UNIVERSITI TEKNOLOGI MARA

MATHEMATICS AND STATISTICS

UNDERGRADUATE RESEARCH PROCEEDINGS 2025

UiTM CAWANGAN NEGERI SEMBILAN

RADIUS OF STARLIKENESS OF A SUBCLASS OF GENERALIZED BI-UNIVALENT FUNCTIONS

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Abstract

In this paper, we let a Taylor series expansion, $S^*(\alpha,\delta,t)$ *of the form* $g(m)=m+a_2m^2+a_3m^3+\ldots+\sum_{z=2}^{\infty}a_zm^z$ *that satisfies the condition* $Re\left(e^{i\alpha}\dfrac{mg'(m)}{g'(m)}\right)>\delta,\ m\in E$ *where* $|\alpha|<\pi,\ \cos\alpha>\delta,\ 0\le\delta<1,\ -1<t\le1$ *and* $g'(m)=\dfrac{m}{(1+tm)(1-m)}$. *The main focus of this article is to determine the radius of starlikeness,* $\Re_{St}$ *by using Poisson Formula for the new generalized class of bi-univalent functions,* $S^*(\alpha,\delta,t)$.

Keywords*:* Radius of starlikness, Bi-univalent, Poisson formula

Introduction

Complex analysis is one of the properties in mathematics that deal with functions that involve complex numbers, including their functions, manipulation, derivatives and other mathematical properties. The geometric function theory is a study of the geometric properties of univalent functions. This paper focuses on the class of analytic univalent functions implied with the complex plane. Let $\mathbb{C}$ be the element of complex number and let $g(m)$ be a complex valued function of the complex variable m . Based on Goodman [1], a function $g(m)$ is univalent in $D\subset\mathbb{C}$ if the function $g(m)$ is injective, which is for all $m_1,m_2\in D$, $g(m_1)=g(m_2)$. Besides, a function $g(m)$ provide one to one mapping onto its image, $g(D)$.

Let Z denoted the class of function of the form

$$g(m)=m+a_2m^2+a_3m^3+\ldots+\sum_{z=2}^{\infty}a_zm^z \tag{1.1}$$

which are analytic in the unit disc, $E=\{m:|m|<1\}$. The function $g(m)$ is also known as normalized univalent function if it satisfies conditions of $g(0)=0$ and $g'(0)=1$ or $g(x)=g'(0)-1=0$ are fixed is denoted by S (Duren, [2])

In this paper, we defined a new generalized class of bi-univalent function $S^*(\alpha,\delta,t)\in S$. In this class, $S^*(\alpha,\delta,t)$ has a Taylor series expansion of the form (1.1) that satisfying the condition

$$Re\left(e^{i\alpha}\frac{mg'(m)}{g'(m)}\right)>\delta \quad (m\in E)$$

where $|\alpha|<\pi,\ cos\,\alpha>\delta,\ 0\le\delta<1,\ g'(m)=\dfrac{m}{(1+tm)(1-m)}$ and $-1<t\le 1$.

Based on $S^*(\alpha,\delta,t)$, the coefficient bound is given following lemma derived from Rathi [3] and the method of proving by Nunokawa and Owa [4].

Lemma 1.1

Suppose that $f\in S^*(\alpha,\delta,t)$ is given by $(1.1.1)$, then sharp inequality

$$|a_n|\le\begin{cases}\dfrac{1}{n}\left(\dfrac{1-t^2+2A_{\alpha\delta}\left[(-1+t)+n(t+1)\right]}{(t+1)^2}\right), & n=2,4,6,...\\[2ex] \dfrac{1}{n}\left(\dfrac{2A_{\alpha\delta}(n-1)+t+1}{(t+1)}\right), & n=3,5,7,...\end{cases}$$

equality is attained for each n while f is extreme point of $S^*(\alpha,\delta,t)$.

Lemma 1.2 (Ikeda and Saigo, [5])

Let $u(m)$ be harmonic in $|m|\le\rho$ and continuous in $|m|\le\rho$. Then $u(m)$ is given by the equation

$$u(m)=u\left(re^{i\theta}\right)=\frac{1}{2\pi}\int \mathrm{Re}\left\{u\left(\rho e^{i\phi}\right)\right\}\frac{\rho^2-r^2}{\rho^2-2\rho r\,cos(\phi-\theta)+r^2}d\phi \tag{1.2}$$

where $0\le r\le\rho$

The equation (1.2) is called the Poisson formula. Thus, by following the method of Nunokawa and Owa [4], the radius of starlikeness will be determined using this formula.

Preliminaries

Based on Lemma 1.1, we are going to find a bound for $\left|\frac{g(m)}{m}\right|$ by using the $|a_n| = \frac{1}{n}\left(\frac{1-t^2+2A_{\alpha\delta}\left[(-1+t)+n(t+1)\right]}{(t+1)^2}\right)$ when n is even.

Theorem 2.1

Let $g \in Z$, *since* $g \in S^*(\alpha,\delta,t)$ *and* $|a_n| = \frac{1}{n}\left(\frac{1-t^2+2A_{\alpha\delta}\left[(-1+t)+n(t+1)\right]}{(t+1)^2}\right)$ *where* $n = 2,4,6,\ldots,$ *then*

$$\log(1-r)\left[\frac{-t^2+2tA_{\alpha\delta}-2A_{\alpha\delta}+1}{r(t+1)^2}\right]+\left[1+\left(\frac{-t^2+4tA_{\alpha\delta}+1}{(t+1)^2}\right)\right]-\left(\frac{1}{1-r}\right)\left[\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{r(t+1)^2}\right]$$
$$+\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{r(t+1)^2} \le \left|\frac{g(m)}{m}\right| \le \log(1-r)\left[\frac{t^2-2tA_{\alpha\delta}+2A_{\alpha\delta}-1}{r(t+1)^2}\right]+\left[1+\frac{t^2-4tA_{\alpha\delta}-1}{(t+1)^2}\right]$$
$$+\left(\frac{1}{1-r}\right)\left[\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{r(t+1)^2}\right]-\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{r(t+1)^2}$$

for $|m| = r < 1.$

Proof.

