UNIVERSITI TEKNOLOGI MARA

POSITIVE SOLUTIONS TO SECOND ORDER BOUNDARY VALUE PROBLEMS

WAN ASHRAF BIN WAN AZMI

Thesis submitted in fulfillment
of the requirements for the degree of
Master of Science

Faculty of Computer & Mathematical Sciences

April 2014

AUTHOR'S DECLARATION

I declare that the work in this thesis was carried out in accordance with the regulations of Universiti Teknologi MARA. It is original and is the result of my own work, unless otherwise indicated or acknowledged as referenced work. This thesis has not been submitted to any other academic institution or non-academic institution for any degree or qualification.

I, hereby, acknowledge that I have been supplied with the Academic Rules and Regulations for Post Graduate, Universiti Teknologi MARA, regulating the conduct of my study and research.

Name of Student	:	Wan Ashraf Bin Wan Azmi
Student I.D. No	:	2010471844
Programme	:	Master of Science
Faculty	:	Computer and Mathematical Sciences
Thesis Title	:	Positive Solutions to Second Order Boundary Value Problems
Signature of Student	:	....
Date	:	April 2014

ABSTRACT

This thesis is concerned with the existence of positive solutions for second order boundary value problems. In particular, firstly we investigate the existence and multiplicity of positive solutions for a singular second order scalar Sturm-Liouville boundary value problem with different values of λ for a function f involve u. Then, we investigate the existence of positive solutions of a Dirichlet boundary value problem where the function f involve u and u'. Lastly, we consider the results of positive solutions for singular Dirichlet second order boundary value problem where the function f involve u and u' in terms of different values of λ. The existence results of positive solutions are proved by applying the Krasnosel'skii fixed point theorem on compression and expansion of cones.

ACKNOWLEDGEMENTS

First and foremost, all praise be to Allah, Lord of the world, the almighty because without his help, it is impossible for me to finish this thesis. I would like to thank my supervisor, Dr. Mesliza Mohamed for her guidance, support and constant help during my graduate study at Universiti Teknologi MARA Perlis. My gratitude goes to the Universiti Teknologi MARA for its financial support. Lastly, thanks to my family because of their constant support during the good and bad days, giving me the spirit to complete my master study.

TABLE OF CONTENTS

CHAPTER ONE

INTRODUCTION

In this chapter, we will look at some definitions and concepts that are of importance for the remainder of the chapters (see Rockafellar (1970), Rudin (1987), Kuttler (2001), Kelley and Peterson (2004), Sista (2004) and Lay (2005)).

1.1 PRELIMINARIES

Definition 1.1.1. A normed space X is a vector space with a norm defined on it. A norm on a vector space X is a real valued function on X whose value at an $x \in X$ is denoted by $\|x\|$ and which has the properties

1. $\|x\| > 0$, $\|x\| = 0$ if and only if $x = 0$,
2. $\|x + y\| \leq \|x\| + \|y\|$,
3. $\|cx\| = |c| \cdot \|x\|$,

where x and y are arbitrary vectors in X and c is any scalar. The normed space is denoted by $(X, \|\cdot\|)$ or simply by X.

Definition 1.1.2. Let X be any nonempty set. A function $d : X \times X \to R$ is called a metric on X if it satisfies the following conditions for all $x, y, z \in X$.

(1) $d(x, y) \geq 0$.

(2) $d(x, y) = 0$ if and only if $x = y$.

(3) $d(x, y) = d(y, x)$.

(4) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

A set X together with a metric d is said to be a metric space. Since a set may have more than one metric define on it, we often identify both and denote the metric space by (X, d). If the particular metric is not important or if it is otherwise identified, we may simply write X.

Definition 1.1.3. A sequences $x_n \in X$ is said to be a Cauchy sequence if for each $\varepsilon > 0$ there exists a number N such that $m, n > N$ implies that $\|x_n - x_m\| < \varepsilon$.

Definition 1.1.4. A metric space is called complete if every Cauchy sequence converges to some element of the metric space.

Definition 1.1.5. A normed linear space, $(X, \|\cdot\|)$ is a Banach space if it is complete. Thus, whenever $\{x_n\}$ is a Cauchy sequence, there exists $x \in X$ such that $\lim_{n\to\infty} \|x - x_n\| = 0$.

Example 1.1.1.

Let $J = [0, 1]$. Then the following spaces are Banach space. $C(J)$ is the space of continuous real-valued functions u on J with the norm $\|u\|_0 = \max\{|u(t)|; t \in J\}$.

Definition 1.1.6. Suppose Y is a closed subset of a complete metric space, X. Then Y is also a complete metric space.

Definition 1.1.7. Let X and Y be two normed linear spaces and let $f : X \to Y$ be linear $(f(ax + by) = af(x) + bf(y)$ for scalars a, b and $x, y \in X)$. The following are equivalent

(a) f is continuous at 0.

(b) f is continuous.

(c) There exists $K > 0$ such that $\|fx\|_Y \leq K \|x\|_X$ for all $x \in X$ (f is bounded).

Definition 1.1.8. Let $f : X \to Y$ be linear and continuous where X and Y are normed linear spaces. We denote the set of all such continuous linear maps by $f(X, Y)$ and define

$$\|f\| = \sup \{\|fx\| : \|x\| < 1\}.$$

If $\|f\| < \infty$, then f is called a bounded linear transformation.

