UNIVERSITI TEKNOLOGI MARA

POSITIVE PERIODIC SOLUTIONS OF CERTAIN SINGULAR NON-AUTONOMOUS DIFFERENCE AND DIFFERENTIAL EQUATIONS

OSMAN BIN OMAR

Thesis submitted in fulfillment
of the requirements for the degree of
Master of Science

Faculty of Computer & Mathematical Sciences

May 2013

AUTHOR'S DECLARATION

I declare that the work in this thesis was carried out in accordance with the regulations of Universiti Teknologi MARA. It is original and is the result of my own work, unless otherwise indicated or acknowledged as referenced work. This thesis has not been submitted to any other academic institution or non-academic institution for any degree or qualification.

I, hereby, acknowledge that I have been supplied with the Academic Rules and Regulations for Post Graduate, Universiti Teknologi MARA, regulating the conduct of my study and research.

Name of Student	:	Osman Omar
Student I.D. No.	:	2010250402
Programme	:	Master of Science
Faculty	:	Faculty of Computer and Mathematical Sciences
Thesis Title	:	Positive Periodic Solutions of Certain Singular Non-Autonomous Difference and Differential Equations
Signature	:	....
Date	:	May 2013

ABSTRACT

This thesis is concerned with the study of singular non-autonomous first order difference equation for a single equation and n-dimensional systems with a positive parameter. The study also involves singular non-autonomous first order differential equations for n-dimensional systems with delay and a positive parameter. Sufficient conditions for the existence and multiplicity of positive periodic solutions for singular first order functional differential and difference equations under various assumptions are presented. First, we employ Kranoselskii fixed point theorem and obtain sufficient conditions for the existence and multiplicity of positive periodic solutions to a scalar singular first order difference equation with a positive parameter. Next, we investigate the existence and multiplicity of positive periodic solutions for singular first order non-autonomous systems of difference equations with a positive parameter by applying the Kranoselskii fixed point theorem. Finally, we apply a fixed point index theorem to study the existence, multiplicity and nonexistence of positive periodic solutions with a positive parameter to nonlinear singular systems of first order functional differential equations.

ACKNOWLEDGEMENTS

All praise to Allah, Lord of the world, the Almighty, with whose gracious help it was possible to accomplish this thesis.

Acknowlegdement is due to the Institute of Graduate Studies of Universiti Teknologi MARA (UiTM) for supporting me in this research.

I would like to thank my supervisor, Dr. Mesliza Mohamed, for her guidance during my study and research at UiTM Perlis. She was always accessible and willing to help me in every way, academically, socially and morally. I can never forget all of her kindness in my life. Words are not enough to express my gratitude.

Not to forget, great appreciation goes to my beloved family for their unflagging love and support throughout my life. It is their encouragement and warm care for both my study and daily life that makes me still energetic even when I was wear out in mind.
In short, this thesis does not belong to me, but to all the people who accompanied me throughout the days.

Thank you very much.

TABLE OF CONTENTS

	Page
AUTHOR'S DECLARATION	ii
ABSTRACT	iii
ACKNOWLEDGEMENTS	iv
TABLE OF CONTENTS	v
CHAPTER ONE: INTRODUCTION	1
1.1 Motivation	1
1.2 Thesis Outline	3
1.3 Preliminaries	4
1.3.1 Fixed Point Theorems	6
CHAPTER TWO: POSITIVE PERIODIC SOLUTIONS OF SINGULAR FIRST ORDER DIFFERENCE EQUATION	8
2.1 Introduction	8
2.2 Preliminary Result	9
2.3 Main Result	15
CHAPTER THREE: POSITIVE PERIODIC SOLUTIONS OF SINGULAR SYSTEMS FIRST ORDER DIFFERENCE EQUATIONS	19
3.1 Introduction	19
3.2 Preliminary Result	20
3.3 Main Result	26
3.4 Application	29

CHAPTER FOUR: POSITIVE PERIODIC SOLUTIONS OF SINGULAR SYSTEMS FIRST ORDER DIFFERENTIAL EQUATIONS **31**

4.1 Introduction 31

4.2 Preliminary Result 33

4.3 Main Result 41

REFERENCES **46**

LIST OF PUBLICATIONS **50**

CHAPTER ONE

INTRODUCTION

1.1 MOTIVATION

A variety of population dynamics and physiological processes can be described as the following equation

$$x'(t) = -a(t)x(t) + \lambda b(t)f(x(t)).$$

Periodic solutions of the type problems have attracted much attention (see (Jiang, Wei and Zhang, 2002), (O'Regan and Wang, 2005), (Wang, 2010)). One of the important question is whether these problems can support positive periodic solutions.

The aim of this thesis is to study the existence and multiplicity of positive periodic solutions for singular non-autonomous first order difference equations for a single equation and n-dimensional systems with a positive parameter. The study also involve the existence, multiplicity and nonexistence of singular non-autonomous first order differential equations for n-dimensional systems with delay and a positive parameter.

