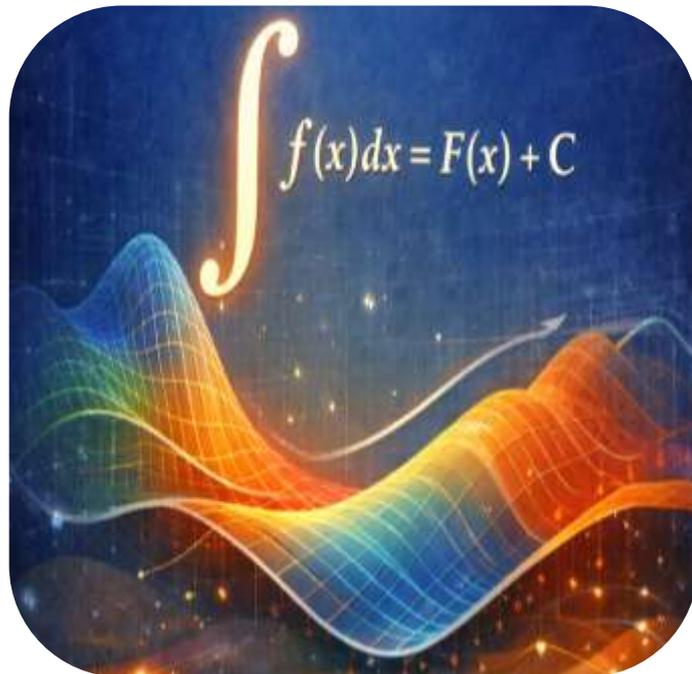


Mastering Integral Methods: A Systematic Approach



Foo Fong Yeng

Ng Set Foong

Shamsatun Nahar Ahmad

Aishah Mahat

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Authors: Foo Fong Yeng, Ng Set Foong, Shamsatun Nahar Ahmad, Aishah Mahat



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Preface

Integration is a fundamental concept in calculus that plays a central role in mathematics, science, engineering, and many applied fields. Mastery of integration techniques is therefore crucial for students seeking a deeper understanding of calculus.

This book is designed to present the techniques of integration in a clear, systematic, and accessible manner. It begins with the concept of the indefinite integral and introduces the basic rules required to evaluate elementary integrals, including algebraic, exponential, and trigonometric functions. These foundational topics prepare the reader for more advanced methods.

Subsequent chapters develop essential techniques such as substitution and integration by parts, emphasizing their conceptual foundations and practical use. The connection between definite integrals and antiderivatives is established through the First Fundamental Theorem of Calculus.

The book then addresses more specialized integration methods, including integrals involving trigonometric functions, trigonometric substitutions, and rational functions. These techniques are systematically introduced such that the readers can handle a wide range of integrals encountered in coursework and applications.

Throughout the text, the progression of topics is arranged to build gradually from basic principles to more sophisticated techniques. This book is suitable for students in introductory and intermediate calculus courses, as well as for self-learners seeking a structured and comprehensive review of integration techniques. It is hoped that this text will serve not only as a practical guide for evaluating integrals, but also as a resource that fosters a deeper appreciation of integration as a powerful and unifying tool in mathematics.

Foo Fong Yeng,

Ng Set Foong,

Shamsatun Nahar Ahmad,

Aishah Mahat

1. Indefinite Integral

This section introduces the concept of the indefinite integral as the inverse process of differentiation. Fundamental properties and interpretations are discussed to build a foundation for all subsequent integration techniques.

1.1 Constant, Power Rules, Reciprocal Functions

This subsection covers the most basic integration rules, including integrals of constants, power functions, and reciprocal functions. These rules are the core for evaluating elementary integrals and serve as building blocks for more advanced methods.

Formula

$$\int a \, dx = ax + C, \quad a, C = \text{constant}$$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

Example 1

$$\begin{aligned} \int 5 \, dx \\ = 5x + C \end{aligned}$$

Example 2

$$\begin{aligned} \int -\frac{1}{2} \, dx \\ = -\frac{1}{2}x + C \end{aligned}$$

Example 3

$$\begin{aligned} \int x^3 \, dx \\ = \frac{x^4}{4} + C \end{aligned}$$

Example 4

$$\begin{aligned}\int 8x^3 dx \\ &= \frac{8x^4}{4} + C \\ &= 2x^4 + C\end{aligned}$$

Example 5

$$\begin{aligned}\int \frac{6}{x^4} dx \\ &= 6 \int x^{-4} dx \\ &= 6 \left[\frac{x^{-3}}{-3} \right] + C \\ &= -2x^{-3} + C \\ &= -\frac{2}{x^3} + C\end{aligned}$$

Example 6

$$\begin{aligned}-\int \frac{5}{2x^3} dx \\ &= -\frac{5}{2} \int x^{-3} dx \\ &= -\frac{5}{2} \left[\frac{x^{-2}}{-2} \right] + C \\ &= \frac{5}{4} x^{-2} + C \\ &= \frac{5}{4x^2} + C\end{aligned}$$

Formula

$$\int \frac{1}{x} dx = \ln x + C$$
$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + C, \quad f(x) = \text{function}$$

Example 7

$$\int \frac{1}{x} dx$$
$$= \ln x + C$$

$$\int \frac{1}{x} dx = \int x^{-1} dx$$
$$= \frac{x^0}{0}$$



Wrong concept

Example 8

$$\int \frac{2}{7x} dx$$
$$= \frac{2}{7} \int \frac{1}{x} dx$$
$$= \frac{2}{7} \ln x + C$$

Example 9

$$\int \frac{1}{x+7} dx$$

let $f(x) = x + 7$ $f'(x) = 1$

$$= \ln|x + 7| + C$$

Example 10

$$\int \frac{1}{2x+7} dx$$

Let $f(x) = 2x + 7$ $f'(x) = 2$

$$= (2) \int \frac{(2) 1}{2x+7} dx$$

$$\begin{aligned} &= 2 \int \frac{2}{2x+7} dx \\ &= 2 \ln |2x+7| + C \end{aligned}$$

Example 11

$$\int \frac{1}{4-3x} dx$$

$$\begin{aligned} \text{Let } f(x) &= 4 - 3x \\ f'(x) &= -3 \end{aligned}$$

$$\begin{aligned} &= (-3) \int \frac{(-3)1}{4-3x} dx \\ &= -3 \int \frac{-3}{4-3x} dx \\ &= -3 \ln |4-3x| + C \end{aligned}$$

Example 12

$$\begin{aligned} &\int \frac{2}{5+3x} dx \\ &= 2 \int \frac{1}{5+3x} dx \end{aligned}$$

$$\begin{aligned} \text{Let } f(x) &= 5 + 3x \\ f'(x) &= 3 \end{aligned}$$

$$\begin{aligned} &= 2(3) \int \frac{(3)1}{5+3x} dx \\ &= 6 \int \frac{3}{5+3x} dx \\ &= 6 \ln |5+3x| + C \end{aligned}$$

1.2 Exponential Functions

Here, special attention is given to natural exponential functions and their unique properties.

