideals, which are completely classified. Thanks to Malcev's and Mubarakzyanov's results (see [8] and [9], respectively) the study of solvable Lie algebras is reduced to the study of nilpotent ones. Thus, the description of finite-dimensional Lie algebras consists of the description of semisimple and nilpotent Lie algebras.

Recently, D. Barnes [3] proved an analogue of Levi-Malcev Theorem for Leibniz algebras. The author proved that a Leibniz algebra is decomposed into a semidirect sum of its solvable radical and a semisimple Lie algebra. Therefore, the main problem of the description of finite-dimensional Leibniz algebras consists of the study of solvable Leibniz algebras. D.Barnes also noted that the non-uniqueness of the semisimple subalgebra S appeared in Levi-Malcev theorem (the minimum dimension of Leibniz algebra where this phenomena appears is six). It is known that in the case of Lie algebras the semisimple Levi quotient is unique up to conjugation via an inner automorphism. The conjugation in the Leibniz algebras case is not true, in general. Recently, in [6] the authors studied the conditions for the semisimple parts of Leibniz algebras in the decomposition to be conjugated. For more on Leibniz algebras and their classification see [2].

2. Methodology

In this paper, we work with vector spaces (and algebras) over the field of complex numbers, although the results can be extended in obvious way to the case of vector spaces over a field F of characteristic is not 2, or even over a commutative ring with unit. By an algebra $(L,\cdot)$, we mean a vector space L over F with a (not necessarily associative) bilinear operation $\cdot: L \times L \to L$. For $x \in L$, $\lambda(x): L \to L$; $y \to x \cdot y$ denotes the left multiplication map. Let $Der(L)$ denote the Lie subalgebra of $gl(L)$ consisting of the derivations on L. Recall that a linear map $d \in gl(L)$ is a derivation of $(L,\cdot)$ if and only if $[d,\lambda](x) = d(\lambda(x))$, for all $x \in L$. Here, we work with a class of algebras in which the left multiplication map has a stronger compatibility relation with derivations. These are Leibniz algebras, introduced by J.-L.Loday (1993) [7] as non-antisymmetric generalizations of Lie algebras.

Remind that a Leibniz algebra L is a vector space over a field F equipped with a bilinear map $\cdot: L \times L \to L$ satisfying the Leibniz identity

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z + y \cdot (x \cdot z), \tag{3}$$

for all $x, y, z \in L$.

Obviously, a Lie algebra is a Leibniz algebra. A Leibniz algebra is a Lie algebra if and only if $x \cdot x = 0 \quad (x \in L)$.

Also, an algebra $(L,\cdot)$ is a Leibniz algebra if and only if $\lambda(x) \subseteq \mathrm{Der}(L)$, or equivalently, $\lambda: (L,\cdot) \to (gl(L), [\cdot,\cdot])$ is a homomorphism. Thus we have a homomorphism $\lambda: (L,\cdot) \to (gl(L), [\cdot,\cdot])$ when $(L,\cdot)$ is a Leibniz algebra.

It is observed that Leibniz algebras are non-antisymmetric, in general. Hence, it is natural to consider the skew-symmetrization of a Leibniz algebra $(L,\cdot)$. This is done through the skew-symmetrized binary operation

$$[[x, y]] = \frac{1}{2}(x \cdot y - y \cdot x) \tag{4}$$

for $x,\ y \in L$. Note that, in general, $(L, [[\cdot,\cdot]])$ is not a Lie algebra. The skew-symmetrisation of a Leibniz algebra $(L,\cdot)$ is an interesting structure in its own right (see [3]).

On the other hand, by the definition of Leibniz algebra, $\lambda(x) \in Der(L, [[\cdot,\cdot]])$, for all $x \in L$, and $\lambda: (L, [[\cdot,\cdot]]) \to Der(L)$ is a homomorphism of anti-commutative algebras.