Definition 1.1.9. If Y is a Banach space, then $f(X, Y)$ is also a Banach space.

1.1.1 Boundary Value Problem

Definition 1.1.10. A boundary value problem for a given differential equation consists of finding a solution of the given differential equation subject to a given set of boundary conditions. A boundary condition is a prescription some combinations of values of the unknown solution and its derivatives at more than one point. There are four kinds of linear boundary conditions:

$$\text{Dirichlet or First kind} : y(a) = \eta_1, y(b) = \eta_2,$$
$$\text{Neumann or Second kind} : y'(a) = \eta_1, y'(b) = \eta_2,$$
$$\text{Robin or Third or Mixed kind} : \alpha_1 y(a) + \alpha_2 y'(a) = \eta_1, \beta_1 y(b) + \beta_2 y'(b) = \eta_2,$$
$$\text{Periodic} : y(a) = y(b), y'(a) = y'(b),$$

where $a, b, \eta_1, \eta_2, \alpha_1, \alpha_2, \beta_1$ and β_2 are all constant.

1.1.2 Convex Set

Definition 1.1.11. A set S in R^n is said to be convex if for each $x_1, x_2 \in S$, the line segment $\lambda x_1 + (1-\lambda)x_2$ for $\lambda \in (0,1)$ belongs to S. This says that all points on a line connecting two points in the set are in the set.

1.1.3 Arzela-Ascoli Theorem

Definition 1.1.12. We say that the sequence of vector functions $\{x_m(t)\}_{m=1}^{\infty}, x_m : I \to R^n$, is uniformly bounded on an interval I provided there is a constant M such that

$$\|x_m(t)\| \leq M,$$

for $m = 1, 2, 3....$, and for all $t \in I$, where $\|\cdot\|$ is any norm on R^n.

Definition 1.1.13. We say that family of vector functions $\{x_\alpha(t)\}$, for α in some index set A, is equicontinuous on an interval I provided given any $\epsilon > 0$ there is a $\delta > 0$ such that

$$\|x_\alpha(t) - x_\alpha(\tau)\| < \epsilon,$$

for all $\alpha \in A$ and for all $t, \tau \in I$ with $|t - \tau| < \delta$.

Theorem 1.1.1. *For $K \subset C[0,1]$, K is compact if and only if K is closed, bounded and equicontinuous.*

Theorem 1.1.2. *Let $K \subset R^n$ be a compact. A subset $F \subset C(K)$ is relatively compact if and only if it is pointwise bounded and equicontinuous, where $C(K)$ denotes the space of all continuous functions on K.*

Theorem 1.1.3. *If a sequence $\{f_m\}_1^{\infty}$ in $C(K)$ is bounded and equicontinuous then it has a uniformly convergent subsequence.*

1.1.4 Fixed Point Theorem

Theorem 1.1.4. (see Krasnosel'skii (1964) and Guo and Lakshmikantham (1988)) (Guo-Krasnosel'skii fixed point theorem).
Let X be a Banach space and let $K \subset X$ be a cone in X. Assume Ω_1, Ω_2 are bounded open subsets of X with $0 \in \Omega_1 \subset \overline{\Omega}_1 \subset \Omega_2$ and let

$$F : K \cap (\overline{\Omega}_2 \backslash \Omega_1) \to K$$

be a completely continuous operator such that either *(i)* $\|Fu\| \leq \|u\|$ *for any* $u \in K \cap \partial\Omega_1$ *and* $\|Fu\| \geq \|u\|$ *for any* $u \in K \cap \partial\Omega_2$

or

(ii) $\|Fu\| \geq \|u\|$ *for any* $u \in K \cap \partial\Omega_1$ *and* $\|Fu\| \leq \|u\|$ *for any* $u \in K \cap \partial\Omega_2$.

Then F *has a fixed point in* $K \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

1.2 MOTIVATING EXAMPLES

In this section we present some examples that motivate our study of second-order boundary value problems.

Example 1.2.1. (Boundary Value Problems Arising in Viscous Flow Behind a Shock Wave, Zheng et al. (2006))

We restrict ourselves to the considerations of perfect gas. It will be assumed that μ (coefficient of viscosity), κ (thermal conductivity) are proportional to T (temperature) and that C_p (specific heat at constant pressure) and P_r (Prandtl number, $\mu C_p/\kappa$) are independent of T. Consider a plane laminar flow with spatial coordinates (x, y), corresponding velocity components (u, v) and $(dp/dx) = 0$. For steady flow, the boundary layer equations for $x > 0$ can be written as Mires (1956), Thompson (1972) and Mires (1995).