To motivate the following study some examples of first order differential with periodic delays appear in some ecological models and periodic population dynamics are shown and how they describe the world around us are now presented.

Example 1.1.1.

The existence of positive periodic solutions of differential equations has been discovered in red blood cells production model, see (Wazewska-Czyzeska and Lasota, 1976). The model

$$N'(t) = -\delta N(t) + Pe^{-\alpha N(t-\tau)}$$

describes the survival of red blood cells in animal. Here $N(t)$ is the number of the red blood cells at time t, δ is the rate of death of the red blood cells, P and α describe the generation of red blood cells per unite time, τ is the time needed to produce red blood cells.

Example 1.1.2.

The existence of positive periodic solutions of functional differential equation also studied in Hematopoiesis model. (Jiang, Wei and Zhang, 2002) considered the scalar equation

$$u'(t) = -\gamma(t)u(t) + p(t)\frac{u^m(t-\tau(t))}{1+u^n(t-\tau(t))}, \quad (1.1.1)$$

where γ, p, τ are continuous periodic positive function with a common period T, and the constant m, n, T are positive. This is a hematopoiesis model which describes the production of red blood cells. In this model, the periodicity of some parameters are assumed, where the periodic variations of the environment play an important role in many biological and ecological systems. (Mackey and Glass, 1987) also used equation (1.1.1), with a continuous function as initial condition, to describe some physiological control systems.

Example 1.1.3.

The existence of positive periodic solutions of difference equation has arisen in harvest population's growth equation. In (Zeng, 2006) considered the population's growth subjets to harvesting. The author assumed that under the catch per unit effort hypothesis, the harvest population's growth equation can be written as

$$\Delta x(k) = x(k)\left[a(k) - \frac{b(k)x(k)}{1+cx(k)}\right] - qEx(k),$$

where $\Delta x(k) = x(k+1) - x(k)$, $k \in \mathbb{Z}, a(k)$ and $b(k) \in C(\mathbb{Z}, (0, +\infty))$ are $\omega-$periodic, c is a positive constant, q and E are positive constant denoting the catch ability coefficient and the harvesting effort, respectively.

Example 1.1.4.

The existence of positive periodic solutions of scalar functional difference equation has derived in the populations dynamcis model. In (Raffoul and Tisdell, 2005) showed that some population models admit the existence of a positive periodic solution. The scalar equation is

$$N(n+1) = \alpha(n)N(n)\left[1 - \frac{1}{N_0(n)\sum_{s=-\infty}^{0} B(s)N(n+s)}\right], \quad n \in \mathbb{Z}. \qquad (1.1.2)$$

where $N(n)$ of a single species whose members compete among themselves for the limited amount of food that is available to sustain the population, α is the intrinsic per capita growth rate and N_0 is the total carrying capacity.

1.2 THESIS OUTLINE

The material of the thesis is organized as follows. Some preliminaries are introduced in Chapter 1 as a preparation for later discussion. It contains some theoretical results (without proofs) to make the presentation self-contained.

In Chapter 2, we investigate the existence and multiplicity of positive periodic solutions for singular first order difference equation

$$x(k+1) = (1 - a(k))x(k) + \lambda b(k)f(x(k)), \;\; k \in \mathbb{Z} \qquad (1.2.1)$$

by using Kranoselskii fixed point theorem.

In Chapter 3, we extend the problem of scalar first order singular difference equation (1.2.1) to singular first order non-autonomous systems of difference equations

$$\Delta\mathbf{x}(k) = -\mathbf{a}(k)\mathbf{x}(k) + \lambda\mathbf{b}(k)\mathbf{f}(\mathbf{x}(k)). \qquad (1.2.2)$$

We use similar method as in Chapter 2 to obtain the existence and multiplicity of positive periodic solutions to (1.2.2) by also using Kranoselskii fixed point theorem.

In Chapter 4, we consider the singular systems of first order functional differential equations

$$\mathbf{u}'(t) = \mathbf{a}(t)\mathbf{g}(\mathbf{u}(t))\mathbf{u}(t) - \lambda\mathbf{b}(t)\mathbf{f}(\mathbf{u}(t-\tau(t))). \qquad (1.2.3)$$

The existence, multiplicity and nonexistence of positive periodic solutions of (1.2.3) are established by using the fixed point index theory.

In order to solve the problems (1.2.1), (1.2.2) and (1.2.3), first we obtain a variation of parameter and display the desired solutions in terms of suitable green function and then try to find a lower and upper estimates for the kernel inside the summation. Once those estimaties are found, we use the fixed point theorem to show the existence, multiplicity and nonexistence of positive periodic solutions.

1.3 PRELIMINARIES

In this section, we state some preliminaries in the form of definitions and theorem that are of importance for the remainder of the thesis. (see (Kuttler, 2001), (Rudin, 1987)).