Given $g \in Z$, since $g \in \mathbb{N}$ satisfying Lemma 4.1.1 for $n = 2,4,6,\ldots$, we have

$$|g(m)| \le |m| + \sum_{n=2}^{\infty}|a_n||m|^n$$

and

$$|g(m)| \le |m| + \sum_{n=2}^{\infty}\left|\frac{1}{n}\left(\frac{1-t^2+2A_{\alpha\delta}\left[(-1+t)+n(t+1)\right]}{(t+1)^2}\right)\right||m|^n \tag{2.1}$$

and the expansion of (2.1),

$$|g(m)| \le |m| + \sum_{n=2}^{\infty}\left|\frac{1}{n}\left(\frac{1}{(t+1)^2}-\frac{t^2}{(t+1)^2}-\frac{2A_{\alpha\delta}}{(t+1)^2}+\frac{2tA_{\alpha\delta}}{(t+1)^2}+\frac{2tnA_{\alpha\delta}}{(t+1)^2}+\frac{2nA_{\alpha\delta}}{(t+1)^2}\right)\right||m|^n$$

produces

$$\left|g(m)\right| \leq |m| + \frac{1}{(t+1)^2}\sum_{n=2}^{\infty}\frac{|m|^n}{n} - \frac{t^2}{(t+1)^2}\sum_{n=2}^{\infty}\frac{|m|^n}{n} - \frac{2A_{\alpha\delta}}{(t+1)^2}\sum_{n=2}^{\infty}\frac{|m|^n}{n}$$

$$+\frac{2tA_{\alpha\delta}}{(t+1)^2}\sum_{n=2}^{\infty}\frac{|m|^n}{n} + \frac{2tA_{\alpha\delta}}{(t+1)^2}\sum_{n=2}^{\infty}|m|^n + \frac{2A_{\alpha\delta}}{(t+1)^2}\sum_{n=2}^{\infty}|m|^n. \quad (2.2)$$

Then, simplification of (2.2), yield

$$\begin{aligned}\left|g(m)\right| \leq |m| &+ \frac{1}{(t+1)^2}\left(-|m| + \sum_{n=1}^{\infty}\frac{|m|^n}{n}\right) - \frac{t^2}{(t+1)^2}\left(-|m| + \sum_{n=1}^{\infty}\frac{|m|^n}{n}\right) \\ &- \frac{2A_{\alpha\delta}}{(t+1)^2}\left(-|m| + \sum_{n=1}^{\infty}\frac{|m|^n}{n}\right) + \frac{2tA_{\alpha\delta}}{(t+1)^2}\left(-|m| + \sum_{n=1}^{\infty}\frac{|m|^n}{n}\right) \\ &+ \frac{2tA_{\alpha\delta}}{(t+1)^2}\left(-1-|m| + \sum_{n=0}^{\infty}|m|^n\right) + \frac{2A_{\alpha\delta}}{(t+1)^2}\left(-1-|m| + \sum_{n=0}^{\infty}|m|^n\right).\end{aligned}$$

Since $\log(1-|m|) = -\sum_{n=1}^{\infty}\frac{|m|^n}{n}$ and $\frac{1}{1-|m|} = \sum_{n=0}^{\infty}|m|^n$, thus

$$\begin{aligned}\left|g(m)\right| \leq |m| &+ \frac{1}{(t+1)^2}\left(-|m| - \log(1-|m|)\right) - \frac{t^2}{(t+1)^2}\left(-|m| - \log(1-|m|)\right) \\ &- \frac{2A_{\alpha\delta}}{(t+1)^2}\left(-|m| - \log(1-|m|)\right) + \frac{2tA_{\alpha\delta}}{(t+1)^2}\left(-|m| - \log(1-|m|)\right) \\ &+ \frac{2tA_{\alpha\delta}}{(t+1)^2}\left(-1-|m| + \frac{1}{1-|m|}\right) + \frac{2A_{\alpha\delta}}{(t+1)^2}\left(-1-|m| + \frac{1}{1-|m|}\right).\end{aligned}$$

Which gives

$$\begin{aligned}\left|g(m)\right| \leq |m| &- \frac{|m|}{(t+1)^2} - \frac{1}{(t+1)^2}\log(1-|m|) + \frac{t^2|m|}{(t+1)^2} + \frac{t^2}{(t+1)^2}\log(1-|m|) \\ &+ \frac{2A_{\alpha\delta}|m|}{(t+1)^2} + \frac{2A_{\alpha\delta}}{(t+1)^2}\log(1-|m|) - \frac{2tA_{\alpha\delta}|m|}{(t+1)^2} - \frac{2tA_{\alpha\delta}}{(t+1)^2}\log(1-|m|) - \frac{2tA_{\alpha\delta}}{(t+1)^2} \\ &- \frac{2tA_{\alpha\delta}|m|}{(t+1)^2} + \frac{2tA_{\alpha\delta}}{(t+1)^2}\left(\frac{1}{1-|m|}\right) - \frac{2A_{\alpha\delta}}{(t+1)^2} - \frac{2A_{\alpha\delta}|m|}{(t+1)^2} + \frac{2A_{\alpha\delta}}{(t+1)^2}\left(\frac{1}{1-|m|}\right). \quad (2.3)\end{aligned}$$

Next, substitute $|m| = r$ into equation (2.3) as

$$|g(m)| \le \log(1-r)\left[\frac{t^2 - 2tA_{\alpha\delta} + 2A_{\alpha\delta} - 1}{(t+1)^2}\right] + r\left[1 + \frac{t^2 - 4tA_{\alpha\delta} - 1}{(t+1)^2}\right]$$
$$+\left(\frac{1}{1-r}\right)\left[\frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{(t+1)^2}\right] - \frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{(t+1)^2}. \quad (2.4)$$

Thus, multiply equation (2.4) with $\frac{1}{r}$ to obtain the equation for $\left|\frac{g(m)}{m}\right|$,

$$\left|\frac{g(m)}{m}\right| \le \log(1-r)\left[\frac{t^2 - 2tA_{\alpha\delta} + 2A_{\alpha\delta} - 1}{r(t+1)^2}\right] + \left[1 + \frac{t^2 - 4tA_{\alpha\delta} - 1}{(t+1)^2}\right]$$
$$+\left(\frac{1}{1-r}\right)\left[\frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{r(t+1)^2}\right] - \frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{r(t+1)^2}$$

for $|m| = r < 1$.