$$\frac{\partial \rho u}{\partial x} + \frac{\partial \rho v}{\partial y} = 0, \tag{1.2.1}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{1}{\rho}\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right), \tag{1.2.2}$$

$$\rho C_p\left(u\frac{\partial T}{\partial x} + v\frac{\partial T}{\partial y}\right) = \frac{\partial}{\partial y}\left(\kappa\frac{\partial u}{\partial y}\right) + \mu\left(\frac{\partial u}{\partial y}\right)^2, \tag{1.2.3}$$

$$P = \rho R T. \tag{1.2.4}$$

The boundary conditions are

$$u(x, 0) = u_w, u(x, \infty) = u_e, \tag{1.2.5}$$

$$v(x, 0) = 0, \tag{1.2.6}$$

$$T(x, 0) = T_w, T(w, \infty) = T_e. \tag{1.2.7}$$

Introduce a stream function ψ and a similarity variable η by the expressions

$$\psi = \sqrt{2u_e x v_w} f(\eta), \eta = \sqrt{\frac{u_e}{2xv_w}} \int_0^Y \frac{T_w}{T(x, y)} dy. \tag{1.2.8}$$

From (1.2.8), we get the velocity components $u = \dfrac{\partial \psi}{\partial y}$ and $v = -\dfrac{\partial \psi}{\partial x}$. Then substitute into (1.2.1)-(1.2.7) to get the momentum and energy equations. By choosing a suitable choice of moving coordinate system and by assuming that the fluid is an ideal gas having the viscosity and thermal conductivity which both proportional to the temperature, the momentums and energy equations are given by
Momentum equations:

$$f'''(\eta) + f(\eta)f''(\eta) = 0, \quad 0 < \eta < +\infty, \tag{1.2.9}$$

$$f(0) = 0, f'(0) = \xi, f'(+\infty) = 1. \tag{1.2.10}$$

Energy equations:

$$\overline{T}''(\eta) + Pr \cdot f(\eta)\overline{T}'(\eta) = -\frac{Pr \cdot u_e^2}{C_{pw}T_e}(f''(\eta))^2, \tag{1.2.11}$$

$$\overline{T}(0) = \lambda, \overline{T}(+\infty) = 1. \tag{1.2.12}$$

Here the prime denotes differentiation with respect to η, $\xi = f'(0) = \dfrac{u_w}{u_e}$, is the velocity ratio parameter, $\lambda = T_w/T_e$ is the temperature ration parameter, $Pr = \mu C_p/k$ is the Prandtl number, and $1 < \xi < 6$ for a shock wave.
By using the following transformation:

$$g(z) = f''(\eta)(\text{dimensionless shear stress}), \tag{1.2.13}$$

$$z = f'(\eta)(\text{dimensionless tangential velocity}), \tag{1.2.14}$$

$$\theta(z) = \overline{T}(\eta)(\text{dimensionless temperature}), \tag{1.2.15}$$

and substituting (1.2.13)-(1.2.15) into (1.2.9)-(1.2.12), in terms of $f''(\eta) < 0, 0 < \eta < +\infty$, $f''(+\infty) = 0$, and $(\gamma - 1)M_e^2 = u_e^2/(C_{p,w} \cdot T_e)$, we arrive at the following singular nonlinear two-point boundary value problems where the momentum equations are given by

$$g(z)g''(z) + z = 0, \quad 1 < z < \xi < 6, \tag{1.2.16}$$

$$g(1) = 0, g'(\xi) = 0. \tag{1.2.17}$$

and energy equations are given by

$$\theta''(z) + (1 - Pr)\theta'(z)g'(z)/g = -Pr(\gamma - 1)M_e^2, \quad 1 < z < \xi < 6, \tag{1.2.18}$$

$$\theta(1) = 1, \theta(\xi) = \lambda. \tag{1.2.19}$$

Clearly, equations (1.2.16)-(1.2.17) are de-coupled and may be considered firstly, the

solutions then may be used to solve equations (1.2.18)-(1.2.19). It may be seen from the derivation process that only the negative solution of equations (1.2.16)-(1.2.17) are physically significant.
Let $t = \xi - z, w(t) = -g(\xi - t)$, then equations (1.2.16)-(1.2.17) are changed into the nonlinear singular two-point boundary value problems

$$w''(t) = \frac{t-\xi}{w(t)}, \quad 0 < t < \xi - 1, \tag{1.2.20}$$

$$w'(0) = 0, w(\xi - 1) = 0.$$

In terms of negative solutions of equations (1.2.16)-(1.2.17), it is seen that only the positive solutions of equation (1.2.20) is physically significant.
The problem is singular at $t = \xi - 1$. Then, it is convenient to consider the boundary conditions without singularities which is

$$w''(t) = \frac{t-\xi}{w(t)}, \quad 0 < t < \xi - 1,$$

$$w'(0) = 0, w(\xi - 1) = 0 = h > 0.$$

Then, sufficient conditions for existence and uniqueness of positive solutions were established.

