MATRIX ALGEBRA AND VECTOR ALGEBRA

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PREFACE

Matrix Algebra and Vector Algebra is an academic book that covers a series of topics in mathematics such as operations on matrices, determinants, inverse matrices, vector in two- and three-dimensional space. With the exposure of these basic concepts in matrix and vector algebra, students would be able to understand and apply the knowledge in solving system of linear equations using matrix and solving simple geometric applications.

The concept of each topic is explained in the beginning of each chapter. Examples are given to illustrate the concept. Detailed steps are shown on how to solve the problems. This module is suitable to be used during lecture session to support teaching and learning processes.

CHAPTER 1

Matrix Algebra

Outline:

1.1 Definition

A **matrix** is a rectangular array of numbers arranged in rows and columns. (plural of matrix: matrices)

Example:

$$\boldsymbol{A} = \begin{array}{cc} \begin{bmatrix} \mathbf{2} & \mathbf{3} & \mathbf{-4} \\ \mathbf{1} & \mathbf{0} & \mathbf{1} \end{bmatrix} & \begin{array}{c} R_1 \\ R_2 \end{array} \\ \begin{array}{ccc} C_1 & C_2 & C_3 \end{array} & \end{array}$$

→ The matrix $\boldsymbol{A}$ has 2 rows (horizontal lines) and 3 columns (vertical lines). So, the **size** of the matrix (the **order** of the matrix) is 2×3 (read “two by three”).

→ The element of the matrix is denoted by a_{ij}, where a_{ij} is located at $\boldsymbol{i^{th}}$ row and $\boldsymbol{j^{th}}$ column.

$$\boldsymbol{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix}$$

1.2 Special Matrices

	Special Matrices	**Example**
1.	Square Matrix ($n \times n$ Matrix)	$\begin{bmatrix} 0 & -1 \\ 2 & 4 \end{bmatrix}$ $\begin{bmatrix} 1 & 0 & 5 \\ 2 & -1 & 3 \\ 4 & 2 & 3 \end{bmatrix}$ (2×2 Matrix) (3×3 Matrix)
2.	Column Matrix (or Column vector) ($m \times 1$ Matrix)	$\begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$ (3×1 Matrix)
3.	Row Matrix (or Row vector) ($1 \times n$ Matrix)	$\begin{bmatrix} 2 & 6 \end{bmatrix}$ (1×2 Matrix)
4.	Null Matrix/ Zero Matrix (All elements are 0)	$\boldsymbol{O} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ $\boldsymbol{O} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
5.	Diagonal Matrix (Elements outside of main diagonal are 0)	$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -4 & 0 \\ 0 & 0 & 1 \end{bmatrix}$
6.	Augmented Matrix	Given $\boldsymbol{A} = \begin{bmatrix} 2 & 0 \\ 1 & 3 \end{bmatrix}$ and $\boldsymbol{B} = \begin{bmatrix} 7 \\ 1 \end{bmatrix}$ Augmented Matrix: $(\boldsymbol{A} \mid \boldsymbol{B}) = \left(\begin{array}{cc\|c} 2 & 0 & 7 \\ 1 & 3 & 1 \end{array}\right)$
7.	Identity Matrix (Elements on main diagonal are 1, outside of main diagonal are 0)	$\boldsymbol{I} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\boldsymbol{I} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Note: $\boldsymbol{AI} = \boldsymbol{A}$ and $\boldsymbol{IA} = \boldsymbol{A}$
8.	Triangular Matrix	Upper Triangular Matrix: $\begin{bmatrix} 2 & -1 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 6 \end{bmatrix}$ Lower Triangular Matrix: $\begin{bmatrix} 1 & 0 & 0 \\ 2 & 5 & 0 \\ 3 & 4 & -1 \end{bmatrix}$
9.	Symmetric Matrix	$\begin{bmatrix} 1 & 4 & -1 \\ 4 & 2 & 5 \\ -1 & 5 & 3 \end{bmatrix}$

1.3 Operations on Matrices

Operations on Matrices

(a) Equality of Matrices

Two matrices are equal if they have the same size and their corresponding entries are equal.

Example 1: Given $A = \begin{bmatrix} x-2 & 0 \\ 3 & y-x \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 3 & 5 \end{bmatrix}$.

Find x and y if $A = B$.

Solution:

$$x - 2 = 1 \qquad\qquad y - x = 5$$

$$x = 3 \qquad\qquad y - 3 = 5$$

$$y = 8$$

(b) Scalar Multiplication

cA, where $c = scalar$

Example 2: Given $A = \begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$. Find $2A$ and $-3A$.

Solution:

$$2A = 2\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix} \qquad\qquad -3A = -3\begin{bmatrix} -1 & 3 \\ 2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 & 6 \\ 4 & 0 \end{bmatrix} \qquad\qquad = \begin{bmatrix} 3 & -9 \\ -6 & 0 \end{bmatrix}$$

(c) Addition and Subtraction

Example 3: Given $A = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$.

Find $A + B$ and $A - 2B$.

Solution:

$$A + B = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 3 & 2 \\ -2 & 3 \end{bmatrix}$$

$$A - 2B = \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} - 2\begin{bmatrix} 2 & 1 \\ 0 & 3 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} 4 & 2 \\ 0 & 6 \end{bmatrix}$$

$$= \begin{bmatrix} -3 & -1 \\ -2 & -6 \end{bmatrix}$$

Operations on Matrices

(d) Matrix Multiplication

Example 4: Given $A = \begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}$, $B = \begin{bmatrix}0 & 3 & -1\\1 & 2 & 0\end{bmatrix}$ and $C = \begin{bmatrix}2\\1\end{bmatrix}$. Find AB and BC .

Solution:

$\boxed{(2 \times 3)}$

2=2

$$AB = \overset{(2\times 2)}{\begin{bmatrix}2 & 1\\0 & 3\end{bmatrix}} \overset{(2\times 3)}{\begin{bmatrix}0 & 3 & -1\\1 & 2 & 0\end{bmatrix}}$$

$$= \begin{bmatrix}2(0)+1(1) & 2(3)+1(2) & 2(-1)+1(0)\\0(0)+3(1) & 0(3)+3(2) & 0(-1)+3(0)\end{bmatrix} = \begin{bmatrix}1 & 8 & -2\\3 & 6 & 0\end{bmatrix}$$

$\boxed{(2 \times 3)}$

3≠2

$$BC = \overset{(2\times 3)}{\begin{bmatrix}0 & 3 & -1\\1 & 2 & 0\end{bmatrix}} \overset{(2\times 1)}{\begin{bmatrix}2\\1\end{bmatrix}}$$

$$= \text{undefined}$$

(e) Transpose of Matrix

Transpose of $A = A^T$

Example 5: Given $A = \begin{bmatrix}2 & 0 & 1\\3 & 4 & 0\end{bmatrix}$. Find A^T .

A is a 2×3 matrix

Solution:

$$A^T = \begin{bmatrix}2 & 0 & 1\\3 & 4 & 0\end{bmatrix}^T$$

$$= \begin{bmatrix}2 & 3\\0 & 4\\1 & 0\end{bmatrix}$$

A^T is a 3×2 matrix

Properties of scalar multiplication, addition and subtraction of matrices

1) $\boldsymbol{A}+\boldsymbol{B}=\boldsymbol{B}+\boldsymbol{A}$
2) $(\boldsymbol{A}+\boldsymbol{B})+\boldsymbol{C}=\boldsymbol{A}+(\boldsymbol{B}+\boldsymbol{C})$
3) $\boldsymbol{A}+\boldsymbol{O}=\boldsymbol{O}+\boldsymbol{A}=\boldsymbol{A}$
4) $k(\boldsymbol{A}+\boldsymbol{B})=k\boldsymbol{A}+k\boldsymbol{B}$, k is scalar
5) $(p+q)\boldsymbol{A}=p\boldsymbol{A}+q\boldsymbol{A}$, p and q are scalar
6) $\boldsymbol{A}+(-\boldsymbol{A})=\boldsymbol{O}$
7) $(0)\boldsymbol{A}=\boldsymbol{O}$, $(-1)\boldsymbol{A}=-\boldsymbol{A}$
8) $p(q\boldsymbol{A})=(pq)\boldsymbol{A}$, p and q are scalar

Properties of matrix multiplication

1) $\boldsymbol{AB}\neq\boldsymbol{BA}$
2) $(\boldsymbol{AB})\boldsymbol{C}=\boldsymbol{A}(\boldsymbol{BC})$
3) $(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{C}=\boldsymbol{AC}+\boldsymbol{BC}$
4) $\boldsymbol{C}(\boldsymbol{A}+\boldsymbol{B})=\boldsymbol{CA}+\boldsymbol{CB}$
5) $\boldsymbol{OA}=\boldsymbol{O}$
6) $\boldsymbol{AB}=\boldsymbol{O}$ does not imply that $\boldsymbol{A}=\boldsymbol{O}$ or $\boldsymbol{B}=\boldsymbol{O}$
7) $\boldsymbol{AB}=\boldsymbol{AC}$ does not imply that $\boldsymbol{B}=\boldsymbol{C}$

Properties of transpose of matrix

1) $(\boldsymbol{A}^{\boldsymbol{T}})^{\boldsymbol{T}}=\boldsymbol{A}$
2) $(k\boldsymbol{A})^{\boldsymbol{T}}=k\,\boldsymbol{A}^{\boldsymbol{T}}$, k is scalar
3) $(\boldsymbol{A}+\boldsymbol{B})^{\boldsymbol{T}}=\boldsymbol{A}^{\boldsymbol{T}}+\boldsymbol{B}^{\boldsymbol{T}}$
4) $(\boldsymbol{AB})^{\boldsymbol{T}}=\boldsymbol{B}^{\boldsymbol{T}}\boldsymbol{A}^{\boldsymbol{T}}$
5) $(\boldsymbol{AB})^{\boldsymbol{T}}\neq\boldsymbol{A}^{\boldsymbol{T}}\boldsymbol{B}^{\boldsymbol{T}}$

1.4 Determinants

The determinant is a number associated with a square matrix.