Formula

$$\int e^x dx = e^x + C$$

$$\int e^{ax} dx = \frac{e^{ax}}{a} + C, \quad a = \text{constant}$$

$$\int e^{ax+b} dx = \frac{e^{ax+b}}{a} + C, \quad a, b = \text{constant}$$

Example 1

$$\begin{aligned} \int e^{5x+2} dx \\ = \frac{e^{5x+2}}{5} + C \end{aligned}$$

Example 2

$$\begin{aligned} \int e^{-x} dx \\ = \frac{e^{-x}}{-1} + C \\ = -e^{-x} + C \end{aligned}$$

Example 3

$$\begin{aligned} \int e^{-2x+5} dx \\ = \frac{e^{-2x+5}}{-2} + C \\ = -\frac{e^{-2x+5}}{2} + C \end{aligned}$$

1.3 Trigonometric Functions

This subsection focuses on the integrals of fundamental trigonometric functions. Common patterns are identified to simplify integration involving sine, cosine, tangent, and related functions.

Formula	
$\int \sin x \, dx = -\cos x + C$	$\int \sin ax \, dx = -\frac{\cos ax}{a} + C$
$\int \cos x \, dx = \sin x + C$	$\int \cos ax \, dx = \frac{\sin ax}{a} + C$
$\int \sec^2 x \, dx = \tan x + C$	$\int \sec^2 ax \, dx = \frac{\tan ax}{a} + C$
$\int \csc^2 x \, dx = -\cot x + C$	$\int \csc^2 ax \, dx = -\frac{\cot ax}{a} + C$
$\int \sec x \tan x \, dx = \sec x + C$	$\int \sec ax \tan ax \, dx = \frac{\sec ax}{a} + C$
$\int \csc x \cot x \, dx = -\csc x + C$	$\int \csc ax \cot ax \, dx = -\frac{\csc ax}{a} + C$

Example 1

$$\begin{aligned} \int \sin x \, dx \\ = -\cos x + C \end{aligned}$$

Example 2

$$\begin{aligned} \int \sin 2x \, dx \\ = -\frac{\cos 2x}{2} + C \end{aligned}$$

Example 3

$$\begin{aligned} \int \sec^2 x \, dx \\ = \tan x + C \end{aligned}$$

Example 4

$$\int \sec^2 4x \, dx$$
$$= \frac{\tan 4x}{4} + C$$

Example 5

$$\int \sec x \tan x \, dx$$
$$= \tan x + C$$

Example 6

$$\int \sec 3x \tan 3x \, dx$$
$$= \frac{\tan 3x}{3} + C$$

2. Integral Algebraic Functions

This section explores the integration of algebraic expressions, including polynomials and rational powers of variables. Emphasis is placed on rewriting expressions into integrable forms and applying previously learned rules efficiently.

Example 1

$$\begin{aligned} & \int \sqrt[3]{x} (2-x)^2 dx \\ &= \int x^{\frac{1}{3}} (2-x)^2 dx \\ &= \int x^{\frac{1}{3}} (4-4x+x^2) dx \\ &= \int 4x^{\frac{1}{3}} - 4x^{\frac{4}{3}} + x^{\frac{7}{3}} dx \\ &= \frac{4x^{\frac{4}{3}}}{\left(\frac{4}{3}\right)} - \frac{4x^{\frac{7}{3}}}{\left(\frac{7}{3}\right)} + \frac{10x^{\frac{10}{3}}}{\left(\frac{10}{3}\right)} + C \\ &= 3x^{\frac{4}{3}} - \frac{12}{7}x^{\frac{7}{3}} + \frac{3}{10}x^{\frac{10}{3}} + C \end{aligned}$$

Example 2

$$\begin{aligned} & \int \frac{(2t+3)(t-3)}{4t} dt \\ &= \int \frac{2t^2 - 6t + 3t - 9}{4t} dt \\ &= \int \frac{2t^2 - 3t - 9}{4t} dt \\ &= \int \frac{2t^2}{4t} - \frac{3t}{4t} - \frac{9}{4t} dt \\ &= \int \frac{1}{2}t - \frac{3}{4} - \frac{9}{4t} dt \\ &= \frac{1}{2} \left(\frac{t^2}{2} \right) - \frac{3}{4}t - \frac{9}{4} \ln t + C \\ &= \frac{t^2}{4} - \frac{3}{4}t - \frac{9}{4} \ln t + C \end{aligned}$$

Example 3

$$\begin{aligned} & \int (e^x - \sec 5x)(e^x + \sec 5x) dx \\ &= \int (e^x)^2 + e^x \sec 5x - e^x \sec 5x - (\sec 5x)^2 dx \\ &= \int e^{2x} - \sec^2 5x dx \\ &= \frac{e^{2x}}{2} - \frac{\tan 5x}{5} + C \end{aligned}$$

Example 4

$$\begin{aligned} & \int (e^x - 1)^2 dx \\ &= \int (e^x)^2 - 2e^x + 1 dx \\ &= \int e^{2x} - 2e^x + 1 dx \\ &= \frac{e^{2x}}{2} - 2e^x + x + C \end{aligned}$$

3. Integral by Substitution

This section introduces the substitution method as a powerful technique for simplifying integrals. By changing variables, complex integrals can be transformed into basic forms.

Example 1

$$\int x \sqrt{7 - x^2} dx$$

$$= \int \sqrt{7 - x^2} x dx$$

Sub $u = 7 - x^2$

$du = -2x dx$

$\frac{du}{-2} = x dx$

$$= \int \sqrt{u} \left(\frac{du}{-2}\right)$$

$$= -\frac{1}{2} \int u^{\frac{1}{2}} du$$

$$= -\frac{1}{2} \left[\frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} \right] + C$$

$$= -\frac{1}{2} \left[\frac{2}{3} u^{\frac{3}{2}} \right] + C$$

$$= -\frac{1}{3} u^{\frac{3}{2}} + C$$

$$= -\frac{1}{3} (7 - x^2)^{\frac{3}{2}} + C$$

$$= -\frac{1}{3} (\sqrt{7 - x^2})^3 + C$$

Example 2

$$\int \frac{5x}{\sqrt{x+2}} dx$$
$$= 5 \int \frac{x}{\sqrt{x+2}} dx$$

Sub $u = x + 2$

$du = dx$

$u - 2 = x$

$$= 5 \int \frac{u-2}{u^{\frac{1}{2}}} du$$

$$= 5 \int \frac{u}{u^{\frac{1}{2}}} - \frac{2}{u^{\frac{1}{2}}} du$$

$$= 5 \int u^{\frac{1}{2}} - 2u^{-\frac{1}{2}} du$$

$$= 5 \left[\frac{u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} - \frac{2u^{\frac{1}{2}}}{\left(\frac{1}{2}\right)} \right] + C$$