Example 1.2.2. (Heat Flow, Webb (2005))

The existence of positive solutions of a nonlocal boundary value problem arises in modelling a thermostat. Consider for example a model for stationary solutions of a heated bar

$$-u'' = g(t)f(t,u), t \in (0,1), \quad u'(0) = 0, \beta u'(1) + u(\eta) = 0.$$

Adding or removing heat dependent on the temperature detected by a sensor at η when a controller at 1, the boundary condition at 0 corresponds to that end being insulated. The study of this problem is via a Hammerstein integral equation of the form

$$u(t) = \int_0^1 k(t,s)g(s)f(s,u(s))ds.$$

The solution is given explicitly by

$$u(t) = \beta \int_0^1 g(s)f(s,u(s))ds + \int_0^\eta (\eta - s)g(s)f(s,u(s))ds - \int_0^t (t-s)g(s)f(s,u(s))ds.$$

Example 1.2.3. (Membrane Response of a Spherical Cap, Agarwal et al. (2003))

The existence criteria for singular boundary value problems are found in modelling the membrane response of a spherical cap. The following boundary value problem models the large deflection membrane response of a spherical cap

$$y'' + \left(\frac{t^2}{32y^2} - \frac{\lambda^2}{8}\right) = 0 \qquad 0 < t < 1, \tag{1.2.21}$$

$$y(0) = 0, 2y'(1) - (1+v)y(1) = 0, \qquad 0 < v < 1, \lambda > 0, \tag{1.2.22}$$

which exists in nonlinear mechanics. The radial stress at points on membrane is denoted by $S_r = y/t$, $(d/d\rho)(\rho S_r)$ is the circumferential stress ($\rho = t^2$), λ is a load geometry parameter and v is the Poisson ratio. The nonlinearity in above problem may change sign.

Motivated by the problem (1.2.21) - (1.2.22), Agarwal et al. (2003) presented existence result for

$$y'' + q(t)f(t, y) = 0, \qquad 0 < t < 1,$$

$$y(0) = y'(1) + \psi(y(1)), \tag{1.2.23}$$

where the nonlinearity f is allowed to change sign, i. e. $f : (0,1) \times (0, \infty) \to R$.

1.3 PROBLEM STATEMENT

This research is mainly concerned with finding the existence and multiplicity of positive solutions to the second order boundary value problem by using some properties of the Green's function and the Krasnosel'skii fixed point theorem on compression and expansion of cones.

1.4 OBJECTIVES OF THE STUDY

Our objectives of this research are :

1) To study the existence and multiplicity of positive solutions to a singular second order Sturm-Liouville boundary value problem by applying Krasnoselskii fixed point theorem.

2) To find sufficient conditions for the existence of positive solutions to the second order Dirichlet boundary value problem by applying Krasnoselskii fixed point theorem.

3) To study the existence and multiplicity of positive solutions to a singular second order Dirichlet boundary value problem by applying Krasnoselskii fixed point theorem.

1.5 THESIS OUTLINE

This thesis is structured as follows. In Chapter 2, we will establish the existence and multiplicity results for singular second order boundary value problem

$$u''(t) + \lambda g(t)f(u(t)) = 0, \qquad t \in [0,1],$$

$$\alpha u(0) - \beta u'(0) = 0,$$
$$\gamma u(1) + \delta u'(1) = 0,$$

where $\alpha > 0$, $\beta > 0$, $\gamma > 0$ and $\delta > 0$ are all constants, λ is a positive parameter and $f(\cdot)$ is singular at $u = 0$. Under suitable conditions, the theorem about the existence and multiplicity of positive solutions are obtained by using fixed-point theorem on cone.

In Chapter 3, we deal with the existence results of positive solutions for second order Dirichlet boundary value problem

$$u''(t) + \lambda a(t)f(u(t), u'(t)) = 0, \qquad t \in [0,1],$$

$$u(0) = u(1) = 0,$$

where $f \in C\left(R^+ \times R, R^+\right)$ and $a \in C\left((0,1), R^+\right)$. We assume that the function f depends on u and u', so our work is new and more general than Erbe and Wang (1994), Lee (1997), Avery and Henderson (2000) and Liu and Yan (2006). We give an appropriate Banach space and construct a cone which we apply the fixed point theorem yielding solutions of the problem.

In Chapter 4, we also establish new result for a second order Dirichlet boundary value problem

$$u''(t) + \lambda a(t)f(u(t), u'(t)) = 0, \qquad t \in [0,1],$$

$$u(0) = u(1) = 0.$$

where f is singular at $u, u' = 0$. Various of λ are determined for which there exist positive solutions of the singular problem. We also use fixed point theorem for operators on a Banach space. We show that all our results are based on the Krasnosel'skii fixed point theorem on compression and expansion of cones.

CHAPTER TWO

POSITIVE SOLUTIONS TO A SINGULAR SECOND ORDER BOUNDARY VALUE PROBLEM

2.1 INTRODUCTION

In this chapter, we consider the following Sturm-Liouville boundary value problem

$$u''(t) + \lambda g(t) f(u(t)) = 0, \qquad t \in [0, 1], \tag{2.1.1}$$

$$\begin{aligned} \alpha u(0) - \beta u'(0) &= 0, \\ \gamma u(1) + \delta u'(1) &= 0, \end{aligned} \tag{2.1.2}$$

where $\alpha > 0, \beta > 0, \gamma > 0$ and $\delta > 0$ are all constants, λ is a positive parameter, $f(\cdot)$ is singular at $u = 0$ and $g \in (C[0, 1], (0, \infty))$.