Let $J = \langle x \cdot x | x \in L \rangle$ be the two-sided ideal of $(L,\cdot)$ generated by all squares. Then J contains all symmetric products $x \cdot y + y \cdot x$, for $x, y \in L$, and since $\lambda(x \cdot x) = [\lambda(x), \lambda(x)]$, for all $x \in L$, we have $J \subseteq ker(\lambda)$.

Let $M \subseteq L$ be any ideal containing J, then since

$$x \cdot y + M = -y \cdot x + (x \cdot y + y \cdot x) + M = -y \cdot x + M, \tag{5}$$

for $x, y \in L$, the Leibniz product in L lifts to a Lie bracket $[\cdot,\cdot]$in L/M. Conversely, if $J \subseteq M \subseteq L$ is an ideal, then the quotient $h = L/M$ is a Lie algebra. In particular, J is the smallest two-sided ideal of L such that L/J is a Lie algebra.

Example 1.2. Let $(\mathcal{G}, [\cdot, \cdot])$ be a Lie algebra, and let V be an $\mathcal{G}$ -module with left action

$$\mathcal{G} \times V \to V : (\xi, x) \mapsto \xi x.$$

The induced left action of $\mathcal{G}$ on $\mathcal{G}\times V$ is just the restricted adjoint action of $\mathcal{G}$ on the semidirect product $\mathcal{G} \ltimes V$:

$$\xi(\eta,y):=[(\xi,0),(\eta,y)]=([\xi,\eta],\xi y) \tag{6}$$

for $\xi,\eta\in \mathcal{G}$, $y\in V$. Define a binary operation $\cdot$ on $L= \mathcal{G}\times V$ as follows

$$(\xi,x)\cdot(\eta,y) =\xi(\eta,y) \text{ for } \xi,\eta\in \mathcal{G}, x,y\in V; \text{ i.e. } (\xi,x)\cdot(\eta,y) =([\xi,\eta],\xi y). \tag{7}$$

Then $(L,\cdot)$ is a Leibniz algebra, and if $\mathcal{G}$ acts nontrivially on V, then $(L, \cdot)$ is not a Lie algebra. The algebra L with this Leibniz algebra structure is called the *hemisemidirect product* of $\mathcal{G}$ with V, and it is denoted by $\mathcal{G} \ltimes_H V$.

Another Leibniz algebra structure on L defined by the skew-symmetric product

$$\{(\zeta, x), (\eta, y)\} = \left([\zeta,\eta], \frac{1}{2}(\zeta y - \eta x)\right) \tag{8}$$

for $\xi,\eta\in \mathcal{G}$, $x,y\in V$. The Leibniz algebra $(L,\{\cdot,\cdot\})$ is called the *demisemidirect product* of G with V, and it is denoted by $\mathcal{G} \ltimes_D V$.

Definition 1.3. Let $(L,\cdot)$ be a Leibniz algebra and $\mathcal{G}$ be a Lie algebra with a derivation action $\mathcal{G}\to\mathrm{Der}(L)$, i.e., $\xi(x\cdot y) =\xi x\cdot y + x\cdot\xi y$ for all $\xi\in \mathcal{G}$, and $x,y\in L$. Let f: $L\to \mathcal{G}$ be an $\mathcal{G}$-equivariant linear map such that the following diagram commutes:

$$\begin{tikzcd} L \arrow[r, "\lambda"] \arrow[d, "f"'] & \mathrm{Der}(L) \\ \mathcal{G} \arrow[ur] & \end{tikzcd} \tag{9}$$

Let $\boldsymbol{h}=\mathcal{G} \ltimes L$ be the semidirect product Lie algebra, L being considered as a Lie algebra with the zero bracket. The triple $(\boldsymbol{h},\mathcal{G},f)$ is said to be an *enveloping algebra* of the Leibniz algebra $(L,\cdot)$.

Proposition 1.4. The algebra $(\boldsymbol{h} \ltimes L, \boldsymbol{h},\pi_{\boldsymbol{h}})$ is an enveloping Lie algebra for the hemisemidirect product $L =\boldsymbol{h} \ltimes_H V$, where $\pi_{\boldsymbol{h}} :L\to \boldsymbol{h}$ is the projection onto the first factor.