Hence, $g(m)$ absolutely converge in E, also $g(m)$ is analytic in E. Therefore, we have

$$|g(m)| \ge |m| - \sum_{n=2}^{\infty} |a_n||m|^n,$$
$$|g(m)| \ge |m| - \sum_{n=2}^{\infty}\left|\frac{1}{n}\left(\frac{1 - t^2 + 2A_{\alpha\delta}\left[(-1+t) + n(t+1)\right]}{(t+1)^2}\right)\right||m|^n, \quad (2.5)$$

and the expansion of (2.5),

$$|g(m)| \ge |m| - \sum_{n=2}^{\infty}\left|\frac{1}{n}\left(\frac{1}{(t+1)^2} - \frac{t^2}{(t+1)^2} - \frac{2A_{\alpha\delta}}{(t+1)^2} + \frac{2tA_{\alpha\delta}}{(t+1)^2} + \frac{2tnA_{\alpha\delta}}{(t+1)^2} + \frac{2nA_{\alpha\delta}}{(t+1)^2}\right)\right||m|^n$$

produces

$$|g(m)| \ge |m| - \frac{1}{(t+1)^2}\sum_{n=2}^{\infty}\frac{|m|^n}{n} + \frac{t^2}{(t+1)^2}\sum_{n=2}^{\infty}\frac{|m|^n}{n} + \frac{2A_{\alpha\delta}}{(t+1)^2}\sum_{n=2}^{\infty}\frac{|m|^n}{n}$$
$$-\frac{2tA_{\alpha\delta}}{(t+1)^2}\sum_{n=2}^{\infty}\frac{|m|^n}{n} - \frac{2tA_{\alpha\delta}}{(t+1)^2}\sum_{n=2}^{\infty}|m|^n - \frac{2A_{\alpha\delta}}{(t+1)^2}\sum_{n=2}^{\infty}|m|^n. \quad (2.6)$$

Then, simplification of (2.6) yields

$$|g(m)| \ge |m| - \frac{1}{(t+1)^2}\left(-|m| + \sum_{n=1}^{\infty}\frac{|m|^n}{n}\right) + \frac{t^2}{(t+1)^2}\left(-|m| + \sum_{n=1}^{\infty}\frac{|m|^n}{n}\right)$$

$$+\frac{2A_{\alpha\delta}}{(t+1)^2}\left(-|m|+\sum_{n=1}^{\infty}\frac{|m|^n}{n}\right)-\frac{2tA_{\alpha\delta}}{(t+1)^2}\left(-|m|+\sum_{n=1}^{\infty}\frac{|m|^n}{n}\right)$$
$$-\frac{2tA_{\alpha\delta}}{(t+1)^2}\left(-1-|m|+\sum_{n=0}^{\infty}|m|^n\right)-\frac{2A_{\alpha\delta}}{(t+1)^2}\left(-1-|m|+\sum_{n=0}^{\infty}|m|^n\right).$$

Since $\log(1-|m|)=-\sum_{n=1}^{\infty}\frac{|m|^n}{n}$ and $\frac{1}{1-|m|}=\sum_{n=0}^{\infty}|m|^n$, thus

$$|g(m)|\geq|m|-\frac{1}{(t+1)^2}\left(-|m|-log(1-|m|)\right)+\frac{t^2}{(t+1)^2}\left(-|m|-log(1-|m|)\right)$$
$$+\frac{2A_{\alpha\delta}}{(t+1)^2}\left(-|m|-log(1-|m|)\right)-\frac{2tA_{\alpha\delta}}{(t+1)^2}\left(-|m|-log(1-|m|)\right)$$
$$-\frac{2tA_{\alpha\delta}}{(t+1)^2}\left(-1-|m|+\frac{1}{1-|m|}\right)-\frac{2A_{\alpha\delta}}{(t+1)^2}\left(-1-|m|+\frac{1}{1-|m|}\right).$$

Which gives

$$|g(m)|\geq|m|+\frac{|m|}{(t+1)^2}+\frac{1}{(t+1)^2}log(1-|m|)-\frac{t^2|m|}{(t+1)^2}$$
$$-\frac{t^2}{(t+1)^2}log(1-|m|)-\frac{2A_{\alpha\delta}|m|}{(t+1)^2}-\frac{2A_{\alpha\delta}}{(t+1)^2}log(1-|m|)$$
$$+\frac{2tA_{\alpha\delta}|m|}{(t+1)^2}+\frac{2tA_{\alpha\delta}}{(t+1)^2}log(1-|m|)+\frac{2tA_{\alpha\delta}}{(t+1)^2}+\frac{2tA_{\alpha\delta}|m|}{(t+1)^2}$$
$$-\frac{2tA_{\alpha\delta}}{(t+1)^2}\left(\frac{1}{1-|m|}\right)+\frac{2A_{\alpha\delta}}{(t+1)^2}+\frac{2A_{\alpha\delta}|m|}{(t+1)^2}-\frac{2A_{\alpha\delta}}{(t+1)^2}\left(\frac{1}{1-|m|}\right)$$

$$|g(m)|\geq\log(1-|m|)\left[\frac{-t^2+2tA_{\alpha\delta}-2A_{\alpha\delta}+1}{(t+1)^2}\right]+|m|\left[1+\left(\frac{-t^2+4tA_{\alpha\delta}+1}{(t+1)^2}\right)\right]$$
$$-\left(\frac{1}{1-|m|}\right)\left[\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{(t+1)^2}\right]+\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{(t+1)^2}. \quad (2.7)$$

Next, substitute $|m|=r$ into equation (2.7) as

$$|g(m)|\geq\log(1-r)\left[\frac{-t^2+2tA_{\alpha\delta}-2A_{\alpha\delta}+1}{(t+1)^2}\right]+r\left[1+\left(\frac{-t^2+4tA_{\alpha\delta}+1}{(t+1)^2}\right)\right]$$

$$-\left(\frac{1}{1-r}\right)\left[\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{(t+1)^2}\right]+\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{(t+1)^2}. \quad (2.8)$$

Thus, multiply equation (2.8) with $\frac{1}{r}$ to obtain the equation for $\left|\frac{g(m)}{m}\right|$,

$$\left|\frac{g(m)}{m}\right| \le log(1-r)\left[\frac{-t^2+2tA_{\alpha\delta}-2A_{\alpha\delta}+1}{r(t+1)^2}\right]+\left[1+\left(\frac{-t^2+4tA_{\alpha\delta}+1}{(t+1)^2}\right)\right]$$
$$-\left(\frac{1}{1-r}\right)\left[\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{r(t+1)^2}\right]+\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{r(t+1)^2}$$

for $|m| = r < 1$ as required.