Definition 1.3.1. A (real) complex normed space is a (real) complex vector space X together with a map : $X \to \mathbb{R}$, called the norm and denoted $\|\cdot\|$, such that

(i) $\|x\| > 0$, for all $x \in X$, and $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\alpha x\| = |\alpha| \cdot \|x\|$, for all $x \in X$ and all $\alpha \in \mathbb{R}$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$, for all $x, y \in X$.

Definition 1.3.2. A complete normed space is called a Banach space. A normed space X is a Banach space if every Cauchy sequence in X converges.

Definition 1.3.3. (Kelly and Peterson, 2001). Let $y(t)$ be a function of a real or complex variable k. The difference operator Δ is defined by

$$\Delta y(t) = y(t+1) - y(t).$$

where the domain of x is a set of consecutive integers such as the natural numbers $N = \{1, 2, 3, \ldots\}$. The step size of one unit used in the definition is not really a restriction.

Consider a difference operation with a step size $h > 0$ say, $z(s+h) - z(s)$. Let $y(t) = z(th)$. Then

$$\begin{aligned} z(s+h) - z(s) &= z(th+h) - z(th) \\ &= y(t+1) - y(t) \\ &= \Delta y(t). \end{aligned}$$

Definition 1.3.4. An elementary operator that is often used in conjunction with the difference operator is the shift operator. The shift operator E is defined by

$$Ey(t) = y(t+1).$$

If I denotes the identity operator-that is, $Iy(t) = y(t)$ then we have

$$\Delta = E - I.$$

The fundamental properties of Δ are given in the following theorem.

Theorem 1.3.1. (Kelly and Peterson, 2001).

(a) $\Delta^m(\Delta^n y(t)) = \Delta^{m+n} y(t)$ for all positive integers m and n.

(b) $\Delta(y(t) + z(t)) = \Delta y(t) + \Delta z(t)$.

(c) $\Delta(Cy(t)) = C\Delta y(t)$ if C is a constant.

(d) $\Delta(y(t)z(t)) = y(t)\Delta z(t) + Ez(t)\Delta y(t)$.

(e) $\Delta(\frac{y(t)}{z(t)}) = \frac{z(t)\Delta y(t)}{z(t)Ez(t)}$.

Proof. We consider the product rule (d).

$$\begin{aligned} \Delta(y(t)z(t)) &= y(t+1)z(t+1) - y(t)z(t) \\ &= y(t+1)z(t+1) - y(t)z(t+1) + y(t)z(t+1) - y(t)z(t) \\ &= \Delta y(t)Ez(t) + y(t)\Delta z(t). \end{aligned}$$

The other part are also straightforward. The formulas in Theorem 1.3.1 closely resemble the sum rule, the product rule, and the quotient rule from the differential calculus. However, note the appearance of the shift operator in parts (d) and (e). ■

Definition 1.3.5. (Kelly and Peterson, 2001). Indefinite sum (or antidifference) of $y(t)$, denoted $\sum y(t)$, is any function so that

$$\Delta(\sum y(t)) = y(t)$$

for all t in the domain of y. The reader will recall that the indefinite integral plays a similar role in the differential calculus:

$$\frac{d}{dt}\left(\int y(t)dt\right).$$

The indefinite integral is not unique, for example,

$$\int \cos t dt = \sin t + C,$$

where C is any constant. The indefinite sum is also not unique.

1.3.1 Fixed Point Theorems

Let X be the Banach space and K be closed, nonempty subset of X. K is said to be a cone if (i) $\alpha u + \beta v \in K$ for all $u, v \in K$ and all $\alpha, \beta > 0$ and (ii) $u, -u \in K$ imply $u = 0$. A fixed point of a transformation $T : X \to X$ is a point $x \in X$ such that $Tx = x$.

We now state a theorem given by Kranoselskii in 1964, the Kranoselskii fixed point theorem.

Theorem 1.3.2. (Kranoselskii, 1964). Let X be a Banach space, and let $K \subset X$ be a cone in X. Assume Ω_1, Ω_2 are open subsets of X with $0 \in \Omega_1, \bar{\Omega}_1 \subset \Omega_2$, and let

$$T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$$

be a completely continuous operator such that either

(i) $\|Tx\| \leq \|x\|, x \in K \cap \partial\Omega_1$ and $\|Tx\| \geq \|x\|, x \in K \cap \partial\Omega_2$; or

(ii) $\|Tx\| \geq \|x\|, x \in K \cap \partial\Omega_1$ and $\|Tx\| \leq \|x\|, x \in K \cap \partial\Omega_2$;

Then T has a fixed point in $T : K \cap (\bar{\Omega}_2 \setminus \Omega_1) \to K$.