Remark 1.5. Note that $\pi_{\boldsymbol{h}}$ is $\boldsymbol{h}$-equivariant, i.e., $[\xi,\pi_{\boldsymbol{h}}(\eta,x)]=\pi_{\boldsymbol{h}}(\xi(\eta,x))$. Every Leibniz algebra $(L,\cdot)$ has enveloping Lie algebras. Obviously, the image of the left multiplication operator $\lambda(x)$: $L\to L$ is a Lie subalgebra of $\mathrm{Der}(L)$ since the left multiplication operators are $\mathrm{Der}(L)$–equivariant.

Proposition 1.6. Let $(L,\cdot)$ be a Leibniz algebra and $\boldsymbol{h}$ be a Lie algebra satisfying $\lambda(L)\subseteq \boldsymbol{h} \subseteq\mathrm{Der}(L)$. Then $(\boldsymbol{h} \ltimes L, \boldsymbol{h}, \lambda)$ is an enveloping Lie algebra for $(L,\cdot)$.

Note that Proposition 1.6. reduces the seeking an enveloping Lie algebra of a Leibniz algebra L to the description of subalgebras of $\mathrm{Der}(L)$.

Definition 1.7. *Let* $(L,\cdot)$ *be a Leibniz algebra. We define*

$$L^1 = L, L^k = L \cdot L^{k-1} \quad (k > 1). \tag{10}$$

The series

$$L^1 \supseteq L^2 \supseteq L^3 \supseteq \ldots \tag{11}$$

is said to be the *descending central series* of L. If the series terminates for some positive integer s, then the Leibniz algebra L is said to be *nilpotent*.

Proposition 1.8. *If L is a nilpotent Leibniz algebra, and M is an ideal of L such that $J \subseteq M \subseteq ker(\lambda)$, and $h = L \ltimes M$, then*

(i) h is nilpotent Lie algebra.

(ii) $N = h \ltimes L$ is a nilpotent Lie algebra.

The proposition above associates two Lie algebras h and N to a Leibniz algebra L. Here h is a quotient of L, whereas N is its extension. The corresponding Lie groups could be employed to associate a geometric object to L (see [5]).

Next let us recall a classification result for non-Lie complex Leibniz algebras of dimension two and three (see [1]). We use the convention to denote the j^{th} algebra of dimension i by $L_{i,j}$.

Theorem 1.9. *In dimension two, there are two non-isomorphic nilpotent Leibniz algebras. One of them $L_{2,1}$ is Abelian, another $L_{2,2}$, can be given by the table $e_1 \cdot e_1 = e_2$.*

Theorem 1.10. *In dimension three, there are five isolated and one continious family of pairwise not isomorphic nilpotent Leibniz algebras given by their tables of multiplications in a basis* $\{e_1, e_2, e_3\}$ *as follows*:

$$L_{3,1}: \text{Abelian}$$
$$L_{3,2}: e_1 \cdot e_1 = e_2$$
$$L_{3,3}: e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = -e_3$$
$$L_{3,4}: e_1 \cdot e_1 = e_3, e_2 \cdot e_2 = \alpha e_3, e_1 \cdot e_2 = e_3 (\alpha \in C) \tag{12}$$
$$L_{3,5}: e_1 \cdot e_1 = e_3, e_1 \cdot e_2 = e_3, e_2 \cdot e_1 = e_3$$
$$L_{3,6}: e_1 \cdot e_1 = e_2, e_2 \cdot e_1 = e_3$$

3. Results and Discussions

In this section, we classify the enveloping Lie algebras of Leibniz algebras of dimension two and three. As in each case we have to identify the resulting Lie algebra as one of the known low dimensional Lie algebras [4], by carefully defining the appropriate change of basis.

In dimension two, we have $L_{2,2}: e_1 \cdot e_1 = e_2$.