This completes Theorem 2.1.

Based on Lemma 1.1, in Theorem 2.2, we are going to find a bound for $\frac{g(m)}{m}$ by using the $|a_n| = \frac{1}{n}\left(\frac{2A_{\alpha\delta}(n-1)+t+1}{(t+1)}\right)$ when n is odd.

Theorem 2.2

Let $g \in Z$, *since* $g \in S^*(\alpha,\delta,t)$ *and* $|a_n| = \frac{1}{n}\left(\frac{2A_{\alpha\delta}(n-1)+t+1}{(t+1)}\right)$ *where* $n = 3,5,7,...,$ *then*

$$\log(1-r)\left[\frac{t+1-2A_{\alpha\delta}}{r(t+1)}\right]-\frac{2A_{\alpha\delta}}{(t+1)}\left(\frac{1}{1-r}\right)+2 \le \left|\frac{g(m)}{m}\right| \le \log(1-r)\left[\frac{2A_{\alpha\delta}-t-1}{r(t+1)}\right]+\frac{2A_{\alpha\delta}}{(t+1)}\left(\frac{1}{1-r}\right)$$

for $|m| = r < 1.$

Proof.

Given $g \in Z$, since $g \in \mathbb{N}$ satisfying Lemma 4.1.1 for $n = 3,5,7,...,$ we have

$$|g(m)| \le |m| + \sum_{n=2}^{\infty}|a_n||m|^n,$$

and

$$|g(m)| \le |m| + \sum_{n=2}^{\infty}\left|\frac{1}{n}\left(\frac{2A_{\alpha\delta}(n-1)+t+1}{(t+1)}\right)\right||m|^n \quad (2.9)$$

and the expansion of (2.9),

$$\left|g(m)\right| \le |m| + \sum_{n=2}^{\infty}\left|\frac{1}{n}\left(\frac{2nA_{\alpha\delta}}{(t+1)} - \frac{2A_{\alpha\delta}}{(t+1)} + \frac{t}{(t+1)} + \frac{1}{(t+1)}\right)\right||m|^n$$

produce

$$\left|g(m)\right| \le |m| + \frac{2A_{\alpha\delta}}{(t+1)}\sum_{n=2}^{\infty}|m|^n - \frac{2A_{\alpha\delta}}{(t+1)}\sum_{n=2}^{\infty}\frac{|m|^n}{n} + \frac{t}{(t+1)}\sum_{n=2}^{\infty}\frac{|m|^n}{n} + \frac{1}{(t+1)}\sum_{n=2}^{\infty}\frac{|m|^n}{n} \tag{2.10}$$

Then, simplification of (2.10) yields

$$\left|g(m)\right| \le |m| + \frac{2A_{\alpha\delta}}{(t+1)}\left(-1-|m|+\sum_{n=0}^{\infty}|m|^n\right) - \frac{2A_{\alpha\delta}}{(t+1)}\left(-|m|+\sum_{n=1}^{\infty}\frac{|m|^n}{n}\right)$$
$$+\frac{t}{(t+1)}\left(-|m|+\sum_{n=1}^{\infty}\frac{|m|^n}{n}\right) + \frac{1}{(t+1)}\left(-|m|+\sum_{n=1}^{\infty}\frac{|m|^n}{n}\right)$$

Since $\log\left(1-|m|\right) = -\sum_{n=1}^{\infty}\frac{|m|^n}{n}$ and $\frac{1}{1-|m|} = \sum_{n=0}^{\infty}|m|^n$, thus

$$\left|g(m)\right| \le |m| + \frac{2A_{\alpha\delta}}{(t+1)}\left(-1-|m|+\frac{1}{1-|m|}\right) - \frac{2A_{\alpha\delta}}{(t+1)}\left(-|m|-\log\left(1-|m|\right)\right)$$
$$+\frac{t}{(t+1)}\left(-|m|-\log\left(1-|m|\right)\right) + \frac{1}{(t+1)}\left(-|m|-\log\left(1-|m|\right)\right)$$

Which gives

$$\left|g(m)\right| \le |m| - \frac{2A_{\alpha\delta}}{(t+1)} + \frac{2A_{\alpha\delta}}{(t+1)}\left(\frac{1}{1-|m|}\right) + \frac{2A_{\alpha\delta}}{(t+1)}\left(\log\left(1-|m|\right)\right)$$
$$-\frac{t|m|}{(t+1)} - \frac{t}{(t+1)}\left(\log\left(1-|m|\right)\right) - \frac{|m|}{(t+1)} - \frac{1}{(t+1)}\left(\log\left(1-|m|\right)\right)$$

$$\left|g(m)\right| \le \log\left(1-|m|\right)\left[\frac{2A_{\alpha\delta}}{(t+1)} - \frac{t}{(t+1)} - \frac{1}{(t+1)}\right] + |m|\left[1 - \frac{t}{(t+1)} - \frac{1}{(t+1)}\right]$$
$$+\left(\frac{1}{1-|m|}\right)\left[\frac{2A_{\alpha\delta}}{(t+1)}\right] - \frac{2A_{\alpha\delta}}{(t+1)} \tag{2.11}$$

Next, substitute $|m| = r$ into equation (2.11) as

$$\left|g(m)\right| \le \log(1-r)\left[\frac{2A_{\alpha\delta}-t-1}{(t+1)}\right] + \left(\frac{1}{1-r}\right)\left[\frac{2A_{\alpha\delta}}{(t+1)}\right] - \frac{2A_{\alpha\delta}}{(t+1)}. \tag{2.12}$$

Thus, multiply equation (2.12) with $\frac{1}{r}$ and simplify to obtain the equation for $\left|\frac{g(m)}{m}\right|$,

$$\left|\frac{g(m)}{m}\right| \le \log(1-r)\left[\frac{2A_{\alpha\delta}-t-1}{r(t+1)}\right]+\frac{2A_{\alpha\delta}}{(t+1)}\left(\frac{1}{1-r}\right)$$

for $|m| = r < 1$.