Next, we introduce the fixed point index theorem. First, we recall some concepts and conclusion on the fixed point index in a cone. Assume Ω is bounded open subset in X with the boundary $\partial\Omega$, and let $T : K \cap \bar{\Omega} \to K$ is completely continuous operator such that $Tx \neq x$ for $x \in \partial\Omega \cap K$, then the fixed point index $i(T, K \cap \Omega, K)$ is defined. If $i(T, K \cap \Omega, K) \neq 0$ then T has a fixed point in $K \cap \Omega$.

Theorem 1.3.3. (Deimling, 1985), (Guo and Lakshmikantham, 1988), (Kranoselskii, 1964). Let X be a Banach space and K is a cone in X. For $r > 0$, define $K_r = \{u \in K, \|x\| < r\}$. Assume that $T : \bar{K}_r \to K$ is completely continuous such that $Tx \neq x$ for $x \in \partial K_r = \{u \in K_r : \|x\| = r\}$.

(i) if $\|Tx\| \geq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 0$.

(ii) if $\|Tx\| \leq \|x\|$ for $x \in \partial K_r$, then $i(T, K_r, K) = 1$.

CHAPTER TWO

POSITIVE PERIODIC SOLUTIONS OF SINGULAR FIRST ORDER DIFFERENCE EQUATION

2.1 INTRODUCTION

Let $\mathbb{R}$ denote the real numbers, $\mathbb{Z}$ the integers and $\mathbb{R}_+ = [0, \infty)$, the nonnegative real numbers. Given $a < b$ in $\mathbb{Z}$, let $[a, b] = \{a, a+1, \ldots, b\}$. Let $\mathbb{R}_+^n = \prod_{i=1}^{n} \mathbb{R}_+$.

In this chapter, we investigate the existence and multiplicity of positive periodic solutions for singular first order difference equation

$$x(k+1) = (1 - a(k))x(k) + \lambda b(k) f(x(k)), \;\; k \in \mathbb{Z} \tag{2.1.1}$$

where $\mathbb{Z}$ is the set of integer numbers, $\omega \in \mathbb{N}$ is a fixed integer, $\lambda > 0$ and $b : \mathbb{Z} \to [0, \infty)$, $a(k)$ are $\omega-$periodic and $a(k)$ is continuous with $0 < a(k) < 1$ for all $k \in [0, \omega - 1]$ and $f \in C(\mathbb{R}_+^n \setminus \{0\}, (0, \infty))$.

The study of the existence of positive periodic solutions in difference equations was motivated by the observance of periodic phenomena in mathematical ecological difference models, discrete single-species models and discrete populations models, see for examples, (Gopalsamy and Weng, 1994), (Gurney, Blythe and Nisbet, 1980), (Jiang and Wei, 2002), (Jiang, Wei and Zhang, 2002), (Kelly and Peterson, 2001) and (Weng, 1996). Although most models are described with differential equations, (see (Argawal and O'Regan, 2003)) but the discrete models are more appropriate than the continuous ones when the size of the population is rarely small or the population has non-overlapping generations.

Recently, Kranoselskii fixed point theorem has become an effective tool in proving the existence of periodic solutions. It seems that the Kranoselskii fixed point theorem on compression and expansion of cones is quite effective in dealing with the problem. In fact, by choosing appropriate cones, the singularity of the problem is essentially removed and the associated operator becomes well-defined for certain ranges of functions even there are negative terms.

In (Wang, 2011), the author employed the Kranoselskii fixed point theorem to establish the existence and multiplicity of positive periodic solutions for first non-autonomous singular systems

$$x_i'(t) = -a_i(t)x_i(t) + \lambda b_i(t)f_i(x_1(t), \ldots, x_n(t)),$$

where $i = 1, \ldots, n$. On the other hand, (Zeng, 2006) proved the existence of positive periodic solutions for a class of non-autonomous difference equation

$$\Delta x(k) = -a(k)x(k) + f(k, u(k))$$

where the operator Δ is defined as $\Delta x(k) = x(k+1) - x(k)$. In (Argawal and O'Regan, 2003) and (Chu and Nieto, 2008), the authors showed the existence of periodic solutions for singular first order differential equations.

Inspired by the above work, we consider to carry the work of (Wang, 2011), to the discrete case for scalar difference equations. We shall establish a new result on the existence and multiplicity of positive periodic solutions of equation (2.1.1) by utilizing the well-known theory of Kranoselskii fixed point theorem.

2.2 PRELIMINARY RESULT

In this section we state some preliminaries in the form of lemmas that are essential to proofs our main result.

Let X be the set of all real ω-periodic sequences $x : \mathbb{Z}_+ \to \mathbb{R}_+^n$, endowed with the maximum norm

$$\|x\| = \max_{k\in[0,\omega-1]} |x(k)|.$$

Thus X is a Banach space. Throughout this chapter, we denote the product of $x(k)$ from $k = a$ to $k = b$ with the understanding that $\prod_{k=a}^{b} x(k) := 1$ for all $a > b$. We make the following assumptions:

(H1) $0 < a(k) < 1$ for all $k \in [0, \omega - 1]$.