Then $J_{2,2} = \{\lambda e_2 | \lambda \in C\}$ and $h_{2,2} = L_{2,2}/J_{2,2} = span\{e_1 + J_{2,2} | e_1 \in L_{2,2}\} = span\{\overline{e}_1 | e_1 \in L_{2,2}\}$. Therefore $h_{2,2}$ is 1-dimensional abelian Lie algebra. Now, we consider $N_{2,2} = h_{2,2} \ltimes L_{2,2}$ with basis elements $E_1 = (\overline{e_1}, 0)$, $E_2 = (0, e_1)$, $E_3 = (0, e_2)$. The multiplication table of $N_{2,2}$ is given by

$N_{2,2}$	E_1	E_2	E_3
E_1	0	E_3	0
E_2	$-E_3$	0	0
E_3	0	0	0

For example

$$[E_1, E_1] = [(e_1 + J_{2,2}, 0), (e_1 + J_{2,2}, 0)] = (e_1 \cdot e_1 + J_{2,2}, 0) = (e_2 + J_{2,2}, 0) = 0 \tag{13}$$

and

$$[E_1, E_2] = [(e_1 + J_{2,2}, 0), (0, e_1)] = (0, e_1 \cdot e_1) = (0, e_2) = E_3. \tag{14}$$

Thus, $N_{2,2}$ is a 3-dimensional Lie algebra with the table $[E_1, E_2] = E_3$.

Finally, we have the following table (where the last column identifies the Lie algebra in the Beck-Kolman list (see [4]).

Table 1. Enveloping Lie algebras in dimension two.

$L_{i,j}$	$h_{i,j}$	$N_{i,j} = h_{i,j} \ltimes L_{i,j}$	$N_{i,j}$
$L_{2,2}$	1-dimensional abelian Lie algebra	$[E_1, E_2] = E_3$	g_3

In dimension three, for each $L_{3,k}$, $k = 2,3,4,5,6$, we want to obtain the corresponding Lie algebra $N_{3,k}$. For $L_{3,2}$, one can show that $h_{3,2}$ is 2-dimensional abelian Lie algebra and $N_{3,2}$ is given by

$$[E_1, E_3] = E_4. \tag{15}$$

For $L_{3,3}$, one can prove that $J_{3,3} = \{0\}$, therefore $h_{3,3}$ is given by

$$[e_1, e_2] = e_3, \quad [e_2, e_1] = -e_3, \tag{16}$$

and $N_{3,3}$ by

$$[E_1, E_2] = E_3, \ [E_1, E_5] = E_6, \ [E_2, E_4] = -E_6. \tag{17}$$

Finally, for $L_{3,4}$, one can show that $h_{3,4}$ is 2-dimensional abelian Lie algebra and $N_{3,4}$ is given by

$$[E_1, E_3] = E_5, [E_1, E_4] = E_5, [E_2, E_4] = \alpha E_5 \quad (\alpha \in C^*) \tag{18}$$

Therefore for 3-dimensional Leibniz algebras, we get the following table.

Table 2. Enveloping Lie algebras in dimension three.

$L_{i,j}$	$h_{i,j}$	$N_{i,j}= h_{i,j} \ltimes L_{i,j}$	$N_{i,j}$
$L_{3,2}$	2-dimensional abelian Lie algebra	$[E_1, E_3] = E_4$	$g_3 \oplus C^2$
$L_{3,3}$	$[e_1, e_2] = e_3$, $[e_1, e_2] = -e_3$	$[E_1, E_2] = E_3, [E_1, E_5] = E_6$, $[E_2, E_4] = -E_6$	$g_{6,21}$
$L_{3,4}$	2-dimensional abelian Lie algebra	$[E_1, E_2] = E_3$, $[E_1, E_5] = E_6$, $[E_2, E_4] = \alpha E_5$ $(\alpha \in \mathbb{C}*)$	$g_{5,2}$
$L_{3,5}$	2-dimensional abelian Lie algebra	$[E_1, E_3] = E_5, [E_1, E_4] = E_5$, $[E_2, E_3] = E_5$	$g_{5,2}$
$L_{3,6}$	1-dimensional abelian Lie algebra	$[E_1, E_2] = E_3$	$g_3 \oplus C^2$