The existence of positive solutions of singular boundary value problems of ordinary differential equation has been studied by many researchers such as in Gatica et al. (1989), Daqing (2002), Wang and Liu (2003), Agarwal and Stanek (2007), Erbe et al. (2008), Wang (2011) and Xuan (2011). In Gatica et al. (1989), they proved the existence of positive solution of the problem (2.1.1) - (2.1.2) with $\lambda = 1$ and $g(t) = 1$ using the iterative technique and fixed point theorem for cone for decreasing mappings. Agarwal et al. (1998) obtained the existence of positive solutions for λ on a suitable interval for the Sturm-Liouville boundary value problem

$$(p(t)u'(t))' + \lambda f(t, u(t)) = 0, \qquad 0 < t < 1,$$

$$\begin{aligned} \alpha_1 u(0) - \beta_1 p(0) u'(0) &= 0, \\ \alpha_2 u(1) - \beta_2 p(1) u'(1) &= 0 \end{aligned}$$

by applying the fixed point theorem in a cone. In Wang and Liu (2003), they proved the existence of positive solution to the problem (2.1.1) - (2.1.2) using the Schauder fixed point theorem. Agarwal and Stanek (2007) established the existence criteria for positive solutions of singular boundary value problems for nonlinear second order ordinary and delay differential equations using the Vitali's convergence theorem. The nonlinearities may singular at phase variable and positive solutions pass through the singularities.

In Erbe et al. (2008), they established some criterias for the existence of positive solutions for certain two point boundary value problems for singular nonlinear second order equation on time scale T

$$-(ru^{\Delta})^{\Delta} + qu^{\sigma} = \lambda f(t, u^{\sigma})$$

$$\alpha u(a) - \beta u^{\Delta}(a) = 0,$$
$$\gamma u(\sigma^2(b)) + \delta u^{\Delta}(\sigma(b)) = 0,$$

where f may be singular at one or both end points by applying the Krasnosel'skii fixed point theorem. They assumed either f is continuous on $(a, b) \times R$, if f is singular at b or f is continuous on $(a, b] \times R$ if f is not singular at b.

Xuan (2011) investigated the existence of symmetric positive solutions for the following singular second-order differential equation

$$u''(t) + \lambda a(t) f(t, u(t)) = 0, \qquad 0 < t < 1,$$
$$u(t) = u(1-t), u'(0) - u'(1) = \Sigma_{i=1}^{m-2} b_i u(\xi_i).$$

where $\lambda > 0$ is a positive parameter, $a(t) : (0, 1) \to [0, \infty)$ is continuous, symmetric and may be singular at $t = 0$ or $t = 1$. Wang (2011) considered positive periodic solutions for singular systems of first order problem

$$x_i'(t) = -a_i(t)x_i(t) + b_i(t)f_i(x_1(t), ..., x_n(t)), \qquad i = 1, ..., n, \tag{2.1.3}$$

where λ is positive parameter. The existence and multiplicity of positive periodic solutions of the problem (2.1.3) were established by using the Krasnosels'kii fixed point theorem.

Motivated and inspired by work of Wang (2011), we are concerned with the existence and multiplicity of positive solution to (2.1.1) - (2.1.2) by applying Krasnosel'skii fixed point theorem. This result generalize the work of Henderson and Wang (1997) who considered positive solutions for nonlinear eigenvalue problem (2.1.1) with Dirichlet boundary conditions.

By a positive solution of (2.1.1) - (2.1.2), we mean a solution $u(t)$ such that $u(t) \geq 0$ for $0 \leq t \leq 1$. Let X be the Banach space $C[0, 1]$ endowed with norm

$$\|u\| = \max_{t \in [0,1]} |u(t)|.$$

Let $R = (-\infty, \infty)$, $R_+ = [0, \infty)$. Our basic assumption is:

(A1) $f : (0, \infty) \to (0, \infty)$ is continuous.

2.2 GREEN'S FUNCTION AND BOUNDS

Now we are going to find a positive solution of the second order problem (2.1.1) - (2.1.2). For the convenience of the reader, we shall show that the solution $u(t)$ is of the form $u(t) = \int_0^1 G(t,s)P(s)ds$ where $G(t,s)$ is defined below.

Lemma 2.2.1. *Let $P \in X$, then the solution of the following boundary value problem*

$$u''(t) = -P(t), \qquad 0 \le t \le 1, \tag{2.2.1}$$

$$\alpha u(0) - \beta u'(0) = 0,$$
$$\gamma u(1) + \delta u'(1) = 0,$$

is given by

$$u(t) = \int_0^1 G(t,s)P(s)ds$$

where

$$G(t,s) = \begin{cases} \frac{1}{D}(\gamma + \delta - \gamma t)(\beta + \alpha s), & 0 \le s \le t \le 1, \\ \frac{1}{D}(\gamma + \delta - \gamma s)(\beta + \alpha t), & 0 \le t \le s \le 1, \end{cases} \tag{2.2.2}$$

where $D = \gamma\beta + \alpha\gamma + \alpha\delta > 0$.