Hence, $g(m)$ absolutely converge in E, also $g(m)$ is analytic in E. Therefore, we h

$$|g(m)| \ge |m| - \sum_{n=2}^{\infty}|a_n||m|^n,$$

$$|g(m)| \ge |m| - \sum_{n=2}^{\infty}\left|\frac{1}{n}\left(\frac{2A_{\alpha\delta}(n-1)+t+1}{(t+1)}\right)\right||m|^n \tag{2.13}$$

and expansion of (2.13),

$$|g(m)| \ge |m| - \sum_{n=2}^{\infty}\left|\frac{1}{n}\left(\frac{2nA_{\alpha\delta}}{(t+1)}-\frac{2A_{\alpha\delta}}{(t+1)}+\frac{t}{(t+1)}+\frac{1}{(t+1)}\right)\right||m|^n$$

produces

$$|g(m)| \ge |m| - \frac{2A_{\alpha\delta}}{(t+1)}\sum_{n=2}^{\infty}|m|^n + \frac{2A_{\alpha\delta}}{(t+1)}\sum_{n=2}^{\infty}\frac{|m|^n}{n} - \frac{t}{(t+1)}\sum_{n=2}^{\infty}\frac{|m|^n}{n} - \frac{1}{(t+1)}\sum_{n=2}^{\infty}\frac{|m|^n}{n}. \tag{2.14}$$

Then, simplification of (2.14) yields

$$|g(m)| \ge |m| - \frac{2A_{\alpha\delta}}{(t+1)}\left(-1-|m|+\sum_{n=0}^{\infty}|m|^n\right)+\frac{2A_{\alpha\delta}}{(t+1)}\left(-|m|+\sum_{n=1}^{\infty}\frac{|m|^n}{n}\right)$$
$$-\frac{t}{(t+1)}\left(-|m|+\sum_{n=1}^{\infty}\frac{|m|^n}{n}\right)-\frac{1}{(t+1)}\left(-|m|+\sum_{n=1}^{\infty}\frac{|m|^n}{n}\right)$$

Since $\log(1-|m|) = -\sum_{n=1}^{\infty}\frac{|m|^n}{n}$ and $\frac{1}{1-|m|} = \sum_{n=0}^{\infty}|m|^n$, thus

$$|g(m)| \ge |m| - \frac{2A_{\alpha\delta}}{(t+1)}\left(-1-|m|+\frac{1}{1-|m|}\right)+\frac{2A_{\alpha\delta}}{(t+1)}\left(-|m|-\log(1-|m|)\right)$$
$$-\frac{t}{(t+1)}\left(-|m|-\log(1-|m|)\right)-\frac{1}{(t+1)}\left(-|m|-\log(1-|m|)\right)$$

Which gives

$$\left|g(m)\right| \geq |m| + \frac{2A_{\alpha\delta}}{(t+1)} - \frac{2A_{\alpha\delta}}{(t+1)}\left(\frac{1}{1-|m|}\right) - \frac{2A_{\alpha\delta}}{(t+1)}\left(\log\left(1-|m|\right)\right)$$
$$+\frac{t|m|}{(t+1)} + \frac{t}{(t+1)}\left(\log\left(1-|m|\right)\right) + \frac{|m|}{(t+1)} + \frac{1}{(t+1)}\left(\log\left(1-|m|\right)\right)$$

$$\left|g(m)\right| \geq \log\left(1-|m|\right)\left[-\frac{2A_{\alpha\delta}}{(t+1)} + \frac{t}{(t+1)} + \frac{1}{(t+1)}\right]$$
$$+|m|\left[1+\frac{t}{(t+1)} + \frac{1}{(t+1)}\right] - \left(\frac{1}{1-|m|}\right)\left[\frac{2A_{\alpha\delta}}{(t+1)}\right] + \frac{2A_{\alpha\delta}}{(t+1)} \quad (2.15)$$

Next, substitute $|m| = r$ into equation (2.15) as

$$\left|g(m)\right| \geq \log(1-r)\left[\frac{t-2A_{\alpha\delta}+1}{(t+1)}\right] + 2r - \left(\frac{1}{1-r}\right)\left[\frac{2A_{\alpha\delta}}{(t+1)}\right] + \frac{2A_{\alpha\delta}}{(t+1)}. \quad (2.16)$$

Thus, multiply equation (2.16) with $\frac{1}{r}$ to obtain the equation for $\left|\frac{g(m)}{m}\right|$,

$$\left|\frac{g(m)}{m}\right| \geq \log(1-r)\left[\frac{t-2A_{\alpha\delta}+1}{r(t+1)}\right] + 2 - \left(\frac{1}{1-r}\right)\left[\frac{2A_{\alpha\delta}}{r(t+1)}\right] + \frac{2A_{\alpha\delta}}{r(t+1)}$$
$$\left|\frac{g(m)}{m}\right| \geq \log(1-r)\left[\frac{t+1-2A_{\alpha\delta}}{r(t+1)}\right] - \frac{2A_{\alpha\delta}}{(t+1)}\left(\frac{1}{1-r}\right) + 2$$

for $|m| = r < 1$ as required.

This completes Theorem 2.2.

Now, we shall prove our main result.

Main Results

Now, based on Theorem 2.1 and Theorem 2.2, we will apply these theorem to find the radius of starlikeness, $\Re_{St}$ for the class $S^*(\alpha,\delta,t)$. A similar approach of Nunokawa and Owa [4] and Yahya [6] will be applied to obtain the radius of starlikeness, $\Re_{St}$ for the class of $S^*(\alpha,\delta,t)$ in E.

Lemma 1.2 will be applied to obtain the result for Theorem 3.1 by using bound $|a_n| = \frac{1}{n}\left(\frac{1-t^2+2A_{\alpha\delta}\left[(-1+t)+n(t+1)\right]}{(t+1)^2}\right)$ where $n = 2,4,6,\dots$.

Theorem 3.1

Let $g \in S^*(\alpha,\delta,t)$ and g is univalent and starlike in $|m| < \Re_{St}$, then radius of starlikeness, $\Re_{St}$ is given

$$log\left(1-r-\sqrt{2}r\right)\left[\frac{t^2-2tA_{\alpha\delta}+2A_{\alpha\delta}-1}{\left(r+\sqrt{2}r\right)(t+1)^2}\right]+\left[1+\left(\frac{t^2-4tA_{\alpha\delta}-1}{(t+1)^2}\right)\right]$$
$$+\left(\frac{1}{1-r-\sqrt{2}r}\right)\left(\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{\left(r+\sqrt{2}r\right)(t+1)^2}\right)-\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{\left(r+\sqrt{2}r\right)(t+1)^2}-e.$$

Proof.