Note that in the rows three and four, the enveloping Lie algebras are isomorphic, while the original Leibniz algebras are not isomorphic. The isomorphism

$N_{3,2} \cong g_3 \oplus C^2$ is given by

$$e_1 \mapsto E_1, \ e_2 \mapsto E_3, \ e_3 \mapsto E_4, \ e_4 \mapsto E_2, \ e_5 \mapsto E_5. \tag{19}$$

$N_{3,3} \cong g_{6,21}$ is given by

$$e_1 \mapsto -E_2, e_2 \mapsto E_1, e_3 \mapsto E_4, e_4 \mapsto E_5, e_5 \mapsto E_3, e_6 \mapsto E_6. \tag{20}$$

$N_{3,4} \cong g_{5,2}$ is given by

$$e_1 \mapsto -\alpha E_1, e_2 \mapsto -E_3 + E_4, e_3 \mapsto E_3, e_4 \mapsto E_2, e_5 \mapsto -\alpha E_5. \tag{21}$$

$N_{3,5} \cong g_{5,2}$ is given by

$$e_1 \mapsto -E_1 + E_4, e_2 \mapsto E_1, e_3 \mapsto E_2, e_4 \mapsto E_4, e_5 \mapsto E_5. \tag{22}$$

Also, for the $2-$dimensional abelian Lie algebra $L_{2,1}$, $J_{2,1} = \{0\}$, and $h_{2,1}$ is an abelian Lie algebra. Hence, $N_{2,1} = h_{2,1} \ltimes L_{2,1}$ is $4-$dimensional Lie algebra.

Finally, for the $3-$dimensional abelian Lie algebra $L_{3,1}$, $J_{3,1} = \{0\}$, and $h_{3,1}$ is an abelian Lie algebra. Therefore, $N_{3,1} = h_{3,1} \ltimes L_{3,1}$ is a 6-dimensional Lie algebra.

4. Conclusions and recommendations

4.1. Lie groups

A Lie group is an analytic manifold with a group structure such that the function

$$(x, y) \rightarrow x \cdot y^{-1} \tag{23}$$

is analytic.

If *G* is a Lie group, then the connected component passing through the identity of *G* is a connected Lie subgroup of *G*.

Let $l_a: x \ \rightarrow ax$ be the left translation by *a* in *G*. A vector field *X* on *G* is called left invariant if

$$(l_a)^* X_b = X_{ab} \tag{24}$$

where $l_a^*$is the Jacobian of f_a. As

$$l_a^*[X, Y] = [l_a^* X, l_a^* Y], \tag{25}$$

the set of left invariant vector fields of *G* is a Lie algebra, denoted by *g*, called the Lie algebra of *G*. As the left invariant vector field on *G* is known as soon as its value $X_e$of *X* on the identity *e* of *G* is given, we can identify *g* with the vector tangent space $T_e G$ of *G* in *e*, the bracket on $T_e G$ is deduced to the bracket of *g* as follows

$$[X_e, Y_e] = [X, Y]_e, \text{ for } X,Y \in g \tag{26}$$

4.2. **Correspondence between Lie groups and Lie algebras**.

The exponential mapping exp: $g \rightarrow G$ of the Lie algebra *g* of *G* in the Lie group *G* sends straight lines through the origin in *g* into one-parametric subgroup of *G*. This mapping is defined as follows: let $X \in g$. There exists a unique analytic homomorphism θ of **R** into *G* such that θ(0)=*X*. We have the formula

$$\exp(t+s)X = \exp tX \exp sX \tag{27}$$

for all $s,t \in \mathbf{R}$ and all *X* in *g*.

Remind that the Lie-like group objects are not introduced yet. One hopes that the Lie-like group objects can be found as some combinations of Lie groups of Lie algebras closely related to Leibniz algebras. Therefore, to introduce and study Lie algebras related to Leibniz algebras is of interest. Nowadays plenty of results of classification of different classes of Leibniz algebras are obtained. To use them to construct such Lie algebras is important.

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