(H2) $f : \mathbb{R}_+^n \setminus \{0\} \to (0, \infty)$ is continuous.

Lemma 2.2.1. Assume (H1), (H2) hold. If $x \in X$ then x is a solution of (2.1.1) if and only if

$$x(k) = \sum_{s=k}^{k+\omega-1} G(k,s)\lambda b(s) f(x(s)),$$

where

$$G(k,s) = \frac{\prod_{r=s+1}^{k+\omega-1}(1-a(r))}{1-\prod_{r=0}^{\omega-1}(1-a(r))}, \quad s \in [k, k+\omega-1]. \tag{2.2.1}$$

Note that the denominator in $G(k,s)$ is not zero since $0 < a(k) < 1$ for $k \in [0, \omega - 1]$.

Proof. Multiplying both sides of (2.1.1) by $\prod_{r=0}^{k}(1-a(r))^{-1}$,

$$x(k+1)\prod_{r=0}^{(k+1)-1}(1-a(r))^{-1} - x(k)\prod_{r=0}^{k-1}(1-a(r))^{-1} = \lambda\prod_{r=0}^{k}(1-a(r))^{-1} b(k) f(x(k)).$$

By the discrete product rule (see Kelly and Peterson, 2001), Theorem 1.3.1 (d), we have

$$\Delta\left(x(k)\prod_{r=0}^{k-1}(1-a(r))^{-1}\right) = \lambda\prod_{r=0}^{k}(1-a(r))^{-1} b(k) f(x(k)).$$

Summing the above equation from $s = k$ to $s = k + \omega - 1$, we obtain

$$\sum_{k=k}^{k+\omega-1}\Delta\left(x(k)\prod_{r=0}^{k-1}(1-a(r))^{-1}\right) = \lambda\sum_{k=k}^{k+\omega-1}\prod_{r=0}^{k}(1-a(r))^{-1} b(k) f(x(k)).$$

By the Definition 1.3.5, thus we have

$$x(k+\omega)\prod_{r=0}^{k+\omega-1}(1-a(r))^{-1}-x(k)\prod_{r=0}^{k-1}(1-a(r))^{-1}=\lambda\sum_{k=k}^{k+\omega-1}\prod_{r=0}^{k}(1-a(r))^{-1}b(k)f(x(k)).$$

Since $x(k+\omega)=x(k)$, we obtain

$$x(k)\left[\prod_{r=0}^{k+\omega-1}(1-a(r))^{-1}-\prod_{r=0}^{k-1}(1-a(r))^{-1}\right]=\lambda\sum_{k=k}^{k+\omega-1}\prod_{r=0}^{k}(1-a(r))^{-1}b(k)f(x(k)).$$

Multiplying both sides of the above equation by $\prod_{r=0}^{k+\omega-1}(1-a(r))$, we complete the proof. ■

It is clear that $G(k,s)=G(k+\omega,s+\omega)$ for all $k,s\in\mathbb{Z}$. A direct calculation shows that

$$m:=\frac{\prod_{r=0}^{\omega-1}(1-a(r))}{1-\prod_{r=0}^{\omega-1}(1-a(r))}\leq G(k,s)\leq\frac{1}{1-\prod_{r=0}^{\omega-1}(1-a(r))}:=M. \qquad (2.2.2)$$

Define $\sigma=\prod_{r=0}^{\omega-1}(1-a(r))<1$ satisfying

$$\frac{\sigma}{1-\sigma}\leq G(k,s)\leq\frac{1}{1-\sigma}, \qquad k\leq s\leq k+\omega.$$

Thus, from 2.2.2, we have $\sigma=\dfrac{m}{M}>0$,

$$\|x\|=\max_{k\in[0,\omega-1]}|x(k)|\leq M\sum_{k=0}^{\omega-1}\lambda b(k)f(x(k)).$$

Therefore

$$\begin{aligned}x(k)&\geq m\lambda\sum_{k=0}^{\omega-1}b(k)f(x(k))\\&\geq\frac{m}{M}\lambda\sum_{k=0}^{\omega-1}b(k)f(x(k))\\&\geq\sigma\|x\|.\end{aligned}$$

Now we define a cone

$$K = \left\{ x \in X | k \in [0, \omega - 1], x(k) \geq \frac{m}{M} \|x\| = \sigma \|x\| \right\}.$$

It is clear that K is a cone in X and $\min_{k \in [0, \omega-1]} |x(k)| \geq \sigma \|x\|$ for $x \in K$. For $r > 0$, define $\Omega_r = \{x \in K : \|x\| < r\}$. Note that $\partial\Omega_r = \{x \in K : \|x\| = r\}$. Define a mapping $T : X \to X$ by

$$Tx(k) = \lambda \sum_{s=k}^{k+\omega-1} G(k, s) b(s) f(x(s)), \tag{2.2.3}$$

where $G(k, s)$ is given by (2.2.1). By the nonnegativity of λ, f, a, b, and G, $Tx(k) \geq 0$ on $[0, \omega - 1]$. It is clear that $Tx(k + \omega) = Tx(k)$.