By means of Theorem 2.1, we have $\left|\frac{g(m)}{m}\right| > 0$ in $|m| < R_1 = 0.7475842618$ and therefore, $log\left(\frac{g(m)}{m}\right)$ is harmonic in $|m| < R$. From Lemma 1.2 and based on the harmonic function theory, we defined

$$log\frac{g(m)}{m} = \frac{1}{2\pi}\int_0^{2\pi}\left(log\left|\frac{g(\zeta)}{\zeta}\right|\right)\frac{\zeta+m}{\zeta-m}d\phi$$

where $\zeta = \rho e^{i\phi}, (0 \le \phi \le 2\pi), m = re^{i\theta}, (0 \le \theta \le 2\pi)$, and $0 \le r < \rho \le R_1 = 0.7475842618$.

Logarithmic differentiation approach will be applied and yields

$$\frac{d}{dm}\left(log\frac{g(m)}{m}\right) = \frac{1}{2\pi}\int_0^{2\pi}\left(log\left|\frac{g(\zeta)}{\zeta}\right|\right)\frac{d}{dm}\left(\frac{\zeta+m}{\zeta-m}\right)d\phi$$

and

$$\frac{d}{g(m)}\left[\frac{g'(m)}{m}-\frac{g(m)}{m^2}\right] = \frac{1}{2\pi}\int_0^{2\pi}\left(log\left|\frac{g(\zeta)}{\zeta}\right|\right)\left(\frac{(\zeta-m)+(\zeta+m)}{(\zeta-m)^2}\right)d\phi. \quad (3.1)$$

Then we simplify of (3.1) will results

$$\frac{1}{m}\left[\frac{mg'(m)}{g(m)}-1\right] = \frac{1}{2\pi}\int_0^{2\pi}\left(log\left|\frac{g(\zeta)}{\zeta}\right|\right)\left(\frac{2\zeta}{(\zeta-m)^2}\right)d\phi$$

and

$$\frac{mg'(m)}{g(m)} = 1 + \frac{1}{2\pi}\int_0^{2\pi}\left(log\left|\frac{g(\zeta)}{\zeta}\right|\right)\left(\frac{2m\zeta}{(\zeta - m)^2}\right)d\phi . \tag{3.2}$$

From (3.2), we have

$$\frac{mg'(m)}{g(m)} = 1 + \frac{1}{2\pi}\int_0^{2\pi}\left(\left|log\left|\frac{g(\zeta)}{\zeta}\right|\right|\right)\left(\left|\frac{2m\zeta}{(\zeta - m)^2}\right|\right)d\phi$$

and yields

$$\frac{mg'(m)}{g(m)} = 1 + \frac{1}{2\pi}\int_0^{2\pi}\left(\left|log\left|\frac{g(\zeta)}{\zeta}\right|\right|\right)\left(\frac{2\left|re^{i\theta}\right|\left|\rho e^{i\phi}\right|}{\left|\rho e^{i\phi} - re^{i\theta}\right|^2}\right)d\phi .$$

Let $\left|\rho e^{i\phi}\right| = \rho$ and $\left|re^{i\theta}\right| = r$, we have

$$\frac{mg'(m)}{g(m)} = 1 + \frac{1}{2\pi}\int_0^{2\pi}\left(\left|log\left|\frac{g(\zeta)}{\zeta}\right|\right|\right)\frac{2r\rho}{\left|(\rho\cos\phi - r\cos\theta) + i(\rho\sin\phi - r\sin\theta)^2\right|}d\phi$$

which gives

$$\frac{mg'(m)}{g(m)} = 1 + \frac{1}{2\pi}\int_0^{2\pi}\left(\left|log\left|\frac{g(\zeta)}{\zeta}\right|\right|\right)\frac{2r\rho}{(\rho\cos\phi - r\cos\theta)^2 + (\rho\sin\phi - r\sin\theta)^2}d\phi$$

So that,

$$\frac{mg'(m)}{g(m)} = 1 + \frac{1}{2\pi}\int_0^{2\pi}\left(\left|log\left|\frac{g(\zeta)}{\zeta}\right|\right|\right)\frac{2r\rho}{\rho^2 - 2\rho r\cos(\rho - \theta) + r^2}d\phi$$

From Theorem 2.1, we have

$$log(1-r)\left[\frac{-t^2 + 2tA_{\alpha\delta} - 2A_{\alpha\delta} + 1}{r(t+1)^2}\right] + \left[1 + \left(\frac{-t^2 + 4tA_{\alpha\delta} + 1}{(t+1)^2}\right)\right] - \left(\frac{1}{1-r}\right)\left(\frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{r(t+1)^2}\right)$$
$$+ \frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{r(t+1)^2} < log(1-r)\left[\frac{t^2 - 2tA_{\alpha\delta} + 2A_{\alpha\delta} - 1}{r(t+1)^2}\right] + \left[1 + \left(\frac{t^2 - 4tA_{\alpha\delta} - 1}{(t+1)^2}\right)\right]$$
$$+ \left(\frac{1}{1-r}\right)\left(\frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{r(t+1)^2}\right) - \frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{r(t+1)^2}$$

for $|m| = r < 1$, we derive

$$\operatorname{Re}\left\{\frac{mg'(m)}{g(m)}\right\} \geq 1 - \frac{1}{2\pi}\int_0^{2\pi} max_{|\zeta|=\rho}\left(\left|log\left|\frac{g(\zeta)}{\zeta}\right|\right|\right)\frac{2\rho r}{\rho^2 - 2\rho\cos(\phi-\theta)+r^2}d\phi \tag{3.3}$$

and from (3.3) and integrating with respect to ϕ yield

$$\operatorname{Re}\left\{\frac{mg'(m)}{g(m)}\right\} \geq 1 - \frac{2\rho r}{\rho^2 - r^2} log\left(log(1-\rho)\left[\frac{t^2 - 2tA_{\alpha\delta} + 2A_{\alpha\delta} - 1}{\rho(t+1)^2}\right] + \left[1 + \left(\frac{t^2 - 4tA_{\alpha\delta} - 1}{(t+1)^2}\right)\right] + \left(\frac{1}{1-\rho}\right)\left(\frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{\rho(t+1)^2}\right) - \frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{\rho(t+1)^2}\right) \tag{3.4}$$

where $0 \leq r < \rho \leq R_1 = 0.7475842618$.