Proof. Writing $u''(t) = -P(t)$ and solving the differential equation (2.1.1) using the Laplace transform, we have $L(u''(t)) = -L(P(t))$ where $L(u''(t)) = \int_0^\infty e^{-st}u''(t)dt$. Using integration by part, we get

$$\begin{aligned} L(u''(t)) &= \left[e^{-st}u'(t)\right]_0^\infty + s\int_0^\infty u'(t)e^{-st}dt \\ &= \left[e^{-st}u'(t)\right]_0^\infty + sL\left[u'(t)\right] \\ &= \left[e^{-st}u'(t)\right]_0^\infty + s\left[sL[u(t)] - u(0)\right] \\ &= s^2L[u(t)] - su(0) - u'(0), \end{aligned}$$

which implies $s^2L[u(t)] - su(0) - u'(0) = -L(P(t))$, that is

$$L(u(t)) = \frac{su(0)}{s^2} + \frac{u'(0)}{s^2} - \frac{L(P(t))}{s^2}. \tag{2.2.3}$$

Taking the inverse Laplace both side of (2.2.3),

$$L^{-1}L(u(t)) = L^{-1}\left[\frac{u(0)}{s} + \frac{u'(0)}{s^2} - \frac{L(P(t))}{s^2}\right],$$

we obtain $u(t) = u(0) + tu'(0) - \int_0^t (t-s)P(s)ds$, and $u'(t) = u'(0) - \int_0^t P(s)ds$. Using the boundary conditions (2.1.2), we have

$$\alpha u(0) - \beta u'(0) = \alpha\left(u(0) + (0)u'(0)\right) - \beta u'(0) = 0,$$

which implies,

$$\alpha u(0) - \beta u'(0) = 0. \tag{2.2.4}$$

Likewise,

$$\gamma u(1) + \delta u'(1) = 0,$$

implies

$$\gamma\left(u(0) + (1)u'(0) - \int_0^1 (1-s)P(s)ds\right) + \delta\left(u'(0) - \int_0^1 P(s)ds\right) = 0,$$

which implies that

$$\gamma u(0) + (\gamma + \delta)u'(0) = \int_0^1 \left(\gamma(1-s) + \delta\right)P(s)ds. \tag{2.2.5}$$

Solving (2.2.4) and (2.2.5) for $u(0)$ and $u'(0)$, we have

$$u(0) = \frac{\beta u'(0)}{\alpha},$$

$$u'(0) = \frac{\alpha\int_0^1 \left(\gamma(1-s) + \delta\right)P(s)ds}{\gamma\beta + \alpha(\gamma + \delta)}.$$

Let $D = \gamma\beta + \alpha\gamma + \alpha\delta > 0$, so

$$u(0) = \frac{\beta\int_0^1 \left(\gamma(1-s) + \delta\right)P(s)ds}{D}, \quad u'(0) = \frac{\alpha\int_0^1 \left(\gamma(1-s) + \delta\right)P(s)ds}{D}.$$

Thus,

$$
\begin{aligned}
u(t) &= u(0) + tu'(0) - \int_0^t (t-s)P(s)ds \\
&= \frac{\beta \int_0^1 (\gamma(1-s)+\delta)\, P(s)ds}{D} + \frac{t\alpha \int_0^1 (\gamma(1-s)+\delta)\, P(s)ds}{D} - \int_0^t (t-s)P(s)ds \\
&= \frac{1}{D}\int_0^1 (\gamma - \gamma s + \delta)(\beta + \alpha t)P(s)ds - \int_0^t (t-s)P(s)ds \\
&= \frac{1}{D}\int_0^t (\gamma - \gamma s + \delta)(\beta + \alpha t)P(s)ds + \frac{1}{D}\int_t^1 (\gamma - \gamma s + \delta)(\beta + \alpha t)P(s)ds \\
&\quad - \int_0^t (t-s)P(s)ds \\
&= \frac{1}{D}\int_0^t (\gamma - \gamma s + \delta)(\beta + \alpha t)P(s)ds - \frac{1}{D}\int_0^t (\gamma\beta + \alpha\gamma + \alpha\delta)(t-s)P(s)ds \\
&\quad + \frac{1}{D}\int_t^1 (\gamma - \gamma s + \delta)(\beta + \alpha t)P(s)ds \\
&= \frac{1}{D}\int_0^t \beta(\gamma + \delta - \gamma t) + \alpha s(\gamma + \delta - \gamma t)P(s)ds + \frac{1}{D}\int_t^1 (\gamma - \gamma s + \delta)(\beta + \alpha t)P(s)ds \\
&= \frac{1}{D}\int_0^t (\beta + \alpha s)(\gamma + \delta - \gamma t)P(s)ds + \frac{1}{D}\int_t^1 (\gamma - \gamma s + \delta)(\beta + \alpha t)P(s)ds.
\end{aligned}
$$

Therefore,

$$u(t) = \int_0^1 G(t,s)P(s)ds$$

where $G(t,s)$ is given by (2.2.2). It is clear that $G(t,s) > 0$ if $(t,s) \in (0,1)\times(0,1)$. ■

Lemma 2.2.2. *The function $G(t,s)$ satisfies the homogenous differential equation $-u'' = 0$ and the boundary conditions (2.1.2) for fixed s.*

Proof. Since $G(t,s)$ is polynomial of degree one, then it satisfies $\frac{d^2}{dt^2}G(t,s) = 0$ for all $(t,s) \in [0,1] \times [0,1]$.

Note that differentiation G(t, s) is with respect to t.