Lemma 2.2.2. $T : K \backslash \{0\} \to K$ is well-defined.

Proof. For any $x \in K \backslash \{0\}$, for all $k \in [0, \omega - 1]$ we have

$$\|Tx\| = \max_{k \in [0, \omega-1]} |Tx(k)| \leq M \sum_{s=0}^{\omega-1} \lambda b(s) f(x(s)).$$

Therefore

$$\begin{aligned} Tx(k) &= \lambda \sum_{s=k}^{k+\omega-1} G(k, s) b(s) f(x(s)) \\ &\geq \lambda m \sum_{s=0}^{\omega-1} b(s) f(x(s)) \\ &\geq \frac{m}{M} \|Tx\|. \end{aligned}$$

Hence $Tx(k) \geq \sigma \|Tx\|$. This implies that $T : K \backslash \{0\} \to K$. ■

Lemma 2.2.3. If (H1) and (H2) hold, then the operator $T : K \backslash \{0\} \to K$ is completely continuous.

Proof. Let $x_m(k), x_0(k) \in K \backslash \{0\}$ with $x_m(k) \to x_0(k)$ as $m \to \infty$. From (2.2.3) and since $f(k, x(k))$ is continuous in $x(k)$, as $m \to \infty$, we have

$$|Tx_m(k) - Tx_0(k)| \leq M \sum_{s=0}^{\omega-1} \lambda |b(s)| |f(x_m(s)) - f(x_0(s))| \to 0.$$

Hence $\|Tx_m(k) - Tx_0(k)\| \to 0$, it follows that the operator T is continuous. Further if $K\backslash\{0\} \in X$ is a bounded set, then $\|x\| \leq C_1 = const$ for all $x \in K\backslash\{0\}$. Set $C_2 = \max f(x(k)), x \in K\backslash\{0\}$ then from (2.2.3) we get, for all $x \in K\backslash\{0\}$,

$$\|Tx\| \leq M \sum_{s=k}^{k+\omega-1} \lambda\,|b(s)|\,|f(x(k))| \leq M\omega C_2.$$

This shows that $T(K\backslash\{0\})$ is a bounded set in K. Since K is n-dimensional, $T(K\backslash\{0\})$ is relatively compact in K. Therefore T is a completely continuous operator. ■

For the next following lemmas, we now introduce some notations. For $r > 0$, let

$$\Gamma = \sigma m \sum_{s=0}^{\omega-1} b(s), \quad \chi = M \sum_{s=0}^{\omega-1} b(s),$$
$$C(r) = \max\left\{f(x) : x \in \mathbb{R}_+^n, \|x\| \leq r\right\} > 0.$$

Lemma 2.2.4. Assume that (H1), (H2) hold. If $\eta > 0, x \in K\backslash\{0\}$, and $f(x(k)) \geq \|x(k)\|\,\eta$ for $k \in [0, \omega - 1]$, then $\|Tx\| \geq \lambda\Gamma\eta\,\|x\|$.

Proof. Since $x \in K\backslash\{0\}$ and $f(x(k)) \geq \|x(k)\|\,\eta$ for $k \in [0, \omega - 1]$, we have

$$\begin{aligned} Tx(k) &= \lambda \sum_{s=k}^{k+\omega-1} G(k,s)b(s)f(x(s)) \\ &\geq \lambda m \sum_{s=0}^{\omega-1} b(s)f(x(s)) \\ &\geq \lambda m \sum_{s=0}^{\omega-1} b(s)\,\|x(k)\|\,\eta \\ &\geq \lambda m \sum_{s=0}^{\omega-1} b(s)\sigma\,\|x\|\,\eta. \end{aligned}$$

Thus $\|Tx\| \geq \lambda\Gamma\eta\,\|x\|$. This completes the proof. ■

Let $\hat{f} : [1, \infty) \to \mathbb{R}_+^n$ be the function given by

$$\hat{f}(\theta) = \max\left\{f(x) : x \in \mathbb{R}_+^n, \text{and } 1 \leq \|x\| \leq \theta\right\}.$$

It is easy to see that $\hat{f}(\theta)$ is nondecreasing function on $[1, \infty)$. The following lemma is essentially the same as Lemma 3.6 in (Wang, 2011) and Lemma 2.8 in (Wang, 2003).