$$= 5 \left[\frac{2}{3} u^{\frac{3}{2}} - 4u^{\frac{1}{2}} \right] + C$$

$$= \frac{10}{3} u^{\frac{3}{2}} - 20u^{\frac{1}{2}} + C$$

$$= \frac{10}{3} (x+2)^{\frac{3}{2}} - 20(x+2)^{\frac{1}{2}} + C$$

$$= \frac{10}{3} (\sqrt{x+2})^3 - 20\sqrt{x+2} + C$$

Example 3

$$\int \frac{x^3}{x^4 + 7} dx$$
$$= \int \frac{1}{x^4 + 7} x^3 dx$$

$$\text{Sub } u = x^4 + 7$$

$$du = 4x^3 dx$$

$$\frac{du}{4} = x^3 dx$$

$$= \int \frac{1}{u} \left(\frac{du}{4} \right)$$

$$= \frac{1}{4} \int \frac{1}{u} du$$

$$= \frac{1}{4} \ln u + C$$

$$= \frac{1}{4} \ln |x^4 + 7| + C$$

Example 4

$$\int x(x^2 - 9)^{10} dx$$

$$= \int (x^2 - 9)^{10} x dx$$

$$\text{Sub } u = x^2 - 9$$

$$du = 2x dx$$

$$\frac{du}{2} = x dx$$

$$= \int u^{10} \left(\frac{du}{2} \right)$$

$$= \frac{1}{2} \int u^{10} du$$

$$= \frac{1}{2} \left[\frac{u^{11}}{11} \right] + C$$

$$= \frac{1}{22} u^{11} + C$$

$$= \frac{1}{22} (x^2 - 9)^{11} + C$$

Example 5

$$\int \frac{(\sqrt{3} + e^{2x})^2}{e^{2x}} dx$$

Sub $u = e^{2x}$

$du = 2e^{2x} dx$

$\frac{du}{2e^{2x}} = dx$

$\frac{du}{2u} = dx$

$$= \int \frac{(\sqrt{3} + u)^2}{u} \left(\frac{du}{2u}\right)$$

$$= \int \frac{(\sqrt{3} + u)^2}{2u^2} du$$

$$= \frac{1}{2} \int \frac{(\sqrt{3})^2 + 2\sqrt{3}u + u^2}{u^2} du$$

$$= \frac{1}{2} \int \frac{3 + 2\sqrt{3}u + u^2}{u^2} du$$

$$= \frac{1}{2} \int \frac{3}{u^2} + \frac{2\sqrt{3}u}{u^2} + \frac{u^2}{u^2} du$$

$$= \frac{1}{2} \int 3u^{-2} + \frac{2\sqrt{3}}{u} + 1 du$$

$$= \frac{1}{2} \left[\frac{3u^{-1}}{-1} + 2\sqrt{3} \ln u + u \right] + C$$

$$= \frac{1}{2} \left[-\frac{3}{u} + 2\sqrt{3} \ln u + u \right] + C$$

$$= -\frac{3}{2u} + \frac{2\sqrt{3}}{2} \ln u + \frac{1}{2}u + C$$

$$\begin{aligned}
&= -\frac{3}{2u} + \sqrt{3} \ln u + \frac{1}{2}u + C \\
&= -\frac{3}{2e^{2x}} + \sqrt{3} \ln e^{2x} + \frac{1}{2}e^{2x} + C
\end{aligned}$$

Example 6

$$\begin{aligned}
&\int \frac{x^2}{\sin^2(x^3 + 1)} dx \\
&= \int \frac{1}{\sin^2(x^3 + 1)} x^2 dx
\end{aligned}$$

Sub $u = (x^3 + 1)$
 $du = 3x^2 dx$
 $\frac{du}{3} = x^2 dx$

$$\begin{aligned}
&= \int \frac{1}{\sin^2 u} \left(\frac{du}{3}\right) \\
&= \frac{1}{3} \int \frac{1}{\sin^2 u} du \\
&= \frac{1}{3} \int \csc^2 u du \\
&= \frac{1}{3} [-\cot u] + C \\
&= -\frac{1}{3} \cot u + C \\
&= -\frac{1}{3} \cot(x^3 + 1) + C
\end{aligned}$$

4. Definite Integrals and First Fundamental Theorem of Calculus

This section presents the definite integrals and The First Fundamental Theorem of Calculus.

Definite Integrals

$$\int_a^b f(x) dx$$

- $f(x)$ is called the integrand
- a and b are called the limits of integration
- a is the lower limit and b is the upper limit.

First Fundamental Theorem of Calculus

For a function $f(x)$ continuous over the interval $[a, b]$, with $F(x)$ as its antiderivative, the integral of $f(x)$ over $[a, b]$ is equal to $F(b) - F(a)$,

$$\int_a^b f(x) dx = F(b) - F(a)$$

Example 1

$$\begin{aligned} & \int_4^9 \frac{1 + \sqrt{x}}{\sqrt{x}} dx \\ &= \int_4^9 \frac{1 + x^{\frac{1}{2}}}{x^{\frac{1}{2}}} dx \\ &= \int_4^9 \frac{1}{x^{\frac{1}{2}}} + \frac{x^{\frac{1}{2}}}{x^{\frac{1}{2}}} dx \\ &= \int_4^9 x^{-\frac{1}{2}} + 1 dx \\ &= \left[\frac{x^{\frac{1}{2}}}{\frac{1}{2}} + x \right]_4^9 \end{aligned}$$

$$\begin{aligned}
&= \left[2x^{\frac{1}{2}} + x \right]_4^9 \\
&= \left[2\sqrt{x} + x \right]_4^9 \\
&= [2\sqrt{9} + 9] - [2\sqrt{4} + 4] \\
&= [15] - [8] \\
&= 7
\end{aligned}$$

Example 2

$$\int_{-1}^2 f(x) dx = \begin{cases} e^{3x}, & x \leq 1 \\ \sec^2 3x, & x > 1 \end{cases}$$

$$\begin{aligned}
\int_{-1}^2 f(x) dx &= \int_{-1}^1 e^{3x} dx + \int_1^2 \sec^2 3x dx \\
&= \left[\frac{e^{3x}}{3} \right]_{-1}^1 + \left[\frac{\tan 3x}{3} \right]_1^2 \\
&= \left[\left[\frac{e^3}{3} \right] - \left[\frac{e^{-3}}{3} \right] \right] + \left[\left[\frac{\tan 6}{3} \right] - \left[\frac{\tan 3}{3} \right] \right] \\
&= [6.6952 - 0.0166] + [-0.097 - [-0.0475]] \\
&= 6.6291
\end{aligned}$$