Determinant of a 2 x 2 matrix

Given a matrix A: $\boldsymbol{A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}}$

Determinant of A: $|\boldsymbol{A}| = \begin{vmatrix} \boldsymbol{a} & \boldsymbol{b} \\ \boldsymbol{c} & \boldsymbol{d} \end{vmatrix} = \boldsymbol{ad - bc}$

Note: We may write $|\boldsymbol{A}|$ or $\boldsymbol{det}(\boldsymbol{A})$ to represent determinant of $\boldsymbol{A}$ **.**

Example 1: Given matrix $A = \begin{bmatrix} -2 & 1 \\ 4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$.
Find the determinant of A and B.

Solution:

$$|\boldsymbol{A}| = \begin{vmatrix} -2 & 1 \\ 4 & 3 \end{vmatrix} = \boldsymbol{ad - bc}$$

$$= (-2)(3) - (1)(4)$$

$$= -10$$

$$|\boldsymbol{B}| = \begin{vmatrix} 1 & 2 \\ 2 & 4 \end{vmatrix} = \boldsymbol{ad - bc}$$

$$= (1)(4) - (2)(2)$$

$$= 0$$

Determinant of a 3 x 3 matrix

Finding determinant of a 3 x 3 matrix

- **Method (1): Sarrus' rule**
- **Method (2): Laplace expansion**

Method (1): Sarrus’ rule

Example 1: Given matrix $A = \begin{bmatrix}2 & 4 & 6\\0 & 2 & 7\\1 & 0 & 2\end{bmatrix}$. Find determinant of A using <u>Sarrus’ rule</u>.

Solution:

Determinant of A:

$$|A| = \begin{vmatrix}2 & 4 & 6\\0 & 2 & 7\\1 & 0 & 2\end{vmatrix}\begin{matrix}2 & 4\\0 & 2\\1 & 0\end{matrix}$$

Copy the first and second column to the right

$$= (2 \times 2 \times 2) + (4 \times 7 \times 1) + (6 \times 0 \times 0)$$
$$-(1 \times 2 \times 6) - (0 \times 7 \times 2) - (2 \times 0 \times 4)$$

$$= 24$$

Method (2): Laplace expansion

Example 2: Given matrix $A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 7 \\ 1 & 0 & 2 \end{bmatrix}$. Find determinant of A using <u>Laplace expansion</u>.

Solution:

Determinant of A:

$$|A| = \begin{vmatrix} 2 & 4 & 6 \\ 0 & 2 & 7 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= +2\begin{vmatrix} 2 & 7 \\ 0 & 2 \end{vmatrix} - 4\begin{vmatrix} 0 & 7 \\ 1 & 2 \end{vmatrix} + 6\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix}$$

$$= 2(4-0) - 4(0-7) + 6(0-2)$$

$$= 24$$

Tips (Laplace expansion):
We may use any row or column.

Note: $\begin{bmatrix} \boxed{+ \quad - \quad +} \\ - \quad + \quad - \\ + \quad - \quad + \end{bmatrix}$ $\leftarrow$ Use Row 1

Solution (Alternative):

Determinant of A:

$$|A| = \begin{vmatrix} 2 & 4 & 6 \\ 0 & 2 & 7 \\ 1 & 0 & 2 \end{vmatrix}$$

$$= -4\begin{vmatrix} 0 & 7 \\ 1 & 2 \end{vmatrix} + 2\begin{vmatrix} 2 & 6 \\ 1 & 2 \end{vmatrix} - 0\begin{vmatrix} 2 & 6 \\ 0 & 7 \end{vmatrix}$$

$$= -4(0-7) + 2(4-6) - 0$$

$$= 24$$

Tips (Laplace expansion):
We may use any row or column.

Note: $\begin{bmatrix} + & \boxed{\begin{matrix} - \\ + \\ - \end{matrix}} & + \\ - & & - \\ + & & + \end{bmatrix}$ Use Column 2

Example 3: Find the determinant of Triangular Matrix below.

(a) $\mathbf{A} = \begin{bmatrix} \mathbf{3} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} & -\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{bmatrix}$ (Upper Triangular Matrix)

(b) $\mathbf{B} = \begin{bmatrix} \mathbf{4} & \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{1} & \mathbf{0} \\ -\mathbf{1} & \mathbf{2} & -\mathbf{2} \end{bmatrix}$ (Lower Triangular Matrix)

Solution:

(a) Determinant of A:

Tips (Laplace expansion):
We may use any row or column.

Note: $\begin{bmatrix} \boxed{\begin{matrix} + \\ - \\ + \end{matrix}} & \begin{matrix} - \\ + \\ - \end{matrix} & \begin{matrix} + \\ - \\ + \end{matrix} \end{bmatrix}$ Use Column 1

$$|\boldsymbol{A}| = \begin{vmatrix} \mathbf{3} & \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{2} & -\mathbf{1} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} \end{vmatrix}$$

$$= +\mathbf{3}\begin{vmatrix} \mathbf{2} & -\mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{vmatrix} - \mathbf{0}\begin{vmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{0} & \mathbf{1} \end{vmatrix} + \mathbf{0}\begin{vmatrix} \mathbf{2} & \mathbf{1} \\ \mathbf{2} & -\mathbf{1} \end{vmatrix}$$

$$= 3(2 \times 1 - 0) - 0 + 0$$

$$= 3 \times 2 \times 1$$

$$= \mathbf{6}$$

Determinant of Triangular Matrix
= product of the elements on main diagonal
= $3 \times 2 \times 1$

Solution:

(b) Determinant of B:

$$|\boldsymbol{B}| = \begin{vmatrix} \mathbf{4} & \mathbf{0} & \mathbf{0} \\ \mathbf{3} & \mathbf{1} & \mathbf{0} \\ -\mathbf{1} & \mathbf{2} & -\mathbf{2} \end{vmatrix}$$

$$= 4 \times 1 \times (-2)$$

$$= -\mathbf{8}$$

Determinant of Triangular Matrix
= product of the elements on main diagonal
= $4 \times 1 \times (-2)$

Properties of Determinants (Part 1)

Properties of Determinants (Part 1):

1) $|A^T| = |A|$

2) $|kA| = k^{n}|A| \quad (A \text{ is } n \times n \text{ matrix})$

3) $|AB| = |A|\,|B|$

4) $|A^n| = |A|^n$
$|A^2| = |A|^2$
$|A^3| = |A|^3$

5) $|A^{-1}| = \frac{1}{|A|}$

Example 1:
Given both matrix A and B are 3×3 matrices, where $|A| = 6$ and $|B| = 2$.
Use the properties of determinants to find
(a) $|2A|$
(b) $|4AB|$
(c) $|(A^2B)^T|$
(d) $|A^{-1}B|$

Solution:

(a) $|2A| = 2^3|A|$

$= 8(6)$

$= 48$

Properties:
$|kA| = k^n|A| \qquad (A \text{ is } n \times n \text{ matrix})$

(Since A is 3×3 matrix, $n = 3$)

Solution:

(b) $|4AB| = 4^3|A||B|$

$= 64(6)(2)$

$= 768$

Properties:
$|kA| = k^n|A| \qquad (A \text{ is } n \times n \text{ matrix})$

(Since A is 3×3 matrix, $n = 3$)

Properties:
$|AB| = |A|\,|B|$

Solution:

(c) $\left|(A^2B)^T\right| = \left|A^2B\right|$

Properties:
$|A^T| = |A|$

$= \left|A^2\right||B|$

$= |A|^2|B|$

Properties:
$|A^n| = |A|^n$
$\left|A^2\right| = |A|^2$

$= 6^2(2)$

$= 72$

Solution:

(d) $\left|A^{-1}B\right| = \left|A^{-1}\right||B|$

$= \frac{1}{|A|}|B|$

Properties:
$\left|A^{-1}\right| = \frac{1}{|A|}$

$= \frac{1}{6}(2)$

$= \frac{1}{3}$

Properties of Determinants (Part 2)

Properties of Determinants (Part 2):

1) If we interchange two rows (or two columns), we reverse the sign of the determinant

$\begin{vmatrix} c & d \\ a & b \end{vmatrix} \quad (R_1 \leftrightarrow R_2)$ $= -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$	$\begin{vmatrix} b & a \\ d & c \end{vmatrix} \quad (C_1 \leftrightarrow C_2)$ $= -\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