Lemma 2.2.5. (Wang, 2011), (Wang, 2003) Assume (H2) holds. If $\lim_{x\to\infty} \frac{f(x)}{x}$ exists (which can be infinty) then $\lim_{\theta\to\infty} \frac{\hat{f}(\theta)}{\theta}$ exists and $\lim_{\theta\to\infty} \frac{\hat{f}(\theta)}{\theta} = \lim_{x\to\infty} \frac{f(x)}{x}$.

Lemma 2.2.6. Assume that (H1), (H2) hold. Let $r > \frac{1}{\sigma}$. If there exists an $\varepsilon > 0$ such that $\hat{f}(r) \leq \varepsilon r$, then $\|Tx\| \leq \lambda\chi\varepsilon \|x\|$ for $x \in \partial\Omega_r$.

Proof. From the definition of T for $x \in \partial\Omega_r$, we have

$$\begin{aligned}
\|Tx\| &\leq \lambda M \sum_{s=0}^{\omega-1} b(s) f(x(s)) \\
&\leq \lambda M \sum_{s=0}^{\omega-1} b(s) \hat{f}(r) \\
&\leq \lambda M \sum_{s=0}^{\omega-1} b(s) \varepsilon r \\
&= \lambda M \sum_{s=0}^{\omega-1} b(s) \varepsilon \|x\|.
\end{aligned}$$

This implies that $\|Tx\| \leq \lambda\chi\varepsilon \|x\|$. ■

In views of definition $C(r)$, it follows that

$$0 < f(x(k)) \leq C(r) \quad \text{for } k \in [0, \omega - 1],$$

if $x \in \partial\Omega_r, r > 0$. Thus it is easy to see the following lemma can be shown in similar manner as in Lemma 2.2.6.

Lemma 2.2.7. Assume (H1), (H2) hold. If $x \in \partial\Omega_r, r > 0$ then $\|Tx\| \leq \lambda\chi C(r)$, where χ is defined in Lemma 2.2.6.

Proof. From the definitions of T for $x \in \partial\Omega_r$ we have

$$\begin{aligned}
\|Tx\| &\leq \lambda M \sum_{s=0}^{\omega-1} b(s) f(x(s)) \\
&\leq \lambda M \sum_{s=0}^{\omega-1} b(s) C(r) \\
&\leq \lambda\chi C(r).
\end{aligned}$$

Thus it implies that $\|Tx\| \leq \lambda\chi C(r)$. ■

2.3 MAIN RESULT

In this section, we establish conditions for the existence and multiplicity of positive periodic solutions of (2.1.1).

Theorem 2.3.1. Let (H1), (H2) hold, we assume that $\lim_{x\to 0} f(x) = \infty$.

(a) If $\lim_{x\to\infty} \frac{f(x)}{x} = 0$, then for all $\lambda > 0$ (2.1.1) has a positive periodic solution.

(b) If $\lim_{x\to\infty} \frac{f(x)}{x} = \infty$, then for all small $\lambda > 0$ (2.1.1) has two positive periodic solutions.

(c) There exists a $\lambda_0 > 0$ such that (2.1.1) has a positive periodic solution for $0 < \lambda < \lambda_0$.

Proof. (a). From the assumptions, $\lim_{x\to 0} f(x) = \infty$ there is an $r_1 > 0$ such that

$$f(x) \geq \eta \|x\|$$

for $x \in K\backslash\{0\}$ and $0 < x < r_1$, where $\eta > 0$ is chosen so that

$$\lambda\Gamma\eta > 1,$$

where Γ is defined in Lemma 2.2.4. Let $\Omega_{r_1} = \{x \in K : \|x\| < r_1\}$. If $x \in \partial\Omega_{r_1}$, then

$$f(x(k)) \geq \|x(k)\| \, \eta.$$

Lemma 2.2.4 implies that

$$\|Tx\| \geq \lambda\Gamma\eta \|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_1}. \tag{2.3.1}$$

We now determine Ω_{r_2}. Let $\Omega_{r_1} = \{x \in K : \|x\| < r_2\}$. Note that $\lim_{x\to\infty} \frac{f(x)}{x} = 0$, it follows from Lemma 2.2.5, $\lim_{\theta\to\infty}\frac{\hat{f}(\theta)}{\theta}=0$. Therefore there is an $r_2>\max\left\{2r_1,\frac{1}{\sigma}\right\}$ such that

$$\hat{f}(r_2)\le \varepsilon r_2,$$

where the constant $\varepsilon>0$ satisfies

$$\lambda\varepsilon\chi<1,$$

where χ is defined in Lemma 2.2.6. Thus, we have by Lemma 2.2.6 that

$$\|Tx\|\le \lambda\varepsilon\chi\,\|x\|<\|x\| \quad \text{for } x\in\partial\Omega_{r_2}. \tag{2.3.2}$$

By Lemma 1.3.2 applied to (2.3.1) and (2.3.2), it follows that T has a fixed point in $\bar{\Omega}_{r_2}\backslash\Omega_{r_1}$, which is the desired positive periodic solution of (2.1.1). ■