Example 3

$$\begin{aligned}
&\int_0^3 |4x - 10| dx \\
&= \int_0^{2.5} 10 - 4x dx + \int_{2.5}^3 4x - 10 dx \\
&= [10x - 2x^2]_0^{2.5} + [2x^2 - 10x]_{2.5}^3 \\
&= [[10(2.5) - 2(2.5)^2] - [10(0) - 2(0)^2]] + [[2(3)^2 - 10(3)] - [2(2.5)^2 - 10(2.5)]] \\
&= [[12.5] - [0]] + [[-12] - [-12.5]] \\
&= 13
\end{aligned}$$

Example 4

$$\begin{aligned} & \int_1^2 (x^2 - 1)^4 x \, dx \\ &= \int u^4 \left(\frac{du}{2}\right) \\ &= \frac{1}{2} \int u^4 \, du \\ &= \frac{1}{2} \left[\frac{u^5}{5} \right] \\ &= \frac{1}{10} u^5 \\ &= \left[\frac{1}{10} (x^2 - 1)^5 \right]_1^2 \\ &= \left[\frac{1}{10} (2^2 - 1)^5 \right] - \left[\frac{1}{10} (1^2 - 1)^5 \right] \\ &= [24.3] - [0] \\ &= 24.3 \end{aligned}$$

Example 5

$$\begin{aligned} & \int_{\frac{\pi}{4}}^{\pi} \frac{\sin 2\theta}{\sqrt{\cos 2\theta}} \, d\theta \\ &= \int_{\frac{\pi}{4}}^{\pi} \frac{1}{\sqrt{\cos 2\theta}} \sin 2\theta \, d\theta \\ &= \int \frac{1}{\sqrt{u}} \left(\frac{du}{-2}\right) \\ &= -\frac{1}{2} \int \frac{1}{\sqrt{u}} \, du \\ &= -\frac{1}{2} \int u^{-\frac{1}{2}} \, du \\ &= -\frac{1}{2} \left[\frac{u^{\frac{1}{2}}}{\frac{1}{2}} \right] \end{aligned}$$

$$\begin{aligned}
&= -\frac{1}{2} \left[2u^{\frac{1}{2}} \right] \\
&= -u^{\frac{1}{2}} \\
&= -\sqrt{u} \\
&= \left[-\sqrt{\cos 2\theta} \right]_{\frac{\pi}{4}}^{\pi} \\
&= \left[-\sqrt{\cos 2(\pi)} \right] - \left[-\sqrt{\cos 2\left(\frac{\pi}{4}\right)} \right] \\
&= \left[-\sqrt{-1} \right] - \left[-\sqrt{0} \right] \\
&= -1
\end{aligned}$$

5. Integration by Parts

This section develops the technique of integration by parts based on the product rule of differentiation. It provides strategies for selecting appropriate functions and demonstrates how the method can be applied repeatedly to evaluate more complex integrals.

$$\int u dv = uv - \int v du$$

Example 1

$$\int \ln x dx$$

$$\text{Let } u = \ln x, \quad dv = 1 dx$$

$$u = \frac{1}{x} dx, \quad v = x$$

$$= uv - \int v du$$

$$= [\ln x][x] - \int [x] \left[\frac{1}{x} dx \right]$$

$$= x \ln x - \int 1 dx$$

$$= x \ln x - x + C$$

Example 2

$$\int t^2 e^t dt$$

$$\text{Let } u = t^2, \quad dv = e^t dt$$

$$du = 2t dt, \quad v = e^t$$

$$= t^2 e^t - \int e^t (2t dt)$$

$$= t^2 e^t - 2 \int t e^t dt$$

Let	$u = t,$	$dv = e^t dt$
	$du = dt,$	$v = e^t$

$$= t^2 e^t - 2[te^t - \int e^t dt]$$

$$= t^2 e^t - 2te^t + 2 \int e^t dt$$

$$= t^2 e^t - 2te^t + 2e^t + C$$

Example 3

$$\int e^x \sin x dx$$

Let	$u = \sin x,$	$dv = e^x dx$
	$du = \cos x dx,$	$v = e^x$

$$= e^x \sin x - \int e^x \cos x dx$$

Let	$u = \cos x,$	$dv = e^x dx$
	$du = -\sin x dx,$	$v = e^x$

$$= e^x \sin x - [e^x \cos x - \int e^x (-\sin x dx)]$$

$$= e^x \sin x - [e^x \cos x + \int e^x \sin x dx]$$

$$= e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

$$\text{since } \int e^x \sin x dx = e^x \sin x - e^x \cos x - \int e^x \sin x dx$$

$$\int e^x \sin x dx + \int e^x \sin x dx = e^x \sin x - e^x \cos x$$

$$2 \int e^x \sin x dx = e^x \sin x - e^x \cos x$$

$$\int e^x \sin x dx = \frac{1}{2}[e^x \sin x - e^x \cos x] + C$$

$$\int e^x \sin x dx = \frac{1}{2}e^x \sin x - \frac{1}{2}e^x \cos x + C$$

Example 4

$$\int \sin^{-1} 2x \, dx$$

Let $u = \sin^{-1} 2x,$

$$u = \sin^{-1} w$$

$$du = \frac{1}{\sqrt{1-w^2}} dw$$

$$du = \frac{1}{\sqrt{1-(2x)^2}} (2dx)$$

$$du = \frac{2}{\sqrt{1-4x^2}} dx$$

Sub $w = 2x$ $dw = 2 \, dx$

$$dv = dx$$

$$v = x$$

$$= x \sin^{-1} 2x - \int \frac{2x}{\sqrt{1-4x^2}} dx$$

Sub $b = 1 - 4x^2$

$$db = -8x \, dx$$

$$\frac{db}{-8} = x \, dx$$

$$= x \sin^{-1} 2x - \int \frac{2}{\sqrt{b}} \left(\frac{db}{-8} \right)$$

$$= x \sin^{-1} 2x + \frac{1}{4} \int b^{-\frac{1}{2}} db$$

$$= x \sin^{-1} 2x + \frac{1}{4} \left[\frac{b^{\frac{1}{2}}}{\frac{1}{2}} \right] + C$$

$$= x \sin^{-1} 2x + \frac{1}{4} b^{\frac{1}{2}} + C$$

$$= x \sin^{-1} 2x + \frac{1}{4} (1 - 4x^2)^{\frac{1}{2}} + C$$

$$= x \sin^{-1} 2x + \frac{1}{4} \sqrt{1 - 4x^2} + C$$

6. Integration of Trigonometry Functions

This section extends earlier trigonometric integration techniques to more complicated expressions involving powers and products of trigonometric functions. Useful identities and systematic approaches are introduced to simplify such integrals effectively.

$$\int \sin A \sin B \, dx \qquad \int \cos A \cos B \, dx \qquad \int \sin A \cos B \, dx$$

Note: Angles A and B are in term of x when integral with respect to x . Change the symbol accordingly if integral with respect to θ .