2) If we multiply one row (or one column) of a matrix by k, the determinant is multiplied by k

$\begin{vmatrix} ka & kb \\ c & d \end{vmatrix} = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$	$\begin{vmatrix} ka & b \\ kc & d \end{vmatrix} = k\begin{vmatrix} a & b \\ c & d \end{vmatrix}$

3) If two rows (or two columns) of a matrix are equal, the determinant is zero

$\begin{vmatrix} a & b \\ a & b \end{vmatrix} = 0$	$\begin{vmatrix} a & a \\ c & c \end{vmatrix} = 0$

4) If a matrix has a row (or a column) that is all zero, the determinant is zero

$\begin{vmatrix} a & b \\ 0 & 0 \end{vmatrix} = 0$	$\begin{vmatrix} a & 0 \\ c & 0 \end{vmatrix} = 0$

5) The determinant behaves like a linear function on the rows (or columns) of the matrix

$\begin{vmatrix} a & b \\ c+w & d+u \end{vmatrix}$ $= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & b \\ w & u \end{vmatrix}$	$\begin{vmatrix} a & b+w \\ c & d+u \end{vmatrix}$ $= \begin{vmatrix} a & b \\ c & d \end{vmatrix} + \begin{vmatrix} a & w \\ c & u \end{vmatrix}$

6) Adding or subtracting k times row i (or column i) from row j (or column j) doesn't change the determinant

$\begin{vmatrix} a & b \\ c \pm ka & d \pm kb \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$	$\begin{vmatrix} a \pm kb & b \\ c \pm kd & d \end{vmatrix} = \begin{vmatrix} a & b \\ c & d \end{vmatrix}$

Proof of properties #6:

$$\begin{aligned}
\begin{vmatrix} a & b \\ c \pm ka & d \pm kb \end{vmatrix} &= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \pm \begin{vmatrix} a & b \\ ka & kb \end{vmatrix} \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \pm k\begin{vmatrix} a & b \\ a & b \end{vmatrix} \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix} \pm k(0) \\
&= \begin{vmatrix} a & b \\ c & d \end{vmatrix}
\end{aligned}$$

Example 1:

Given $|A| = \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} = 6.$

Use the properties of determinants to find

(a) $\begin{vmatrix} -2a & -2b & -2c \\ d & e & f \\ 5g-d & 5h-e & 5i-f \end{vmatrix}$

(b) $\begin{vmatrix} d & e & f \\ a & b & c \\ a+3g & b+3h & c+3i \end{vmatrix}$

(c) $\begin{vmatrix} b & c & a \\ e & f & d \\ h & i & g \end{vmatrix}$

Solution:

(a) $\begin{vmatrix} -2a & -2b & -2c \\ d & e & f \\ 5g-d & 5h-e & 5i-f \end{vmatrix}$

$$= \begin{vmatrix} -2a & -2b & -2c \\ d & e & f \\ 5g & 5h & 5i \end{vmatrix} - \begin{vmatrix} -2a & -2b & -2c \\ d & e & f \\ d & e & f \end{vmatrix}$$

two rows are equal, the determinant is zero

$$= (-2)(5)\begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix} - 0$$

$$= (-2)(5)(6)$$

$$= -60$$

Solution:

(b) $\begin{vmatrix} d & e & f \\ a & b & c \\ a+3g & b+3h & c+3i \end{vmatrix}$

two rows are equal, the determinant is zero

$= \begin{vmatrix} d & e & f \\ a & b & c \\ a & b & c \end{vmatrix} + \begin{vmatrix} d & e & f \\ a & b & c \\ 3g & 3h & 3i \end{vmatrix}$

$= 0 + (3) \begin{vmatrix} d & e & f \\ a & b & c \\ g & h & i \end{vmatrix}$

Interchange two rows, $R_1 \leftrightarrow R_2$

$= (3)(-1) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$= (3)(-1)(6)$

$= -18$

Solution:

(c) $\begin{vmatrix} b & c & a \\ e & f & d \\ h & i & g \end{vmatrix}$

Interchange two columns, $C_1 \leftrightarrow C_3$

$= (-1) \begin{vmatrix} a & c & b \\ d & f & e \\ g & i & h \end{vmatrix}$

Interchange two columns, $C_2 \leftrightarrow C_3$

$= (-1)(-1) \begin{vmatrix} a & b & c \\ d & e & f \\ g & h & i \end{vmatrix}$

$= (-1)(-1)(6)$

$= 6$

1.5 Inverse Matrices

Inverse of a 2 x 2 matrix

Given a matrix A: $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

Inverse of matrix A: $A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$ **where** $|A| = ad - bc$

Note: The matrix A is invertible when the determinant, $|A| \neq 0$.

Example 1: Given matrix $A = \begin{bmatrix} 4 & 1 \\ 2 & 0 \end{bmatrix}$.

(a) Find inverse of A.

(b) Find matrix M if $AM = \begin{bmatrix} 2 & 0 \\ 1 & -6 \end{bmatrix}$.

(a) Find inverse of A.

Solution:

$$A^{-1} = \frac{1}{|A|}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} \quad \text{where } |A| = 4(0) - 1(2) = -2$$

$$= \frac{1}{-2}\begin{bmatrix} 0 & -1 \\ -2 & 4 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -2 \end{bmatrix}$$

(b) Find matrix M if $AM = \begin{bmatrix} 2 & 0 \\ 1 & -6 \end{bmatrix}$.

Solution:

Properties of Inverse of Matrix:

$A^{-1}A = I$

Note:

$IM = M$

$$AM = \begin{bmatrix} 2 & 0 \\ 1 & -6 \end{bmatrix}$$

$$A^{-1}AM = A^{-1}\begin{bmatrix} 2 & 0 \\ 1 & -6 \end{bmatrix}$$

Multiply A^{-1} on both sides

$$IM = A^{-1}\begin{bmatrix} 2 & 0 \\ 1 & -6 \end{bmatrix}$$

$$M = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -2 \end{bmatrix}\begin{bmatrix} 2 & 0 \\ 1 & -6 \end{bmatrix}$$

Substitute:

$A^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ 1 & -2 \end{bmatrix}$

$$M = \begin{bmatrix} \frac{1}{2} & -3 \\ 0 & 12 \end{bmatrix}$$

Properties of Inverse of Matrix:

1) $(\boldsymbol{AB})^{-1} = \boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$
   $(\boldsymbol{ABC})^{-1} = \boldsymbol{C}^{-1}\boldsymbol{B}^{-1}\boldsymbol{A}^{-1}$

2) $(\boldsymbol{A}^{-1})^{-1} = \boldsymbol{A}$

3) $(\boldsymbol{cA})^{-1} = \frac{\mathbf{1}}{\boldsymbol{c}}\boldsymbol{A}^{-1}$

4) $(\boldsymbol{A}^{\boldsymbol{T}})^{-1} = (\boldsymbol{A}^{-1})^{\boldsymbol{T}}$

5) If $\boldsymbol{AB} = \boldsymbol{I}$ and $\boldsymbol{BA} = \boldsymbol{I}$, then $\boldsymbol{B} = \boldsymbol{A}^{-1}$.

6) $\boldsymbol{A}\,\boldsymbol{A}^{-1} = \boldsymbol{I}$
   $\boldsymbol{A}^{-1}\boldsymbol{A} = \boldsymbol{I}$

7) If $\boldsymbol{A}$ is a square matrix and $|\boldsymbol{A}| \neq \mathbf{0}$, then $\boldsymbol{A}$ has an inverse.
   $\boldsymbol{A}$ is called a **nonsingular matrix** or invertible matrix.

8) If $\boldsymbol{A}$ is a square matrix and $|\boldsymbol{A}| = \mathbf{0}$, then $\boldsymbol{A}$ does not have an inverse.
   $\boldsymbol{A}$ is called a **singular matrix**.

9) If $\boldsymbol{A}$ is a non-square matrix,
   $|\boldsymbol{A}|$ does not exist.
   $\boldsymbol{A}$ does not have an inverse.

Inverse of a 3 x 3 matrix

- **Finding inverse of a 3 x 3 matrix**
  - **Method (1): Adjoint Method**
  - **Method (2): Elementary Row Operations (ERO)**

Method (1): Adjoint Method

Example 1: Given matrix $A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 7 \\ 1 & 0 & 2 \end{bmatrix}$. Find inverse of A using <u>Adjoint Method</u>.