Proof. (b). Fix two numbers $0<r_3<r_4$, there exists a λ_0 such that

$$\lambda_0<\frac{r_3}{\chi C(r_3)},\quad \lambda_0<\frac{r_4}{\chi C(r_4)},$$

where $\chi C(r)$ defined in Lemma 2.2.7. Thus, in Lemma 2.2.7 implies that, for $0<\lambda<\lambda_0$,

$$\begin{aligned}\|Tx\|&\le \lambda\chi C(r_j)\\ &\le \frac{r_j}{\chi C(r_j)}\chi C(r_j)=r_j=\|x\|\,.\end{aligned}$$

Thus

$$\|Tx\|\le\|x\| \ \text{ for } x\in\partial\Omega_{r_j},\quad (j=3,4). \tag{2.3.3}$$

On the other hand, in view of the assumptions $\lim_{x\to\infty}\frac{f(x)}{x}=\infty$ and $\lim_{x\to 0}f(x)=\infty$, there are positive numbers $0<r_2<r_3<r_4<\hat{H}$ such that

$$f(x)\ge\eta\,\|x\|$$

for $x \in K\backslash\{0\}$ and $0 < x \le r_2$ or $x \ge \hat{H}$ where $\eta > 0$ is chosen so that

$$\lambda\Gamma\eta > 1.$$

Thus if $x \in \partial\Omega_{r_2}$, then

$$f(x) \ge \eta \|x\|.$$

Let $r_1 = \max\left\{2r_4, \frac{\hat{H}}{\sigma}\right\}$ if $x \in \partial\Omega_{r_1}$, then

$$\min_{k\in[0,\omega-1]} x(k) \ge \sigma\|x\| = \sigma r_1 \ge \hat{H},$$

which implies that

$$f(x) \ge \eta \|x\|.$$

Thus Lemma 2.2.4 implies that

$$\|Tx\| \ge \lambda\Gamma\eta\|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_1}, \tag{2.3.4}$$

and

$$\|Tx\| \ge \lambda\Gamma\eta\|x\| > \|x\| \quad \text{for } x \in \partial\Omega_{r_2}. \tag{2.3.5}$$

It follows from Lemma 1.3.2 applied to (2.3.3), (2.3.4) and (2.3.5), T has two fixed points x_1 and x_2 such that $x_1 \in \bar{\Omega}_{r_3}\backslash\Omega_{r_2}$ and $x_2 \in \bar{\Omega}_{r_1}\backslash\Omega_{r_4}$, which are the desired distinct positive periodic solutions of (2.1.1) for $\lambda < \lambda_0$ satisfying

$$r_2 < \|x_1\| < r_3 < r_4 < \|x_2\| < r_1.$$

■

Proof. (c). Choose a number $r_3 > 0$. By Lemma 2.2.7 we infer that there exists a $\lambda_0 = \frac{r_3}{\chi C(r_3)} > 0$ such that

$$\|Tx\| < \|x\| \text{ for } x \in \partial\Omega_{r_3} \qquad 0 < \lambda < \lambda_0. \tag{2.3.6}$$

On the other hand, in view of assumption $\lim_{x\to 0} f(x) = \infty$, there exists a positive number $0 < r_2 < r_3$ such that

$$f(x) \geq \eta \|x\|$$

for $x \in K \backslash \{0\}$ and $0 < x < r_2$ where $\eta > 0$ is chosen so that

$$\lambda \Gamma \eta > 1,$$

where Γ is defined in Lemma 2.2.4. Thus if $x \in \partial\Omega_{r_2}$, then

$$f(x) \geq \eta \|x\|.$$

Lemma 2.2.4 implies that

$$\|Tx\| \geq \lambda\Gamma\eta \|x\| > \|x\|, \qquad \text{for } x \in \partial\Omega_{r_2}. \tag{2.3.7}$$

It follows from Lemma 1.3.2 applied to (2.3.6) and (2.3.7), that T has a fixed point $x \in \bar{\Omega}_{r_3} \backslash \Omega_{r_2}$. The fixed point $x \in \bar{\Omega}_{r_3} \backslash \Omega_{r_2}$ is the desired positive periodic solution of (2.1.1). ■

Remark 2.3.1.

If the right function in equation (2.1.1) is of the form $f(x(k - \tau(k)))$, we can apply the same method to obtain similar result as Theorem 2.3.1. The result extended the work of (Raffoul, 2005) where he considered the existence of positive periodic solutions of scalar nonlinear functional difference equation

$$x(n+1) = a(n)x(n) + h(n)f(x(n - \tau(n))),$$

where $a(n)$, $h(n)$ and $\tau(n)$ are $T-$periodic for T is an integer with $T \geq 1$ under the assumptions that $a(n)$, $f(x)$ and $h(n)$ are nonnegative with $0 < a(n) < 1$ for all $n \in [0, T-1]$.