Use Product to Sum Formula and Even/Odd Formula to solve the question

Product to Sum Formula

$$\sin A \sin B = \frac{1}{2} [\cos(A - B) - \cos(A + B)]$$

$$\cos A \cos B = \frac{1}{2} [\cos(A - B) + \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2} [\sin(A + B) + \sin(A - B)]$$

Even/Odd Formulas

$$\sin(-\theta) = -\sin(\theta) \qquad \csc(-\theta) = -\csc(\theta)$$

$$\cos(-\theta) = \cos(\theta) \qquad \sec(-\theta) = \sec(\theta)$$

$$\tan(-\theta) = -\tan(\theta) \qquad \cot(-\theta) = -\cot(\theta)$$

Example 1

$$\begin{aligned} & \int \sin 5x \cos 3x \, dx && A = 5x, \quad B = 3x \\ &= \int \frac{1}{2} [\sin(5x + 3x) + \sin(5x - 3x)] \, dx \\ &= \frac{1}{2} \int \sin 8x + \sin 2x \, dx \\ &= \frac{1}{2} \left[-\frac{\cos 8x}{8} - \frac{\cos 2x}{2} \right] + C \\ &= -\frac{1}{16} \cos 8x - \frac{1}{4} \cos 2x + C \end{aligned}$$

Example 2

$$\begin{aligned} & \int \sin 3\theta \sin 4\theta \, d\theta && A = 3\theta, \quad B = 4\theta \\ &= \int \frac{1}{2} [\cos(3\theta - 4\theta) - \cos(3\theta + 4\theta)] \, d\theta \\ &= \frac{1}{2} \int \cos(-\theta) - \cos(7\theta) \, d\theta \\ &= \frac{1}{2} \int \cos(\theta) - \cos(7\theta) \, d\theta \\ &= \frac{1}{2} \left[\sin(\theta) - \frac{\sin(7\theta)}{7} \right] + C \\ &= \frac{1}{2} \sin(\theta) - \frac{1}{14} \sin(7\theta) + C \end{aligned}$$

$$\int \sin^m A \cos^n A \, dx \quad \int \sin^m A \, dx \quad \int \cos^n A \, dx$$

Note: Angle A is in term of x when integral with respect to x . Change the symbol accordingly if integral with respect to θ .

Case (1) Either m or n is odd

If $\cos^n A$, n is odd.

Step (1) split off a factor of $\cos A$

Step (2) apply $\cos^2(A) = 1 - \sin^2(A)$

Step (3) sub $u = \sin A$

If $\sin^m A$, m is odd.

Step (1) split off a factor of $\sin A$

Step (2) apply $\sin^2(A) = 1 - \cos^2(A)$

Step (3) sub $u = \cos A$

Note: The identity is Pythagorean Identity

$$\sin^2(A) + \cos^2(A) = 1$$

Case (2) both m and n are even

If $\sin^m A \cos^n A$, m and n are even.

Step (1) apply $\sin^2(A) = \frac{1}{2}[1 - \cos(2A)]$

$$\cos^2(A) = \frac{1}{2}[1 + \cos(2A)]$$

Note: The identities are Half Angle Formulas

They are just some guidelines, there are cases where you might need to trial and error with the identities, integral by parts, and integration by substitutions

Example 1

$$\begin{aligned} & \int \sin^3 5x \, dx \\ &= \int \sin^2 5x \sin 5x \, dx \\ &= \int [1 - \cos^2 5x] \sin 5x \, dx \end{aligned}$$

Sub $u = \cos 5x$
 $du = -5 \sin 5x \, dx$
 $\frac{du}{-5} = \sin 5x \, dx$

$$\begin{aligned} &= \int [1 - u^2] \left[\frac{du}{-5} \right] \\ &= -\frac{1}{5} \int 1 - u^2 \, du \\ &= -\frac{1}{5} \left[u - \frac{u^3}{3} \right] + C \\ &= -\frac{1}{5} \cos 5x + \frac{1}{15} \cos^3 5x + C \end{aligned}$$

Example 2

$$\begin{aligned} & \int \cos^2 3x \sin^3 3x \, dx \\ &= \int \cos^2 3x \sin^2 3x \sin 3x \, dx \\ &= \int \cos^2 3x [1 - \cos^2 3x] \sin 3x \, dx \end{aligned}$$

Sub $u = \cos 3x$
 $du = -3 \sin 3x \, dx$
 $\frac{du}{-3} = \sin 3x \, dx$

$$= \int u^2 [1 - u^2] \left[-\frac{du}{3} \right]$$

$$\begin{aligned}
&= -\frac{1}{3} \int u^2 - u^4 du \\
&= -\frac{1}{3} \left[\frac{u^3}{3} - \frac{u^5}{5} \right] + C \\
&= -\frac{1}{9} u^3 + \frac{1}{15} u^5 + C \\
&= -\frac{1}{9} \cos^3 3x + \frac{1}{15} \cos^5 3x + C
\end{aligned}$$

Example 3

$$\begin{aligned}
&\int \sin^2 7x \cos^2 7x dx \\
&= \int \left(\frac{1}{2} [1 - \cos 14x] \right) \left(\frac{1}{2} [1 + \cos 14x] \right) dx \\
&= \frac{1}{4} \int [1 - \cos 14x][1 + \cos 14x] dx \\
&= \frac{1}{4} \int 1 - \cos^2 14x dx \\
&= \frac{1}{4} \int 1 - \left(\frac{1}{2} [1 + \cos 28x] \right) dx \\
&= \frac{1}{4} \int 1 - \left(\frac{1}{2} + \frac{1}{2} \cos 28x \right) dx \\
&= \frac{1}{4} \int \frac{1}{2} - \frac{1}{2} \cos 28x dx \\
&= \frac{1}{4} \left[\frac{1}{2} x - \frac{1}{56} \sin 28x \right] + C \\
&= \frac{1}{8} x - \frac{1}{224} \sin 28x + C
\end{aligned}$$

$$\int \tan^m A \sec^n A \, dx \quad \int \tan^m A \, dx \quad \int \sec^n A \, dx$$

Note: Angle A is in term of x when integral with respect to x . Change the symbol accordingly if integral with respect to θ .

Case (1) n is even

If $\sec^n A$, n is even

Step (1) split off a factor of $\sec^2 A$

Step (2) apply $\sec^2(A) = \tan^2(A) + 1$

Step (3) sub $u = \tan A$

Case (2) m is odd

If $\tan^m A$, m is odd.

Step (1) split off a factor of $\sec A \tan A$

Step (2) apply $\tan^2 A = \sec^2 A - 1$

Step (3) sub $u = \sec A$

Note: The identity is Pythagorean Identity

$$\sec^2(A) = \tan^2(A) + 1$$

Case (3) m is even and n is odd

If $\tan^m A \sec^n A$, m is even, n is odd.