Solution:

Determinant of A:

$$|A| = \begin{vmatrix} 2 & 4 & 6 \\ 0 & 2 & 7 \\ 1 & 0 & 2 \end{vmatrix}$$

Use Sarrus' rule or Laplace expansion to find determinant

$$= (2 \times 2 \times 2) + (4 \times 7 \times 1) + (6 \times 0 \times 0) \\ -(1 \times 2 \times 6) - (0 \times 7 \times 2) - (2 \times 0 \times 4)$$

$$= 24$$

Cofactor of A:

$$cof(A) = \begin{bmatrix} +\begin{vmatrix} 2 & 7 \\ 0 & 2 \end{vmatrix} & -\begin{vmatrix} 0 & 7 \\ 1 & 2 \end{vmatrix} & +\begin{vmatrix} 0 & 2 \\ 1 & 0 \end{vmatrix} \\ -\begin{vmatrix} 4 & 6 \\ 0 & 2 \end{vmatrix} & +\begin{vmatrix} 2 & 6 \\ 1 & 2 \end{vmatrix} & -\begin{vmatrix} 2 & 4 \\ 1 & 0 \end{vmatrix} \\ +\begin{vmatrix} 4 & 6 \\ 2 & 7 \end{vmatrix} & -\begin{vmatrix} 2 & 6 \\ 0 & 7 \end{vmatrix} & +\begin{vmatrix} 2 & 4 \\ 0 & 2 \end{vmatrix} \end{bmatrix}$$

Note 1: $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$

Note 2: $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$

$$= \begin{bmatrix} 4 & 7 & -2 \\ -8 & -2 & 4 \\ 16 & -14 & 4 \end{bmatrix}$$

Adjoint of A:

$$\boldsymbol{adj(A) = [\ cof(A)\ ]^T}$$

$$= \begin{bmatrix} \mathbf{4} & \mathbf{7} & \mathbf{-2} \\ \mathbf{-8} & \mathbf{-2} & \mathbf{4} \\ \mathbf{16} & \mathbf{-14} & \mathbf{4} \end{bmatrix}^{\boldsymbol{T}}$$

$$= \begin{bmatrix} \mathbf{4} & \mathbf{-8} & \mathbf{16} \\ \mathbf{7} & \mathbf{-2} & \mathbf{-14} \\ \mathbf{-2} & \mathbf{4} & \mathbf{4} \end{bmatrix}$$

Inverse of A:

$$\boldsymbol{A^{-1} = \frac{1}{|A|} adj(A)}$$

$$= \frac{\mathbf{1}}{\mathbf{24}} \begin{bmatrix} \mathbf{4} & \mathbf{-8} & \mathbf{16} \\ \mathbf{7} & \mathbf{-2} & \mathbf{-14} \\ \mathbf{-2} & \mathbf{4} & \mathbf{4} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\mathbf{1}}{\mathbf{6}} & -\frac{\mathbf{1}}{\mathbf{3}} & \frac{\mathbf{2}}{\mathbf{3}} \\ \frac{\mathbf{7}}{\mathbf{24}} & -\frac{\mathbf{1}}{\mathbf{12}} & -\frac{\mathbf{7}}{\mathbf{12}} \\ -\frac{\mathbf{1}}{\mathbf{12}} & \frac{\mathbf{1}}{\mathbf{6}} & \frac{\mathbf{1}}{\mathbf{6}} \end{bmatrix}$$

Method (2): Elementary Row Operations (ERO)

Rules of ERO:

1) Interchange two rows (example: $R_1 \leftrightarrow R_3$)

2) Multiply a row by a non-zero constant k (example: $R_3 : kR_3$)

3) Replace a row by the sum of itself and a constant multiple of another row (example: $R_3 : R_3 \pm kR_1$)

To find inverse of A using ERO:

→ Transform $[\,A\mid I\,]$ to $[\,I\mid A^{-1}\,]$

Example 1: Given matrix $A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 7 \\ 1 & 0 & 2 \end{bmatrix}$. Find inverse of A using <u>Elementary Row Operations (ERO).</u>

Solution:

Write in $[\,A\mid I\,]$

$$[\,A\mid I\,] = \left(\begin{array}{ccc|ccc} 2 & 4 & 6 & 1 & 0 & 0 \\ 0 & 2 & 7 & 0 & 1 & 0 \\ 1 & 0 & 2 & 0 & 0 & 1 \end{array}\right)$$

Start, get '1' here.
(interchange Row 1 and Row 3)

$$R_1 \leftrightarrow R_3 \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 2 & 7 & 0 & 1 & 0 \\ 2 & 4 & 6 & 1 & 0 & 0 \end{array}\right)$$

Get first '0' here.
Get second '0' here.

$$R_3 : R_3 - 2R_1 \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 2 & 7 & 0 & 1 & 0 \\ 0 & 4 & 2 & 1 & 0 & -2 \end{array}\right)$$

Get third '0' here

$$R_3 : R_3 - 2R_2 \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 2 & 7 & 0 & 1 & 0 \\ 0 & 0 & -12 & 1 & -2 & -2 \end{array}\right)$$

Get '1' here

$$R_3 : -\frac{1}{12}R_3 \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 2 & 7 & 0 & 1 & 0 \\ 0 & 0 & 1 & -\frac{1}{12} & \frac{1}{6} & \frac{1}{6} \end{array}\right)$$

Get fourth '0' here

$$R_2 : R_2 - 7R_3 \left(\begin{array}{ccc|ccc} 1 & 0 & 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & \frac{7}{12} & -\frac{1}{6} & -\frac{7}{6} \\ 0 & 0 & 1 & -\frac{1}{12} & \frac{1}{6} & \frac{1}{6} \end{array}\right)$$

Get fifth ‘0’ here

$$\boldsymbol{R_1: R_1 - 2R_3}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 2 & 0 & \frac{7}{12} & -\frac{1}{6} & -\frac{7}{6} \\ 0 & 0 & 1 & -\frac{1}{12} & \frac{1}{6} & \frac{1}{6} \end{array}\right)$$

Get ‘1’ here.
Get sixth ‘0’ here.

$$\boldsymbol{R_2: \frac{1}{2}R_2}\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{1}{6} & -\frac{1}{3} & \frac{2}{3} \\ 0 & 1 & 0 & \frac{7}{24} & -\frac{1}{12} & -\frac{7}{12} \\ 0 & 0 & 1 & -\frac{1}{12} & \frac{1}{6} & \frac{1}{6} \end{array}\right) = [\,\boldsymbol{I} \mid \boldsymbol{A^{-1}}\,]$$

Inverse of A:

$$\boldsymbol{A^{-1}} = \begin{bmatrix} \frac{1}{6} & -\frac{1}{3} & \frac{2}{3} \\ \frac{7}{24} & -\frac{1}{12} & -\frac{7}{12} \\ -\frac{1}{12} & \frac{1}{6} & \frac{1}{6} \end{bmatrix}$$

1.6 Solving system of linear equations

Solving system of linear equations

- **Method (1): Inverse Matrix Method**
- **Method (2): Cramer's Rule**
- **Method (3): Gauss Elimination Method**

Method (1): Inverse Matrix Method

Example 1: Solve the system of linear equations using <u>Inverse Matrix Method</u>.

$$\begin{aligned} 2x + 4y + 6z &= 10 \\ 2y + 7z &= 4 \\ x \qquad + 2z &= 1 \end{aligned}$$

Solution:

Write in matrix equation $AX = B$

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 7 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 1 \end{bmatrix}$$

$$A\,X = B$$

coefficient matrix

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 7 \\ 1 & 0 & 2 \end{bmatrix}$$

unknown vector

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

constant vector

$$B = \begin{bmatrix} 10 \\ 4 \\ 1 \end{bmatrix}$$

Note:

1) If B is a **non-zero matrix**, then the system is called **non-homogeneous system**. The system of linear equations in Example 1 above is a non-homogeneous system because $B = \begin{bmatrix} 10 \\ 4 \\ 1 \end{bmatrix}$ is a **non-zero matrix**.

2) If B is a **zero matrix**, $B = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then the system is called **homogeneous system**.

First, find the inverse of matrix A:

$$A^{-1} = \frac{1}{24}\begin{bmatrix} 4 & -8 & 16 \\ 7 & -2 & -14 \\ -2 & 4 & 4 \end{bmatrix}$$

Recall: We have found A^{-1} using Adjoint Method.

(See the steps in the topic of Adjoint Method)

From the matrix equation:

$$A\,X = B$$

$$X = A^{-1}\,B$$

$$X = \frac{1}{24}\begin{bmatrix} 4 & -8 & 16 \\ 7 & -2 & -14 \\ -2 & 4 & 4 \end{bmatrix}\begin{bmatrix} 10 \\ 4 \\ 1 \end{bmatrix}$$

$$= \frac{1}{24}\begin{bmatrix} 24 \\ 48 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$$

→ $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$

Answer: $x = 1$, $y = 2$, $z = 0$

Method (2): Cramer's Rule

Example 1: Solve the system of linear equations using <u>Cramer's Rule</u>.

$$\begin{aligned} 2x + 4y + 6z &= 10 \\ 2y + 7z &= 4 \\ x \qquad + 2z &= 1 \end{aligned}$$

Solution:

Write in matrix equation $AX = B$

$$\begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 7 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 4 \\ 1 \end{bmatrix}$$

$$A\,X = B$$

coefficient matrix

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 0 & 2 & 7 \\ 1 & 0 & 2 \end{bmatrix}$$

unknown vector

$$X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

constant vector

$$B = \begin{bmatrix} 10 \\ 4 \\ 1 \end{bmatrix}$$

Find determinant of A:

$$|A| = \begin{vmatrix} 2 & 4 & 6 \\ 0 & 2 & 7 \\ 1 & 0 & 2 \end{vmatrix}$$

Use Sarrus' rule or Laplace expansion to find determinant

$$= (2 \times 2 \times 2) + (4 \times 7 \times 1) + (6 \times 0 \times 0)$$
$$-(1 \times 2 \times 6) - (0 \times 7 \times 2) - (2 \times 0 \times 4)$$

$$= 24$$

Find determinant of A_1:

$$|A_1| = \begin{vmatrix} 10 & 4 & 6 \\ 4 & 2 & 7 \\ 1 & 0 & 2 \end{vmatrix}$$

Replace B in first column of matrix A

$$= (10 \times 2 \times 2) + (4 \times 7 \times 1) + (6 \times 4 \times 0)$$
$$-(1 \times 2 \times 6) - (0 \times 7 \times 10) - (2 \times 4 \times 4)$$

$$= 24$$

Find determinant of A_2:

$$|A_2| = \begin{vmatrix} 2 & 10 & 6 \\ 0 & 4 & 7 \\ 1 & 1 & 2 \end{vmatrix}$$

Replace B in second column of matrix A

$$= (2 \times 4 \times 2) + (10 \times 7 \times 1) + (6 \times 0 \times 1)$$
$$-(1 \times 4 \times 6) - (1 \times 7 \times 2) - (2 \times 0 \times 10)$$

$$= 48$$

Find determinant of A_3:

$$|A_3| = \begin{vmatrix} 2 & 4 & 10 \\ 0 & 2 & 4 \\ 1 & 0 & 1 \end{vmatrix}$$

Replace B in third column of matrix A

$$= (2 \times 2 \times 1) + (4 \times 4 \times 1) + (10 \times 0 \times 0)$$
$$-(1 \times 2 \times 10) - (0 \times 4 \times 2) - (1 \times 0 \times 4)$$

$$= 0$$

Then,

$$x = \frac{|A_1|}{|A|} = \frac{24}{24} = 1$$

$$y = \frac{|A_2|}{|A|} = \frac{48}{24} = 2$$

$$z = \frac{|A_3|}{|A|} = \frac{0}{24} = 0$$

Answer: $x = 1$, $y = 2$, $z = 0$

Method (3): Gauss Elimination method

Example 1: Solve the system of linear equations using <u>Gauss Elimination Method</u>.

$$\begin{aligned} 3x+6y-5z&=0\\ 2x+4y-3z&=1\\ x+y+2z&=9 \end{aligned}$$

Solution:

Write in Augmented Matrix

$$\left[\begin{array}{ccc|c} 3 & 6 & -5 & 0\\ 2 & 4 & -3 & 1\\ 1 & 1 & 2 & 9 \end{array}\right]$$

Start, get '1' here

(interchange Row 1 and Row 3)

$$R_1 \leftrightarrow R_3 \quad \left[\begin{array}{ccc|c} \enclose{circle}{1} & 1 & 2 & 9\\ 2 & 4 & -3 & 1\\ 3 & 6 & -5 & 0 \end{array}\right]$$

Then, get first '0' here

$$R_2 : R_2 - 2R_1 \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9\\ \enclose{circle}{0} & 2 & -7 & -17\\ 3 & 6 & -5 & 0 \end{array}\right]$$

Get second '0' here

$$R_3 : R_3 - 3R_1 \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9\\ 0 & 2 & -7 & -17\\ \enclose{circle}{0} & 3 & -11 & -27 \end{array}\right]$$

Get '1' here

$$R_2 : \frac{1}{2}R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9\\ 0 & \enclose{circle}{1} & -\frac{7}{2} & -\frac{17}{2}\\ 0 & 3 & -11 & -27 \end{array}\right]$$

Get third '0' here

$$R_3 : R_3 - 3R_2 \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9\\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2}\\ 0 & \enclose{circle}{0} & -\frac{1}{2} & -\frac{3}{2} \end{array}\right]$$

Get ‘1’ here

$$R_3 : -2R_3 \quad \left[\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

Last step is to get <u>Row Echelon Form</u>: $\left[\begin{array}{ccc|c} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & 0 & 1 & \bullet \end{array}\right]$ or $\left[\begin{array}{ccc|c} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & 0 & 0 & \bullet \end{array}\right]$

Write into system of linear equation:

$$x+y+2z=9 \qquad ---(1)$$

$$y-\frac{7}{2}z=-\frac{17}{2} \qquad ---(2)$$

$$z=3 \qquad ---(3)$$

Substitute $z=3$ **into equation (2):** $y-\frac{7}{2}(3)=-\frac{17}{2}$

$$y=2$$

Substitute $y=2$ **and** $z=3$ **into equation (1):** $x+y+2z=9$

$$x+2+2(3)=9$$

$$x=1$$

Answer: $x=1,\ y=2,\ z=3$

→ The system has a unique solution

→ The system is consistent

Summary:

	Type of solution		
	A unique solution	No solution	Infinitely many solutions
Last row in row echelon form	$\left[\begin{array}{ccc|c} 0 & 0 & \neq 0 & \in R \end{array}\right]$	$\left[\begin{array}{ccc|c} 0 & 0 & =0 & \neq 0 \end{array}\right]$	$\left[\begin{array}{ccc|c} 0 & 0 & =0 & =0 \end{array}\right]$
Consistency of the system	The system is consistent	The system is inconsistent	The system is consistent

Type of solution and consistency of the system

Type of solution	**Last row in row echelon form**	**Example**
A unique solution	$\begin{bmatrix}0 & 0 & \neq 0 & \in R\end{bmatrix}$ R is real number. Real number includes all +ve, -ve, 0.	**Example 1: last step (row echelon form)** $\left[\begin{array}{ccc\|c}1 & 1 & 2 & 9\\ 0 & 1 & -\frac{7}{2} & -\frac{17}{2}\\ 0 & 0 & 1 & 3\end{array}\right]$ $\neq 0$ $\in R$ $x+y+2z=9 \quad ---(1)$ $y-\frac{7}{2}z=-\frac{17}{2} \quad ---(2)$ $z=3 \quad ---(3)$ Answer: $x=1,\ y=2,\ z=3$ → The system has a unique solution → The system is consistent
No Solution	$\begin{bmatrix}0 & 0 & =0 & \neq 0\end{bmatrix}$	**Example 2: last step (row echelon form)** $\left[\begin{array}{ccc\|c}1 & 3 & 1 & 18\\ 0 & 1 & 6 & 13\\ 0 & 0 & 0 & 4\end{array}\right]$ $=0$ $\neq 0$ $x+3y+z=18 \quad ---(1)$ $y+6z=13 \quad ---(2)$ $0z=4 \quad ---(3)$ 0 = 4 (not logical) Answer: x, y, z = **no solution** → The system has no solution → The system is inconsistent
Infinitely many solutions	$\begin{bmatrix}0 & 0 & =0 & =0\end{bmatrix}$	**Example 3: last step(row echelon form)** $\left[\begin{array}{ccc\|c}1 & 2 & 1 & 8\\ 0 & 1 & 3 & 13\\ 0 & 0 & 0 & 0\end{array}\right]$ $=0$ $=0$ $x+2y+z=8 \quad ---(1)$ $y+3z=13 ---(2)$ Let z = t, where t $\in R$, t is any real number. Equation (2): $y+3t=13$ $y=13-3t$ Equation (1): $x+2y+z=8$ $x+2(13-3t)+(t)=8$ $x=-18+5t$ Answer: $\begin{bmatrix}x\\ y\\ z\end{bmatrix}=\begin{bmatrix}-18+5t\\ 13-3t\\ t\end{bmatrix}$ → The system has infinitely many solutions → The system is consistent

Type of solution and consistency of the system

Example 4:

$$\begin{aligned} x+2y+3z&=1\\ 2x+5y+3z&=-3\\ x+\quad (a+2)z&=b-3 \end{aligned}$$

Find the value of **a** and **b** such that the system has

(a) a unique solution
(b) no solution
(c) infinitely many solutions

Solution:

Write in Augmented Matrix

$$\left[\begin{array}{ccc|c} 1 & 2 & 3 & 1\\ 2 & 5 & 3 & -3\\ 1 & 0 & a+2 & b-3 \end{array}\right]$$

Start, get '1' here

Get first '0' here

$$R_2 : R_2-2R_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1\\ \enclose{circle}{0} & 1 & -3 & -5\\ 1 & 0 & a+2 & b-3 \end{array}\right]$$

Get second '0' here

$$R_3 : R_3-R_1 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1\\ 0 & 1 & -3 & -5\\ \enclose{circle}{0} & -2 & a-1 & b-4 \end{array}\right]$$

Get third '0' here

$$R_3 : R_3+2R_2 \quad \left[\begin{array}{ccc|c} 1 & 2 & 3 & 1\\ 0 & 1 & -3 & -5\\ 0 & \enclose{circle}{0} & a-7 & b-14 \end{array}\right]$$

(a) a unique solution:

$$\begin{array}{lcl} a-7\neq 0 & \text{and} & b-14\in R\\ a\neq 7 & & b\in R \end{array}$$

a unique solution:

$$\left[\begin{array}{ccc|c} 0 & 0 & \enclose{circle}{\neq 0} & \enclose{circle}{\in R} \end{array}\right]$$

(b) no solution:

$$\begin{array}{lcl} a-7=0 & \text{and} & b-14\neq 0\\ a=7 & & b\neq 14 \end{array}$$

no solution:

$$\left[\begin{array}{ccc|c} 0 & 0 & \enclose{circle}{=0} & \enclose{circle}{\neq 0} \end{array}\right]$$

(c) infinitely many solutions:

$$\begin{array}{lcl} a-7=0 & \text{and} & b-14=0\\ a=7 & & b=14 \end{array}$$

infinitely many solutions:

$$\left[\begin{array}{ccc|c} 0 & 0 & \enclose{circle}{=0} & \enclose{circle}{=0} \end{array}\right]$$

1.7 Eigenvalues and eigenvectors

Tips:
To find eigenvalue λ
→ Solve characteristic equation $|\boldsymbol{A} - \lambda\boldsymbol{I}| = 0$
where $\boldsymbol{I}$ is Identity matrix

Example 1: Given $A = \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix}$

(a) Find all eigenvalues of $\boldsymbol{A}$.
(b) Find the eigenvector corresponding to the smallest eigenvalue.
(c) Find the normal eigenvector for $\lambda = 1$.