Based on $\operatorname{Re}\left\{\frac{mg'(m)}{g(m)}\right\} \geq 0$ in $|m| < \Re_{St}$ and from (3.4), we see that

$$\frac{2\rho r}{\rho^2 - r^2} log\left(log(1-\rho)\left[\frac{t^2 - 2tA_{\alpha\delta} + 2A_{\alpha\delta} - 1}{\rho(t+1)^2}\right] + \left[1 + \left(\frac{t^2 - 4tA_{\alpha\delta} - 1}{(t+1)^2}\right)\right] + \left(\frac{1}{1-\rho}\right)\left(\frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{\rho(t+1)^2}\right) - \frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{\rho(t+1)^2}\right) = 1$$

and

$$log\left(log(1-\rho)\left[\frac{t^2 - 2tA_{\alpha\delta} + 2A_{\alpha\delta} - 1}{\rho(t+1)^2}\right] + \left[1 + \left(\frac{t^2 - 4tA_{\alpha\delta} - 1}{(t+1)^2}\right)\right] + \left(\frac{1}{1-\rho}\right)\left(\frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{\rho(t+1)^2}\right) - \frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{\rho(t+1)^2}\right) = \frac{\rho^2 - r^2}{2\rho r}. \tag{3.5}$$

Putting $\rho = \left(1+\sqrt{2}\right)r$ in (3.5), we have

$$log\left(log\left(1-\left(1+\sqrt{2}\right)r\right)\left[\frac{t^2 - 2tA_{\alpha\delta} + 2A_{\alpha\delta} - 1}{\left(\left(1+\sqrt{2}\right)r\right)(t+1)^2}\right] + \left[1 + \left(\frac{t^2 - 4tA_{\alpha\delta} - 1}{(t+1)^2}\right)\right] + \left(\frac{1}{1-\left(1+\sqrt{2}\right)r}\right)\left(\frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{\left(\left(1+\sqrt{2}\right)r\right)(t+1)^2}\right) - \frac{2tA_{\alpha\delta} + 2A_{\alpha\delta}}{\left(\left(1+\sqrt{2}\right)r\right)(t+1)^2}\right) = \frac{\left(\left(1+\sqrt{2}\right)r\right)^2 - r^2}{2\left(\left(1+\sqrt{2}\right)r\right)r}$$

and

$$log\left(log\left(1-r-\sqrt{2}r\right)\left[\frac{t^2-2tA_{\alpha\delta}+2A_{\alpha\delta}-1}{\left(r+\sqrt{2}r\right)(t+1)^2}\right]+\left[1+\left(\frac{t^2-4tA_{\alpha\delta}-1}{(t+1)^2}\right)\right]\right.$$

$$\left.+\left(\frac{1}{1-r-\sqrt{2}r}\right)\left(\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{\left(r+\sqrt{2}r\right)(t+1)^2}\right)-\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{\left(r+\sqrt{2}r\right)(t+1)^2}\right)=1$$

which gives

$$log\left(1-r-\sqrt{2}r\right)\left[\frac{t^2-2tA_{\alpha\delta}+2A_{\alpha\delta}-1}{\left(r+\sqrt{2}r\right)(t+1)^2}\right]+\left[1+\left(\frac{t^2-4tA_{\alpha\delta}-1}{(t+1)^2}\right)\right]$$

$$+\left(\frac{1}{1-r-\sqrt{2}r}\right)\left(\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{\left(r+\sqrt{2}r\right)(t+1)^2}\right)-\frac{2tA_{\alpha\delta}+2A_{\alpha\delta}}{\left(r+\sqrt{2}r\right)(t+1)^2}=e$$

$|m|<\Re_{St}$.

This completes Theorem 3.1.

Next, Lemma 1.2 will be applied to obatin the result for Theorem 3.2 by using bound $|a_n|=\frac{1}{n}\left(\frac{2A_{\alpha\delta}(n-1)+t+1}{(t+1)}\right)$ where $n=3,5,7,\ldots$.

Theorem 3.2

Let $g\in S^*(\alpha,\delta,t)$, and g is univalent and starlike in $|m|<\Re_{St}$, then radius of starlikeness, $\Re_{St}$ is given

$$\frac{2A_{\alpha\delta}}{(t+1)}\left(\frac{1}{1-r-\sqrt{2}r}\right)+\left(\frac{2A_{\alpha\delta}-t-1}{\left(r+\sqrt{2}r\right)(t+1)}\right)log\left(1-r-\sqrt{2}r\right)-e$$

Proof.

Through Theorem 2.2, we have $\left|\frac{g(m)}{m}\right|>0$ in $|m|<R_1=0.7475842618$ and therefore, $log\left(\frac{g(m)}{m}\right)$ is harmonic in $|m|<R$. From Lemma 1.2 and based on the harmonic function theory, we defined

$$log\frac{g(m)}{m}=\frac{1}{2\pi}\int_0^{2\pi}\left(log\left|\frac{g(\zeta)}{\zeta}\right|\right)\frac{\zeta+m}{\zeta-m}d\phi$$

where $\zeta=\rho e^{i\phi},(0\le\phi\le 2\pi),m=re^{i\theta},(0\le\theta\le 2\pi)$, and
$0\le r<\rho\le R_1=0.7475842618$.

Logarithmic differentiation approach will be applied and yields

$$\frac{d}{dm}\left(log\frac{g(m)}{m}\right)=\frac{1}{2\pi}\int_0^{2\pi}\left(log\left|\frac{g(\zeta)}{\zeta}\right|\right)\frac{d}{dm}\left(\frac{\zeta+m}{\zeta-m}\right)d\phi$$

and

$$\frac{d}{g(m)}\left[\frac{g'(m)}{m}-\frac{g(m)}{m^2}\right]=\frac{1}{2\pi}\int_0^{2\pi}\left(log\left|\frac{g(\zeta)}{\zeta}\right|\right)\left(\frac{(\zeta-m)+(\zeta+m)}{(\zeta-m)^2}\right)d\phi. \tag{3.6}$$