Step (1) Reduce the integrand to powers of $\sec x$ alone,

Step (2) Use integration by parts if necessary

They are just some guidelines, there are cases where you might need to trial and error with the identities, integral by parts, and integration by substitutions

Example 1

$$\begin{aligned} & \int \tan^3 x \, dx \\ &= \int \tan^2 x \tan x \, dx \\ &= \int (\sec^2 x - 1) \tan x \, dx \\ &= \int (\tan x \sec^2 x - \tan x) \, dx \\ &= \int \tan x \sec^2 x \, dx - \int \tan x \, dx \end{aligned}$$

$$\begin{aligned} \text{Sub } & u = \tan x \\ & du = \sec^2 x \, dx \end{aligned}$$

$$\begin{aligned} &= \int u \, du - \ln |\sec x| + C \\ &= \frac{u^2}{2} - \ln |\sec x| + C \\ &= \frac{1}{2} \tan^2 x - \ln |\sec x| + C \end{aligned}$$

Example 1 (Alternative)

$$\begin{aligned} & \int \tan^3 x \, dx \\ &= \int \tan^2 x \tan x \, dx \\ &= \int (\sec^2 x - 1) \tan x \, dx \\ &= \int (\tan x \sec^2 x - \tan x) \, dx \\ &= \int \tan x \sec^2 x \, dx - \int \tan x \, dx \end{aligned}$$

Sub $u = \tan x$ $du = \sec^2 x dx$
--

$$\begin{aligned}
 &= \int u du - [-\ln |\cos x|] + C \\
 &= \frac{u^2}{2} + \ln |\cos x| + C \\
 &= \frac{1}{2} \tan^2 x + \ln |\cos x| + C
 \end{aligned}$$

Example 2

$$\begin{aligned}
 &\int \sec^3 5x dx \\
 &= \int \sec 5x \sec^2 5x dx
 \end{aligned}$$

$u = \sec 5x$	$dv = \sec^2 5x dx$
$du = 5 \sec 5x \tan 5x dx$	$v = \frac{\tan 5x}{5}$

$$\begin{aligned}
 &= [\sec 5x][\tan 5x] - \int \left[\frac{\tan 5x}{5} \right] [\sec 5x \tan 5x dx] \\
 &= \sec 5x \tan 5x - \frac{1}{5} \int \tan^2 5x \sec 5x dx \\
 &= \sec 5x \tan 5x - \frac{1}{5} \int [\sec^2 5x - 1] \sec 5x dx \\
 &= \sec 5x \tan 5x - \frac{1}{5} \int \sec^3 5x dx + \frac{1}{5} \int \sec 5x dx \\
 &= \sec 5x \tan 5x - \frac{1}{5} \int \sec^3 5x dx + \frac{1}{25} \ln |\sec 5x + \tan 5x|
 \end{aligned}$$

Since

$$\begin{aligned}
 \int \sec^3 5x dx &= \sec 5x \tan 5x - \frac{1}{5} \int \sec^3 5x dx + \frac{1}{25} \ln |\sec 5x + \tan 5x| \\
 \frac{6}{5} \int \sec^3 5x dx &= \sec 5x \tan 5x + \frac{1}{25} \ln |\sec 5x + \tan 5x| \\
 \int \sec^3 5x dx &= \frac{5}{6} \sec 5x \tan 5x + \frac{1}{30} \ln |\sec 5x + \tan 5x| + C
 \end{aligned}$$

7. Integration of Trigonometry Substitutions

This section presents trigonometric substitution as a method for evaluating integrals containing radical expressions. By exploiting geometric relationships and trigonometric identities, difficult algebraic integrals can be reduced to manageable trigonometric forms

Case (1)

$$\sqrt{a^2 - x^2}$$

Step (1) sub $x = a \sin \theta$

Step (2) apply $\cos^2 \theta = 1 - \sin^2 \theta$

Case (2)

$$\sqrt{a^2 + x^2}$$

Step (1) sub $x = a \tan \theta$

Step (2) apply $\sec^2 \theta = \tan^2 \theta + 1$

Case (3)

$$\sqrt{x^2 - a^2}$$

Step (1) sub $x = a \sec \theta$

Step (2) apply $\tan^2 \theta = \sec^2 \theta - 1$

Example 1

$$\int \frac{4x}{\sqrt{x^2 - 4}} dx$$

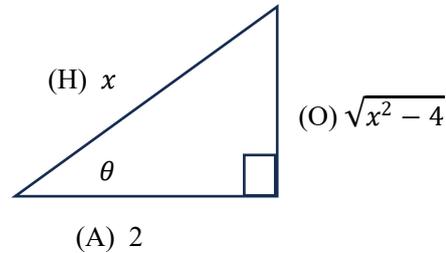
$$\text{Sub } x = 2 \sec \theta$$

$$dx = 2 \sec \theta \tan \theta d\theta$$

$$= \int \frac{4(2 \sec \theta)}{\sqrt{4 \sec^2 \theta - 4}} 2 \sec \theta \tan \theta d\theta$$

$$\begin{aligned}
&= \int \frac{16 \sec^2 \theta \tan \theta}{\sqrt{4 \sec^2 \theta - 4}} d\theta \\
&= \int \frac{16 \sec^2 \theta \tan \theta}{\sqrt{4(\sec^2 \theta - 1)}} d\theta \\
&= \int \frac{16 \sec^2 \theta \tan \theta}{\sqrt{4 \tan^2 \theta}} d\theta \\
&= \int \frac{16 \sec^2 \theta \tan \theta}{2 \tan \theta} d\theta \\
&= \int 8 \sec^2 \theta d\theta \\
&= 8 \tan \theta + C
\end{aligned}$$

<p>Since $x = 2 \sec \theta$</p> $\frac{x}{2} = \sec \theta$ $\frac{2}{x} = \cos \theta$



$$\begin{aligned}
&= 8 \frac{\sqrt{x^2 - 4}}{2} + C \\
&= 4\sqrt{x^2 - 4} + C
\end{aligned}$$