Solution:
(a) Find all eigenvalues of $\boldsymbol{A}$.

Let λ = eigenvalue of $\boldsymbol{A}$
Solve $|\boldsymbol{A} - \lambda\boldsymbol{I}| = 0$

$$\left| \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \right| = 0$$

$$\left| \begin{pmatrix} 0 & 0 & -2 \\ 1 & 2 & 1 \\ 1 & 0 & 3 \end{pmatrix} - \begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{pmatrix} \right| = 0$$

$$\begin{vmatrix} 0-\lambda & 0 & -2 \\ 1 & 2-\lambda & 1 \\ 1 & 0 & 3-\lambda \end{vmatrix} = 0$$

Use Sarrus' rule or Laplace expansion to find determinant

$$(-\lambda)(2-\lambda)(3-\lambda)+0+0-(1)(-2)(2-\lambda)-0-0=0$$

$$-\lambda(2-\lambda)(3-\lambda)+2(2-\lambda)=0$$

$$(2-\lambda)[-\lambda(3-\lambda)+2]=0$$

$$(2-\lambda)(\lambda^2-3\lambda+2)=0$$

$$(2-\lambda)(\lambda-1)(\lambda-2)=0$$

$$\lambda = 2 \text{ and } \lambda = 1$$

Solution:

(b) Find the eigenvector corresponding to the smallest eigenvalue.

Solve $\left(\begin{array}{ccc|c} 0-\lambda & 0 & -2 & 0 \\ 1 & 2-\lambda & 1 & 0 \\ 1 & 0 & 3-\lambda & 0 \end{array}\right)$

$$\left(\begin{array}{ccc|c} 0-1 & 0 & -2 & 0 \\ 1 & 2-1 & 1 & 0 \\ 1 & 0 & 3-1 & 0 \end{array}\right)$$

Substitute smallest eigenvalue $\lambda = 1$

$$\left(\begin{array}{ccc|c} -1 & 0 & -2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$$

$R_1 : (-1)R_1$ $\left(\begin{array}{ccc|c} \boxed{1} & 0 & 2 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$

Start, get '1' here

$R_2 : R_2 - R_1$ $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ \boxed{0} & 1 & -1 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right)$

Get first '0'

$R_3 : R_3 - R_1$ $\left(\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ \boxed{0} & 0 & 0 & 0 \end{array}\right)$

Get second '0'

Get third '0'

This is Row echelon form: $\left[\begin{array}{ccc|c} 1 & \bullet & \bullet & \bullet \\ 0 & 1 & \bullet & \bullet \\ 0 & 0 & 0 & \bullet \end{array}\right]$

$$x + \quad 2z = 0 ---(1)$$
$$y - z = 0 ---(2)$$

Let z = t, where t $\in R$, t is any real number.

Substitute z = t into Equation (2): $y - z = 0$

$$y - t = 0$$
$$y = t$$

Substitute z = t into Equation (1): $x + 2z = 0$

$$x + 2t = 0$$
$$x = -2t$$

Eigenvector for $\boldsymbol{\lambda = 1}$:

$$\nu = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -2t \\ t \\ t \end{bmatrix} = t\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

Solution:

(c) Find the normal eigenvector for $\lambda = 1$.

Normal eigenvector $= \dfrac{1}{\sqrt{(-2)^2 + 1^2 + 1^2}}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \dfrac{1}{\sqrt{6}}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{-2}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}$

CHAPTER 2

Vector Algebra

Outline:

2.1 Introduction – Definition

2.2 Vector in two and three dimensional space

2.3 Product of Two Vectors and Product of Three Vectors
  - (A) Scalar product
  - (B) Vector product
  - (C) Scalar triple product
  - (D) Vector triple product

2.4 Simple geometric applications
  - (A) Equation of line
  - (B) Equation of plane

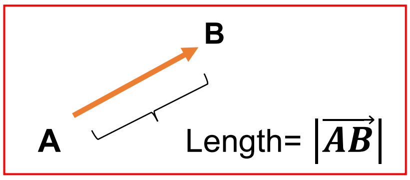
2.1 Introduction – Definition

1. A **vector** is a quantity **with magnitude** and **a direction**.
   eg. Force, velocity, wind movement (100km/h East, 100km/h West)

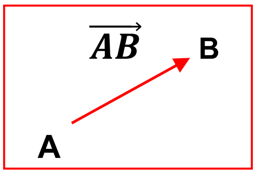
2. A **scalar** is a quantity **with magnitude** and **<u>no direction</u>**.
   eg. Area (50 m^2), Length (3m), temperature (37 ^oC)

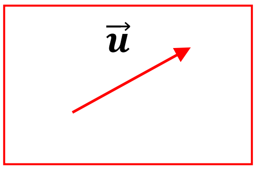
Vector	**Scalar**
Magnitude and direction	Magnitude
Example: Force, velocity, wind movement	Example: Area, Length, temperature
Example: 100km/h East, 100km/h West	Example: 50 m^2, 3m, 37 ^oC

3. A vector is represented by a directed line segment or arrow
   eg. $\overrightarrow{AB}$ = vector from A to B
   $|\overrightarrow{AB}|$ = magnitude = length

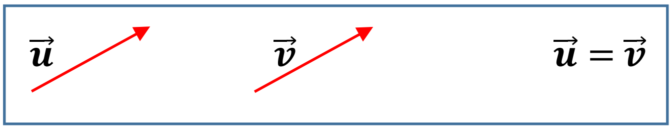


4. Notation of vector: eg. $\overrightarrow{AB}$ or $\vec{u}$ or $\vec{a}$





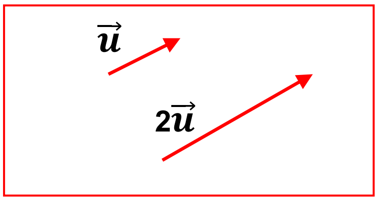
5. Vectors with same length and same direction are equivalent.

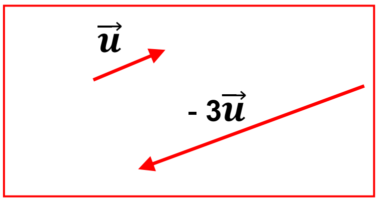


6. Basic vector operations: Addition, Subtraction, scalar multiplication

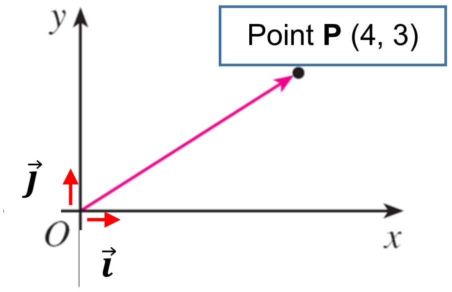
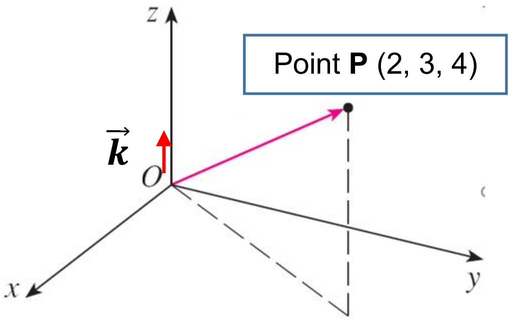
   Scalar multiplication = $k\,\vec{u}$

   If $k > 0$**, same direction** **If** $k < 0$**, opposite direction**

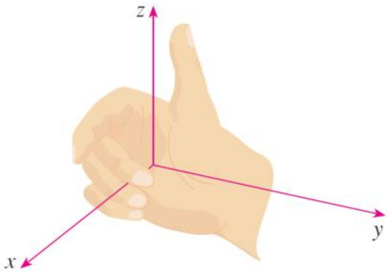




2.2 Vector in two and three dimensional space

Vector in two dimensional space	**Vector in three dimensional space**
 $\vec{i}$ is unit vector (with length 1) having same direction as x-axis, $\vec{j}$ is unit vector (with length 1) having same direction as y-axis	 $\vec{k}$ is unit vector (with length 1) having same direction as z-axis
Position vector $= \overrightarrow{OP}$ $= 4\vec{i} + 3\vec{j}$ $= \langle 4, 3 \rangle$ $= \begin{bmatrix} 4 \\ 3 \end{bmatrix}$	**Position vector** $= \overrightarrow{OP}$ $= 2\vec{i} + 3\vec{j} + 4\vec{k}$ $= \langle 2, 3, 4 \rangle$ $= \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$