For $0 \leq t \leq s \leq 1$, $G'(t,s) = \frac{1}{D}\alpha(\gamma + \delta - \gamma s)$ so that

$$\alpha G(0,s) - \beta G'(0,s) = 0.$$

Also for $0 \leq s \leq t \leq 1$, $G'(t,s) = -\frac{1}{D}\gamma(\beta + \alpha s)$ so that

$$\gamma G(1,s) + \delta G'(1,s) = 0.$$

■

Lemma 2.2.3. *For any fixed $s \in [0,1]$, the function $G(t,s)$ is continuous for every $t \in [0,1]$.*

Proof. $G(t, s)$ is continuous everywhere on $[0, 1] \times [0, 1]$ since it is continuous at the point $t = s$. Hence, the proof is complete. ■

Lemma 2.2.4. $\frac{d}{dt}G(t, s)$ *has a jump discontinuity with a jump of factor* -1 *at the point* $t = s$.

Proof. We show that limit of $\frac{d}{dt}G(t, s)$ as t approaches s from above differ from its limit as t approaches s from below by -1.

$$\begin{aligned} G'(s^+, s) - G'(s^-, s) &= \lim_{t\to s^+} G'(t, s) - \lim_{t\to s^-} G'(t, s) \\ &= \frac{1}{D}(-\gamma(\beta + \alpha s)) - \alpha(\gamma + \delta - \gamma s) \\ &= \frac{-\gamma\beta - \alpha\gamma - \alpha\delta}{\gamma\beta + \alpha\gamma + \alpha\delta} \\ &= -\frac{(\gamma\beta + \alpha\gamma + \alpha\delta)}{\gamma\beta + \alpha\gamma + \alpha\delta} \\ &= -1. \end{aligned}$$

■

Lemma 2.2.5. *Define*

$$\sigma = \min\left\{\frac{G(0, s)}{G(s, s)}, \frac{G(1, s)}{G(s, s)}\right\},$$

then $0 < \sigma < 1$.

Proof. Since $G(t, s) > 0$ for all $(t, s) \in [0, 1] \times [0, 1], \sigma > 0$.
Case (i) If $s = 0, \sigma = \min\left\{1, \frac{G(1,0)}{G(0,0)}\right\}$, which implies

$$\sigma = \frac{G(1,0)}{G(0,0)} = \frac{\delta}{\gamma + \delta} < 1.$$

Case (ii) If $s = 1, \sigma = \min\left\{1, \frac{G(0,1)}{G(1,1)}\right\}$, which implies

$$\sigma = \frac{G(0,1)}{G(1,1)} = \frac{\beta}{\beta + \alpha} < 1.$$

Hence, the proof is complete. ■

Lemma 2.2.6. *For all* s, $t \in [0, 1]$,

$$\sigma G(s, s) \leq G(t, s) \leq G(s, s)$$

where

$$0 < \sigma = \min\left\{\frac{\gamma + \delta - \gamma t}{\gamma + \delta - \gamma s}, \frac{\beta + \alpha t}{\beta + \alpha s}\right\} < 1.$$

Proof. For $0 \le s \le t \le 1$,

$$\frac{G(t,s)}{G(s,s)} = \frac{(\gamma+\delta-\gamma t)(\beta+\alpha s)}{(\gamma+\delta-\gamma s)(\beta+\alpha s)} = \frac{\gamma+\delta-\gamma t}{\gamma+\delta-\gamma s}.$$

For $0 \le t \le s \le 1$,

$$\frac{G(t,s)}{G(s,s)} = \frac{(\gamma+\delta-\gamma s)(\beta+\alpha t)}{(\gamma+\delta-\gamma s)(\beta+\alpha s)} = \frac{\beta+\alpha t}{\beta+\alpha s}.$$

Case (i) For $0 \le s \le t \le 1$, $G'(t,s) = -\frac{\gamma}{D}(\beta+\alpha s) < 0$, which implies that $G(t,s)$ is a decreasing function of t, so that $G(t,s) \le G(s,s)$ and also for $t \le 1$, $\frac{G(t,s)}{G(s,s)} \ge \frac{G(1,s)}{G(s,s)} \ge \sigma$ which implies

$$\sigma G(s,s) \le G(t,s).$$

Case (ii) For $0 \le t \le s \le 1$, $G'(t,s) = \frac{\alpha}{D}(\gamma+\delta-\gamma s) > 0$ implies that $G(t,s)$ is an increasing function of t, so that $G(t,s) \le G(s,s)$ and also for $t \ge 0$, $\frac{G(t,s)}{G(s,s)} \ge \frac{G(0,s)}{G(s,s)} \ge \sigma$ and so we have

$$\sigma G(s,s) \le G(t,s).$$

Therefore, $\sigma G(s,s) \le G(t,s) \le G(s,s)$ for $0 \le t,s \le 1$. ∎

2.3 PRELIMINARY RESULTS

In this section, we will show some lemmas that are useful in the proving the existence and multiplicity of positive solutions for the problem (2.1.1) - (2.1.2). Define the cone K in X by

$$K = \{u \in X : u(t) > 0, t \in [0,1] \quad \text{and} \quad \min_{t\in[0,1]} u(t) \ge \sigma \|u\|\}$$

where σ is defined in Lemma 2.2.6.
Let $\Omega_r = \{u \in K : \|u\| < r\}$ for $r > 0$. It is clear that $\partial\Omega_r = \{u \in K : \|u\| = r\}$. Transform the problem (2.1.1) - (2.1.2) into a fixed point problem. Consider the operator $T : K \to X$ defined by

$$Tu(t) = \lambda \int_0^1 G(t,s)g(s)f(u(s))\,ds, \qquad t \in [0,1].$$

Lemma 2.3.1. (see Wang (2011)) *Assume (A1) holds. Then $u \in K$ is a positive fixed point of T if and only if u is a positive solution of (2.1.1) - (2.1.2).*

In the next lemma, we show that $T : K \to K$ is completely continuous.