Example 2

Solve

$$\int \frac{\sqrt{25x^2 - 4}}{x} dx$$

by substituting $x = \frac{2}{5} \sec \theta$

$$\int \frac{\sqrt{25x^2 - 4}}{x} dx$$

<p>Sub $x = \frac{2}{5} \sec \theta$</p> $dx = \frac{2}{5} \sec \theta \tan \theta d\theta$
--

$$= \int 2 \tan \theta (\tan \theta d\theta)$$

$$= 2 \int \tan^2 \theta d\theta$$

$$= 2 \int \sec^2 \theta - 1 d\theta$$

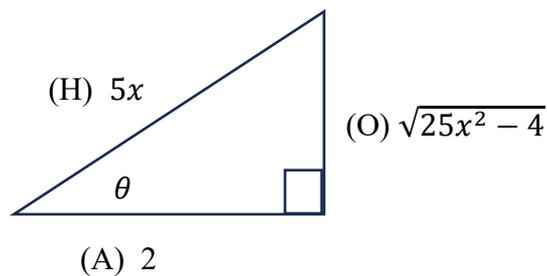
$$= 2[\tan \theta - \theta] + C$$

$$= 2 \tan \theta - 2\theta + C$$

Since $x = \frac{2}{5} \sec \theta$

$$\frac{5x}{2} = \sec \theta$$

$$\cos \theta = \frac{2}{5x}$$



Then $\tan \theta = \frac{\sqrt{25x^2 - 4}}{2}$ and $\theta = \tan^{-1}\left(\frac{\sqrt{25x^2 - 4}}{2}\right)$

$$= 2 \left[\frac{\sqrt{25x^2 - 4}}{2} \right] - 2 \left[\tan^{-1} \left(\frac{\sqrt{25x^2 - 4}}{2} \right) \right] + C$$

$$= \sqrt{25x^2 - 4} - 2 \left[\tan^{-1} \left(\frac{\sqrt{25x^2 - 4}}{2} \right) \right] + C$$

8. Integration of Improper Rational and Proper Rational Functions

This section focuses on integrating improper rational and proper functions. Techniques such as algebraic manipulation and decomposition are introduced to prepare integrals for evaluation.

8.1 Integration of Improper Rational Function

This subsection discusses improper rational functions and demonstrates how polynomial division is used to rewrite them as a sum of a polynomial and a proper rational function, simplifying the integration process.

$$\int \frac{P(x)}{Q(x)} dx$$

$$\deg(P(x)) \geq \deg(Q(x))$$

$$Eg: \int \frac{x^3 + x}{x - 1} dx, \int \frac{x^2 + 2}{x^2 + 3x + 2} dx$$

Step (1) Use Long Division before integral

Formula:

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

Example 1

$$\begin{aligned} & \int \frac{x^2 + 3x + 5}{x + 1} dx \\ &= \int x + 2 + \frac{3}{x + 1} dx \\ &= \frac{x^2}{2} + 2x + 3 \ln|x + 1| + C \end{aligned}$$

$$\begin{array}{r} \quad \quad \quad x + 2 \\ x + 1 \) \ x^2 + 3x + 5 \\ \underline{(-) \quad x^2 + x} \\ \quad \quad \quad 2x + 5 \\ \underline{(-) \quad \quad 2x + 2} \\ \quad \quad \quad 3 \end{array}$$

Example 2

$$\begin{aligned} & \int \frac{x^3 - 1}{x - 1} dx \\ &= \int x^2 + x + 1 dx \\ &= \frac{x^3}{3} + \frac{x^2}{2} + x + C \end{aligned}$$

Note:

$$\begin{array}{r} x^3 - 1 = x^3 + 0x^2 + 0x - 1 \\ \quad \quad \quad x^2 + x + 1 \\ x - 1 \overline{) x^3 + 0x^2 + 0x - 1} \\ (-) \quad \quad \quad x^3 - x^2 \\ \hline \quad \quad \quad x^2 + 0x \\ (-) \quad \quad \quad x^2 - x \\ \hline \quad \quad \quad x - 1 \\ (-) \quad \quad \quad x - 1 \\ \hline \quad \quad \quad \underline{\quad} \end{array}$$

8.2 Integration of Proper Rational Function: Q(x) cannot be factored

This subsection addresses the integration of proper rational functions whose denominators contain irreducible quadratic factors. Methods for expressing the integrand in suitable partial fraction forms are developed and applied.

$$\int \frac{P(x)}{Q(x)} dx$$

$$\deg(P(x)) < \deg(Q(x))$$

$$\text{Eg: } \int \frac{x}{x^2 + 9} dx, \int \frac{2}{x^2 - 2x - 3} dx$$

$x^2 + 9$ cannot be factored

$x^2 - 2x + 3$ cannot be factored

Ensure the denominator is in the form of $x^2 + a^2$, else use Completing the square to convert it

Formula:

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

Example 1

$$\begin{aligned} & \int \frac{2x + 9}{4x^2 + 9} dx \\ &= \int \frac{2x + 9}{4 \left[x^2 + \frac{9}{4} \right]} dx \\ &= \frac{1}{4} \int \frac{2x + 9}{\left[x^2 + \frac{9}{4} \right]} dx \end{aligned}$$

$$\begin{aligned}
&= \frac{1}{4} \int \frac{2x}{\left[x^2 + \frac{9}{4}\right]} dx + \frac{1}{4} \int \frac{9}{\left[x^2 + \frac{9}{4}\right]} dx \\
&= \frac{1}{4} \ln \left|x^2 + \frac{9}{4}\right| + \frac{9}{4} \int \frac{1}{\left[x^2 + \left(\frac{3}{2}\right)^2\right]} dx \\
&= \frac{1}{4} \ln \left|x^2 + \frac{9}{4}\right| + \frac{9}{4} \left[\frac{1}{\left(\frac{3}{2}\right)} \tan^{-1} \left(\frac{x}{\left(\frac{3}{2}\right)} \right) \right] + C \\
&= \frac{1}{4} \ln \left|x^2 + \frac{9}{4}\right| + \frac{9}{4} \left[\frac{2}{3} \tan^{-1} \left(\frac{2x}{3} \right) \right] + C \\
&= \frac{1}{4} \ln \left|x^2 + \frac{9}{4}\right| + \frac{3}{2} \tan^{-1} \left(\frac{2x}{3} \right) + C
\end{aligned}$$

Example 2

$$\int \frac{2x + 5}{x^2 + 2x + 4} dx$$

Completing the square

$$\begin{aligned}
x^2 + 2x + 4 &= x^2 + 2x + \left(\frac{2}{2}\right)^2 - \left(\frac{2}{2}\right)^2 + 4 \\
&= x^2 + 2x + (1)^2 - (1)^2 + 4 \\
&= (x + 1)^2 + 3
\end{aligned}$$

$$= \int \frac{2x + 5}{(x + 1)^2 + 3} dx$$

Sub $u = x + 1$ \Rightarrow $x = u - 1$ $du = dx$
--

$$= \int \frac{2(u - 1) + 5}{u^2 + 3} du$$

$$= \int \frac{2u + 3}{u^2 + 3} du$$

$$= \int \frac{2u}{u^2 + 3} du + \int \frac{3}{u^2 + 3} du$$

$$\begin{aligned} &= \ln |u^2 + 3| + 3 \int \frac{1}{u^2 + (\sqrt{3})^2} du \\ &= \ln |u^2 + 3| + 3 \left[\frac{1}{\sqrt{3}} \tan^{-1} \left(\frac{u}{\sqrt{3}} \right) \right] + C \\ &= \ln |(x + 1)^2 + 3| + \frac{3}{\sqrt{3}} \tan^{-1} \left(\frac{x + 1}{\sqrt{3}} \right) + C \\ &= \ln |x^2 + 2x + 4| + \frac{3}{\sqrt{3}} \tan^{-1} \left(\frac{x + 1}{\sqrt{3}} \right) + C \end{aligned}$$