Tips:
The direction of the *z*-axis is determined by the right-hand rule



Example 1: Given points A (2, 0, 4) and B (4, 1, 2). Find

(a) vector $\overrightarrow{AB}$

(b) magnitude of $\overrightarrow{AB}$

(c) unit vector of $\overrightarrow{AB}$

Find (a) vector $\overrightarrow{AB}$

Solution:

(a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= \langle 4, 1, 2\rangle - \langle 2, 0, 4\rangle$$

$$= \langle 2, 1, -2\rangle$$

$$= 2\vec{i} + \vec{j} - 2\vec{k}$$

☼ **Remember:**

If given
Point A (2, 0, 4),
Then
Position vector

$$\overrightarrow{OA} = \langle 2, 0, 4\rangle$$

$$= \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

$$= 2\vec{i} + 0\vec{j} + 4\vec{k}$$

Alternative answer:

(a) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$$= \begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ -2 \end{bmatrix}$$

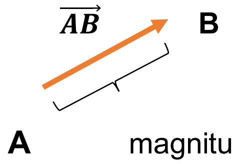
$$= 2\vec{i} + \vec{j} - 2\vec{k}$$

(b) magnitude of $\overrightarrow{AB}$

Solution:

$$\left|\overrightarrow{AB}\right| = \sqrt{2^2 + 1^2 + (-2)^2}$$

$$= \sqrt{9} = 3$$



magnitude = Length

(c) unit vector of $\overrightarrow{AB}$

Solution:

unit vector of $\overrightarrow{AB}$:

$$\frac{1}{\left|\overrightarrow{AB}\right|}\overrightarrow{AB} = \frac{1}{3}(2\vec{i} + \vec{j} - 2\vec{k})$$

$$= \frac{2}{3}\vec{i} + \frac{1}{3}\vec{j} - \frac{2}{3}\vec{k}$$

unit vector of $\overrightarrow{AB}$

has length 1 unit,

has direction same as vector $\overrightarrow{AB}$

Example 2: Given vector $\vec{u} = \langle 3, -4, 0 \rangle$. Find

(a) all direction cosines of $\vec{u}$
(b) all direction angles of $\vec{u}$

Solution:

magnitude of $\vec{u}$

$$|\vec{u}| = \sqrt{3^2 + (-4)^2 + 0^2}$$

$$= 5$$

(a) direction cosines of $\vec{u}$:

$$\cos\alpha = \frac{x}{|\vec{u}|}$$

$$= \frac{3}{5}$$

$$\cos\beta = \frac{y}{|\vec{u}|}$$

$$= \frac{-4}{5} = -\frac{4}{5}$$

$$\cos\gamma = \frac{z}{|\vec{u}|}$$

$$= \frac{0}{5} = 0$$

(b) direction angles of $\vec{u}$:

$$\alpha = \cos^{-1}\left(\frac{3}{5}\right)$$

$$= 53.13^\circ$$

$$\beta = \cos^{-1}\left(-\frac{4}{5}\right)$$

$$= 143.13^\circ$$

$$\gamma = \cos^{-1}(0)$$

$$= 90^\circ$$

Note: range of angles α, β, γ :

$$0^\circ \leq \alpha, \beta, \gamma \leq 180^\circ$$

or $0 \leq \alpha, \beta, \gamma \leq \pi$

2.3 Product of Two Vectors and Product of Three Vectors

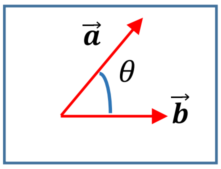
Product of Two Vectors

- **(A) Scalar product (Dot product)** $\vec{a} \cdot \vec{b}$
- **(B) Vector product (Cross product)** $\vec{a} \times \vec{b}$

Product of Three Vectors

- **(C) Scalar triple product** $\vec{a} \cdot (\vec{b} \times \vec{c})$
- **(D) Vector triple product** $\vec{a} \times (\vec{b} \times \vec{c})$

(A) Scalar product (Dot product) $\vec{a} \cdot \vec{b}$

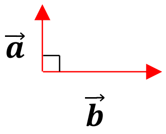


Definition: $\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\,cos\theta \qquad (0 \le \theta \le \pi)$

Calculation: **Given** $\vec{a} = a_1\,\vec{\imath} + a_2\,\vec{\jmath} + a_3\,\vec{k}\ ,\ \vec{b} = b_1\,\vec{\imath} + b_2\,\vec{\jmath} + b_3\,\vec{k}$

Then $\vec{a} \cdot \vec{b} = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = a_1\,b_1 + a_2\,b_2 + a_3\,b_3$

Note: Two vectors $\vec{a}$ and $\vec{b}$ are perpendicular (orthogonal) if $\vec{a} \cdot \vec{b} = 0$.



Example 1: Given $\vec{a} = \vec{\imath} + 3\,\vec{\jmath} + 6\,\vec{k}\ ,\ \vec{b} = \ 7\,\vec{\jmath} + \vec{k}$. Find $\vec{a} \cdot \vec{b}$.

Solution:

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} 1 \\ 3 \\ 6 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$$

$$= (1)(0) + (3)(7) + (6)(1)$$

$$= 27$$

Example 2: Given $\vec{a} = 2\,\vec{\imath} + \vec{\jmath} + \vec{k}$, $\vec{b} = \vec{\imath} - \vec{\jmath} - \vec{k}$.

Show that $\vec{a}$ **and** $\vec{b}$ **are perpendicular.**

Solution:

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$$

$$= (\mathbf{2})(\mathbf{1}) + (\mathbf{1})(-\mathbf{1}) + (\mathbf{1})(-\mathbf{1})$$

$$= \mathbf{0}$$

Since $\vec{a} \cdot \vec{b} = 0$ **, then** $\vec{a}$ **and** $\vec{b}$ **are perpendicular.**

Example 3: Given $\vec{a} = \langle \mathbf{1}, \mathbf{4}, -\mathbf{4} \rangle$, $\vec{b} = \langle \mathbf{0}, -\mathbf{7}, \mathbf{3} \rangle$ **. Find**

(a) $\vec{a} \cdot \vec{b}$

(b) $|\vec{a}|$ **and** $|\vec{\mathrm{b}}|$

(c) angle between $\vec{a}$ **and** $\vec{b}$

Solution:

(a)

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} 1 \\ 4 \\ -4 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ -7 \\ 3 \end{bmatrix}$$

$$= (\mathbf{1})(\mathbf{0}) + (\mathbf{4})(-\mathbf{7}) + (-\mathbf{4})(\mathbf{3})$$

$$= -\mathbf{40}$$

(b) $$|\vec{a}| = \sqrt{\mathbf{1}^2 + \mathbf{4}^2 + (-\mathbf{4})^2} \quad = \sqrt{\mathbf{33}}$$

$$|\vec{b}| = \sqrt{\mathbf{0}^2 + (-\mathbf{7})^2 + \mathbf{3}^2} \quad = \sqrt{\mathbf{58}}$$

(c) $$\vec{a} \cdot \vec{b} = |\vec{a}|\,|\vec{b}|\, cos\theta$$

$$-40 = \sqrt{\mathbf{33}}\,\sqrt{\mathbf{58}}\, cos\theta$$

$$cos\theta = \frac{-40}{\sqrt{\mathbf{33}}\,\sqrt{\mathbf{58}}}$$

$$cos\theta = -0.9143$$

$$\theta = cos^{-1}(-0.9143) = \mathbf{156.1}^{\circ}$$

Example 4: Given $\vec{a} = \langle 1, 1, 2 \rangle$, $\vec{b} = \langle 0, 3, 4 \rangle$. Find

(a) vector projection of $\vec{a}$ onto $\vec{b}$

(b) scalar projection of $\vec{a}$ onto $\vec{b}$

(c) vector component of $\vec{a}$ orthogonal to $\vec{b}$

Solution:

(a)

$$\vec{a} \cdot \vec{b} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 0 \\ 3 \\ 4 \end{bmatrix} = (1)(0) + (1)(3) + (2)(4) = \boxed{11}$$

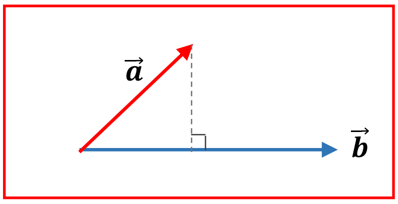
$$|\vec{b}| = \sqrt{0^2 + 3^2 + 4^2} = \boxed{5}$$