Lemma 2.3.2. *Assume (A1) holds. Then $T(K) \subset K$ and $T : K \to K$ is completely continuous.*

Proof. Let $u \in K$, then $Tu(t) > 0$ on $[0, 1]$ and

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s)g(s)f\left(u(s)\right)ds \\ &\geq \sigma \int_0^1 G(s,s)\lambda g(s)f(u(s))ds \\ &\geq \lambda\sigma \int_0^1 \max_{t\in[0,1]} G(t,s)g(s)f(u(s))ds \\ &\geq \sigma\lambda \max_{t\in[0,1]} \int_0^1 G(t,s)g(s)f(u(s))ds \\ &= \sigma \left\|Tu\right\|. \end{aligned}$$

Thus $Tu \in K$ if $u \in K$ and hence $T(K) \subset K$. A standard argument can be used to show that $T : K \to K$ is completely continuous. Let $u \in K$ and $\epsilon > 0$ be given. By continuity of f, there exists $\delta > 0$ such that for any $y \in (0, \infty)$ with $|u(t) - y| < \delta$, $t \in [0, 1]$, then $|f(u(t))-f(y)| < \epsilon$. Let $w \in K$ with $\|u-w\| < \delta$, then $|w(t)-u(t)| < \delta$ for all $t \in [0, 1]$.

$$\begin{aligned} |(Tu)(t) - (Tw)(t)| &= \lambda \int_0^1 G(t,s)g(s)(f(u(s)) - f(w(s))ds \\ &\leq \epsilon\lambda \int_0^1 G(t,s)g(s)ds. \end{aligned}$$

Thus $\|Tu - Tw\| \leq \epsilon\lambda \int_0^1 G(t,s)g(s)ds$ and T is continuous. Let $\{u_n\}$ be a bounded sequence in K. Since f is continuous, there exists $N > 0$ such that $|f(u_n(t))| \leq N$ for all n where $t \in [0, 1]$.

$$\begin{aligned} |(Tu_n)(t)| &= |\lambda \int_0^1 G(t,s)g(s)f(u_n(s))ds| \\ &\leq \lambda \int_0^1 |G(s,s)g(s)f(u_n(s))|ds \\ &\leq N\lambda \int_0^1 G(s,s)g(s)ds. \end{aligned}$$

By choosing subsequences, there exists $\{Tu_{n_j}\}$ which converges uniformly on $[0, 1]$. Hence T is completely continuous mapping. ■

Define $f^* : [1, \infty) \to R^+$ be the function given by

$$f^*(\theta) = \max\{f(u) : u \in R_+, 1 \leq |u| \leq \theta\}.$$

It is easy to see that $f^*(\theta)$ is nondecreasing function on $[1, \infty)$. Our next lemma gives some relationships between the functions f and f^*. The following lemma is essentially the same as Lemma 2.8 in Wang (2003).

Lemma 2.3.3. (see Wang (2003)) *Assume (A1) holds. Then* $f^*_\infty = f_\infty$.

In the next two lemmas, we get lower and upper estimates on operator T. Define

$$\Gamma = \sigma^2 \int_0^1 G(s,s)g(s)ds. \tag{2.3.1}$$

Lemma 2.3.4. *Assume (A1) holds and let* $\eta > 0$ *be given. If* $u \in K$ *and* $f(u(t)) \geq u(t)\eta$ *for* $t \in [0,1]$, *then*

$$\|Tu\| \geq \lambda\Gamma\eta \|u\|.$$

Proof. From the definitions of Tu and K, it follows that

$$\begin{aligned} Tu(t) &= \lambda \int_0^1 G(t,s)g(s)f(u(s))\,ds \\ &\geq \lambda\sigma \int_0^1 G(s,s)g(s)f(u(s))ds \\ &\geq \lambda\sigma \int_0^1 G(s,s)\eta u(s)g(s)ds \\ &\geq \lambda\sigma^2\eta \|u\| \int_0^1 G(s,s)g(s)ds \\ &= \lambda\Gamma\eta \|u\|, \end{aligned}$$

where Γ is defined in (2.3.1). Hence $\|Tu\| \geq \lambda\Gamma\eta \|u\|$.
This completes the proof. ■

Lemma 2.3.5. *Assume (A1) holds. Let* $r > 0$ *and if there exists* $\epsilon > 0$ *such that* $f^*(r) \leq \epsilon r$, *for any* $r > 0$, *then*

$$\|Tu\| \leq \lambda\epsilon \|u\| \int_0^1 G(s,s)g(s)ds \quad \text{for} \quad u \in \partial\Omega_r.$$