8.3 Integration of Proper Rational Function: Q(x) can be factored

This subsection focuses on proper rational functions with factorizable denominators. The method of partial fractions is systematically applied to decompose the integrand into simpler terms that can be integrated directly.

$$\int \frac{P(x)}{Q(x)} dx$$

$$\deg(P(x)) < \deg(Q(x))$$

$$\text{Eg: } \int \frac{p(x)}{x^2 - 9} dx, \int \frac{P(x)}{x^2 - 2x - 3} dx$$

$$x^2 - 9 = (x + 3)(x - 3)$$

$$x^2 - 2x - 3 = (x - 3)(x + 1)$$

Use Decomposition Method

Formula:

$$\int \frac{1}{x^2 + a^2} dx = \frac{1}{a} \tan^{-1} \left(\frac{x}{a} \right) + C$$

$$\int \frac{f'(x)}{f(x)} dx = \ln|f(x)| + c$$

Decomposition of rational function

1) $Q(x)$ is a product of distinct linear factors.

$$\frac{P(x)}{Q(x)} = \frac{A}{ax + b} + \frac{B}{cx + d} + \frac{C}{ex + f} \dots$$

E.g.,

$$\frac{P(x)}{(x + 2)(2x - 5)x} = \frac{A}{(x + 2)} + \frac{B}{(2x - 5)} + \frac{C}{x}$$

2) $Q(x)$ is a product of repeated linear factors.

$$\frac{P(x)}{Q(x)} = \frac{A}{ax + b} + \frac{B}{(ax + b)^2} + \dots + \frac{W}{\gamma x + \omega}$$

E.g.,

$$\frac{P(x)}{(x + 2)(x + 2)x} = \frac{A}{(x + 2)} + \frac{B}{(x + 2)^2} + \frac{C}{x}$$

E.g.,

$$\frac{P(x)}{x^2(x - 3)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{(x - 3)}$$

3) $Q(x)$ contains irreducible quadratic factor

$$\frac{P(x)}{Q(x)} = \frac{A}{ax + b} + \frac{Bx + c}{cx^2 + dx + e} + \dots$$

E.g.,

$$\frac{P(x)}{(x + 2)(x^2 + 2x + 7)} = \frac{A}{(x + 2)} + \frac{Bx + C}{(x^2 + 2x + 7)}$$

4) $Q(x)$ contains combination of (1) to (3).

Example 1

$$\int \frac{x}{(x+2)^2} dx$$

Decomposition

$Q(x) = (x+2)^2$ has repeated factors

$$\frac{x}{(x+2)^2} = \frac{A}{(x+2)} + \frac{B}{(x+2)^2}$$

multiply $(x+2)^2$

$$x = A(x+2) + B$$

$$x = Ax + 2A + B$$

Compare LHS & RHS

$$x \quad 1 = A$$

$$\text{const} \quad 0 = 2A + B \text{----(1)}$$

Sub $A = 1$ into (1)

$$0 = 2(1) + B$$

$$B = -2$$

Then

$$\frac{x}{(x+2)^2} = \frac{1}{(x+2)} - \frac{2}{(x+2)^2}$$

$$= \int \frac{1}{(x+2)} dx - \int \frac{2}{(x+2)^2} dx$$

$$\text{Sub } u = (x+2) \\ du = dx$$

$$= \int \frac{1}{(x+2)} dx - \int \frac{2}{(u)^2} du$$

$$= \int \frac{1}{(x+2)} dx - 2 \int u^{-2} du$$

$$\begin{aligned}
&= \ln|x+2| - 2 \left[\frac{u^{-1}}{-1} \right] + C \\
&= \ln|x+2| + 2u^{-1} + C \\
&= \ln|x+2| + \frac{2}{u} + C \\
&= \ln|x+2| + \frac{2}{x+2} + C
\end{aligned}$$

Example 2

$$\int \frac{1-5x}{x^4+x^2} dx$$

Decomposition

$Q(x) = x^4 + x^2 = x^2(x^2 + 1)$ has repeated factors & irreducible quadratic

$$\frac{1-5x}{x^2(x^2+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

multiply $x^2(x^2+1)$

$$1-5x = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$$

$$1-5x = (Ax^3 + Ax) + (Bx^2 + B) + (Cx^3 + Dx^2)$$

Compare LHS & RHS

$$\begin{array}{l}
x^3 \quad 0 = A + C \\
C = -A \quad \text{---(1)}
\end{array}$$

$$\begin{array}{l}
x^2 \quad 0 = B + D \\
D = -B \quad \text{---(2)}
\end{array}$$

x

$$-5 = A$$

Sub $A = -5$ into (1)

$$C = -(-5) = 5$$

 $const$

$$1 = B$$

Sub $B = 1$ into (2)

$$D = -1$$

$$= \int \frac{-5}{x} dx + \int \frac{1}{x^2} dx + \int \frac{5x - 1}{x^2 + 1} dx$$

$$= -5 \ln x + \int x^{-2} dx + \int \frac{5x}{x^2 + 1} dx - \int \frac{1}{x^2 + 1} dx$$

$$= -5 \ln x + \left(\frac{x^{-1}}{-1} \right) + 5 \left(\frac{1}{2} \right) \int \frac{(2)x}{x^2 + 1} dx - \int \frac{1}{x^2 + 1^2} dx$$

$$= -5 \ln x - \frac{1}{x} + \frac{5}{2} \int \frac{2x}{x^2 + 1} dx - \frac{1}{1} \tan^{-1} \left(\frac{x}{1} \right) + C$$

$$= -5 \ln x - \frac{1}{x} + \frac{5}{2} \ln |x^2 + 1| - \tan^{-1}(x) + C$$

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Integration is one of the cornerstones of calculus and a critical tool across mathematics, science, and engineering. However, integration is not always straightforward, many integrals cannot be solved by simple formulas, and systematic techniques are required to handle a wide variety of functions.

This book is designed to guide the reader to understand integration methods in a clear and logical manner. Each chapter is structured to build intuition and problem-solving skills, with examples progressing from simple to challenging.

This book is intended for students, instructors, and self-learners who wish to master the techniques of integration comprehensively. By the end of the text, readers will be able to apply the integration methods with confidence.



Universiti Teknologi MARA Cawangan Johor
Kampus Pasir Gudang
Jalan Purnama, Bandar Seri Alam
81750 Masai

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