Then we simplify of (3.6) will results

$$\frac{1}{m}\left[\frac{mg'(m)}{g(m)}-1\right]=\frac{1}{2\pi}\int_0^{2\pi}\left(log\left|\frac{g(\zeta)}{\zeta}\right|\right)\left(\frac{2\zeta}{(\zeta-m)^2}\right)d\phi$$

and

$$\frac{mg'(m)}{g(m)}=1+\frac{1}{2\pi}\int_0^{2\pi}\left(log\left|\frac{g(\zeta)}{\zeta}\right|\right)\left(\frac{2m\zeta}{(\zeta-m)^2}\right)d\phi. \tag{3.7}$$

From (3.7), we have

$$\frac{mg'(m)}{g(m)}=1+\frac{1}{2\pi}\int_0^{2\pi}\left(\left|log\left|\frac{g(\zeta)}{\zeta}\right|\right|\right)\left(\left|\frac{2m\zeta}{(\zeta-m)^2}\right|\right)d\phi$$

and yields

$$\frac{mg'(m)}{g(m)}=1+\frac{1}{2\pi}\int_0^{2\pi}\left(\left|log\left|\frac{g(\zeta)}{\zeta}\right|\right|\right)\left(\frac{2\left|re^{i\theta}\right|\left|\rho e^{i\phi}\right|}{\left|\rho e^{i\phi}-re^{i\theta}\right|^2}\right)d\phi.$$

Let $\left|\rho e^{i\phi}\right|=\rho$ and $\left|re^{i\theta}\right|=r$, we have

$$\frac{mg'(m)}{g(m)}=1+\frac{1}{2\pi}\int_0^{2\pi}\left(\left|log\left|\frac{g(\zeta)}{\zeta}\right|\right|\right)\frac{2r\rho}{\left|(\rho\cos\phi-r\cos\theta)+i(\rho\sin\phi-r\sin\theta)^2\right|}d\phi$$

which gives

$$\frac{mg'(m)}{g(m)}=1+\frac{1}{2\pi}\int_0^{2\pi}\left(\left|log\left|\frac{g(\zeta)}{\zeta}\right|\right|\right)\frac{2r\rho}{(\rho\cos\phi-r\cos\theta)^2+(\rho\sin\phi-r\sin\theta)^2}d\phi .$$

From Theorem 2.2, we have

$$log(1-r)\left[\frac{2A_{\alpha\delta}-t-1}{r(t+1)}\right]+\frac{2A_{\alpha\delta}}{(t+1)}\left(\frac{1}{1-r}\right)+2<log(1-r)\left(\frac{2A_{\alpha\delta}-t-1}{r(t+1)}\right)+\frac{2A_{\alpha\delta}}{(t+1)}\left(\frac{1}{1-r}\right)$$

for $|m|=r<1$, we derive

$$\operatorname{Re}\left\{\frac{mg'(m)}{g(m)}\right\}\geq 1-\frac{1}{2\pi}\int_0^{2\pi} max_{|\zeta|=\rho}\left(\left|log\left|\frac{g(\zeta)}{\zeta}\right|\right|\right)\frac{2\rho r}{\rho^2-2\rho\cos(\phi-\theta)+r^2}d\phi \tag{3.8}$$

and from (3.8) and integrating with respect to ϕ yield

$$\operatorname{Re}\left\{\frac{mg'(m)}{g(m)}\right\}\geq 1-\frac{2\rho r}{\rho^2-r^2}log\left(log(1-\rho)\left(\frac{2A_{\alpha\delta}-t-1}{\rho(t+1)}\right)+\frac{2A_{\alpha\delta}}{(t+1)}\left(\frac{1}{1-\rho}\right)\right) \tag{3.9}$$

where $0\leq r<\rho\leq R_1=0.7475842618$.

Based on $\operatorname{Re}\left\{\frac{mg'(m)}{g(m)}\right\}\geq 0$ in $|m|<\Re_{St}$ and from (3.9), we see that

$$log\left(log(1-\rho)\left(\frac{2A_{\alpha\delta}-t-1}{\rho(t+1)}\right)+\frac{2A_{\alpha\delta}}{(t+1)}\left(\frac{1}{1-\rho}\right)\right)=\frac{\rho^2-r^2}{2\rho r}. \tag{3.10}$$

Putting $\rho=(1+\sqrt{2})r$ in (3.10), we have

$$log\left(log\left(1-(1+\sqrt{2})r\right)\left(\frac{2A_{\alpha\delta}-t-1}{\left((1+\sqrt{2})r\right)(t+1)}\right)+\frac{2A_{\alpha\delta}}{(t+1)}\left(\frac{1}{1-(1+\sqrt{2})r}\right)\right)=\frac{\left((1+\sqrt{2})r\right)^2-r^2}{2\left((1+\sqrt{2})r\right)r}$$

and

$$log\left(log(1-r-\sqrt{2}r)\left(\frac{2A_{\alpha\delta}-t-1}{(r+\sqrt{2}r)(t+1)}\right)+\frac{2A_{\alpha\delta}}{(t+1)}\left(\frac{1}{1-r-\sqrt{2}r}\right)\right)=1$$

which gives

$$\frac{2A_{\alpha\delta}}{(t+1)}\left(\frac{1}{1-r-\sqrt{2}r}\right)+\left(\frac{2A_{\alpha\delta}-t-1}{\left(r+\sqrt{2}r\right)(t+1)}\right)l\,og\left(1-r-\sqrt{2}r\right)=e$$

in $|m|<\Re_{St}$.

This completes Theorem 3.2.

Conclusion

In conclusion, there is a purpose of this paper, which is to determine the radius of starllikenss, $\Re_{St}$ using the Poisson formula. We believe that we have achieved the objective that we highlighted.

References

[1] Goodman, A. W. (1983). Univalent functions. Polygonal Publishing House, Washington, New Jersey, Vol. I & II.

[2] Duren, P. L. (1983). Univalent Function.*Springer Verlag New York Inc*

[3] Rathi, S. (2015) Properties of titled univalent analytics function of order δ, Unpublished Master Thesis, Universiti Teknologi MARA.

[4] Nunokawa, M., & Owa, S. (2000). On some inverse properties for univalent functions. Kyoto University Koukyuroku, 1164, 73-76.

[5] Ikeda, & Saigo, M. (2000). Various inverse problems for univalent functions. RIMS Kokyu Record, 1164, 22-30.

[6] Yahya, A., & Soh, S. C. (2017). Radius of starlikeness of certain class of close-to-convex function. International Journal of Pure and Applied Mathematical, 10, 341-348. http://dx.doi.org/10.12732/ijpam.v112i2.11