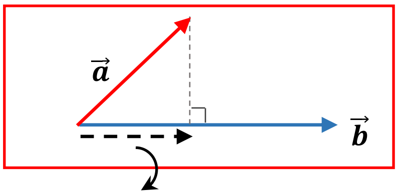
$$\text{vector projection of } \vec{a} \text{ onto } \vec{b} = \frac{(\vec{a} \cdot \vec{b})}{|\vec{b}|^2} \vec{b}$$

$$= \frac{11}{(5)^2} \langle 0, 3, 4 \rangle$$

$$= \frac{11}{25} \langle 0, 3, 4 \rangle$$

$$= \langle 0, \frac{33}{25}, \frac{44}{25} \rangle$$





Vector projection of $\vec{a}$ onto $\vec{b}$

(b) scalar projection of $\vec{a}$ onto $\vec{b}$ $= \frac{(\vec{a} \cdot \vec{b})}{|\vec{b}|}$

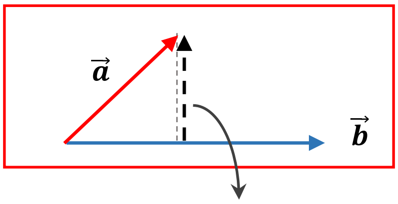
$$= \frac{11}{5}$$

(c) vector component of $\vec{a}$ orthogonal to $\vec{b}$

$$= \vec{a} - \frac{(\vec{a} \cdot \vec{b})}{|\vec{b}|^2} \vec{b}$$

$$= \langle 1, 1, 2 \rangle - \langle 0, \frac{33}{25}, \frac{44}{25} \rangle$$

$$= \langle 1, -\frac{8}{25}, \frac{6}{25} \rangle$$



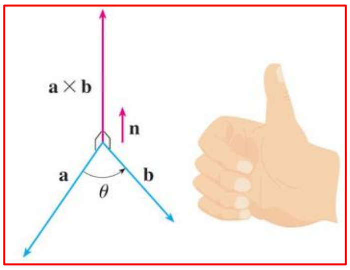
Vector component of $\vec{a}$ orthogonal to $\vec{b}$

(B) Vector product (Cross product) $\vec{a}\times\vec{b}$

Definition: $\vec{a}\times\vec{b} = |\vec{a}|\,|\vec{b}|\,sin\theta\,\vec{n}$ $\quad(0\le\theta\le\pi)$

Calculation: **Given** $\vec{a} = a_1\,\vec{i} + a_2\,\vec{j} + a_3\,\vec{k}\,,\ \vec{b} = b_1\,\vec{i} + b_2\,\vec{j} + b_3\,\vec{k}$

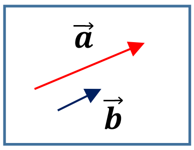
$$\text{Then }\ \vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix}$$



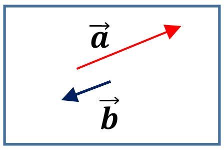
The right-hand rule gives the direction of $\vec{a}\times\vec{b}$.

Note 1: $\vec{b}\times\vec{a} = -\,\vec{a}\times\vec{b}$

Note 2: Two vectors $\vec{a}$ and $\vec{b}$ are parallel if $\vec{a}\times\vec{b} = 0$



or



Example 1: Given vectors $\vec{a} = \vec{i} - 3\,\vec{j} + 6\,\vec{k}\,,\ \vec{b} = 7\,\vec{j} + 2\,\vec{k}$ **. Find** $\vec{a}\times\vec{b}$ **.**

Solution:

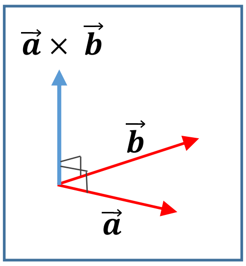
$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \mathbf{1} & \mathbf{-3} & \mathbf{6} \\ \mathbf{0} & \mathbf{7} & \mathbf{2} \end{vmatrix}$$

Use Laplace expansion to find determinant

$$= \vec{i}\begin{vmatrix} -3 & 6 \\ 7 & 2 \end{vmatrix} \ominus \vec{j}\begin{vmatrix} 1 & 6 \\ 0 & 2 \end{vmatrix} + \vec{k}\begin{vmatrix} 1 & -3 \\ 0 & 7 \end{vmatrix}$$

$$= \vec{i}(-6-42) - \vec{j}(2-0) + \vec{k}(7-0)$$

$$= \mathbf{-48}\,\vec{i}\ - \mathbf{2}\,\vec{j} + \mathbf{7}\,\vec{k}$$



Example 2: Given points A (4, 5, 0) , B (1, - 1, 0) and C (2, 3, 1). Find

(a) $\overrightarrow{CA}$ and $\overrightarrow{CB}$

(b) vector perpendicular to both $\overrightarrow{CA}$ and $\overrightarrow{CB}$

(c) unit vector perpendicular to both $\overrightarrow{CA}$ and $\overrightarrow{CB}$

Solution:

(a) $\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC}$

$$= \langle 4, 5, 0\rangle - \langle 2, 3, 1\rangle$$

$$= \langle 2, 2, -1\rangle$$

$$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC}$$

$$= \langle 1, -1, 0\rangle - \langle 2, 3, 1\rangle$$

$$= \langle -1, -4, -1\rangle$$

Solution:

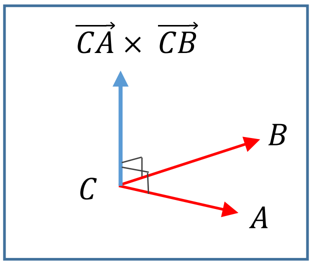
(b) vector perpendicular to both $\overrightarrow{CA}$ and $\overrightarrow{CB}$

$$= \overrightarrow{CA} \times \overrightarrow{CB} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 2 & -1 \\ -1 & -4 & -1 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 2 & -1 \\ -4 & -1 \end{vmatrix} \ominus \vec{j}\begin{vmatrix} 2 & -1 \\ -1 & -1 \end{vmatrix} + \vec{k}\begin{vmatrix} 2 & 2 \\ -1 & -4 \end{vmatrix}$$

$$= \vec{i}(-2-4) - \vec{j}(-2-1) + \vec{k}(-8+2)$$

$$= -6\,\vec{i} + 3\,\vec{j} - 6\,\vec{k}$$



Solution:

(c) unit vector perpendicular to both $\overrightarrow{CA}$ and $\overrightarrow{CB}$

$$= \frac{1}{|\overrightarrow{CA} \times \overrightarrow{CB}|}\,\overrightarrow{CA} \times \overrightarrow{CB}$$

$$= \frac{1}{9}\langle -6, 3, -6\rangle$$

$$= \langle -\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}\rangle$$

$\overrightarrow{CA} \times \overrightarrow{CB} = -6\,\vec{i} + 3\,\vec{j} - 6\,\vec{k}$

$\rightarrow |\overrightarrow{CA} \times \overrightarrow{CB}|$

$= \sqrt{(-6)^2 + 3^2 + (-6)^2}$

$= 9$

Example 3: Given vectors $\vec{a} = \langle 1, 0, 2\rangle$, $\vec{b} = \langle 0, 1, 3\rangle$ **. Find**

(a) Area of parallelogram formed by $\vec{a}$ **and** $\vec{b}$

(b) Area of triangle formed by $\vec{a}$ **and** $\vec{b}$

Solution:

(a) Area of parallelogram formed by $\vec{a}$ **and** $\vec{b}$

$$= |\vec{a} \times \vec{b}|$$

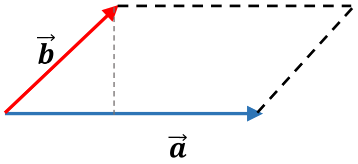
$$= \sqrt{(-2)^2 + (-3)^2 + 1^2}$$

$$= \sqrt{14} \text{ units}^2$$

First, find

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= -2\,\vec{i} - 3\,\vec{j} + \vec{k}$$



Parallelogram

Solution:

(b) Area of triangle formed by $\vec{a}$ **and** $\vec{b}$

$$= \frac{1}{2}|\vec{a} \times \vec{b}|$$

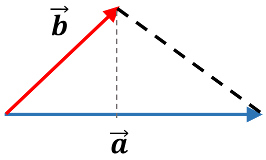
$$= \frac{1}{2}\sqrt{(-2)^2 + (-3)^2 + 1^2}$$

$$= \frac{1}{2}\sqrt{14} \text{ units}^2$$

First, find

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 0 & 2 \\ 0 & 1 & 3 \end{vmatrix}$$

$$= -2\,\vec{i} - 3\,\vec{j} + \vec{k}$$



Triangle

(C) Scalar Triple Product $\vec{a}\cdot(\vec{b}\times\vec{c})$

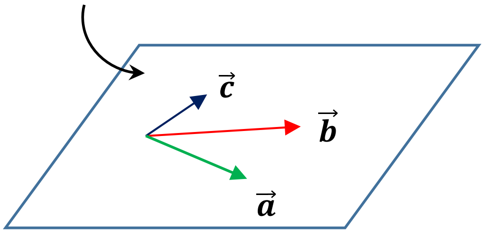
Calculation: Given $\vec{a}=\langle a_1,a_2,a_3\rangle$, $\vec{b}=\langle b_1,b_2,b_3\rangle$, $\vec{c}=\langle c_1,c_2,c_3\rangle$

Then Scalar Triple Product

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix}$$

Note 1: $\vec{a}\cdot(\vec{b}\times\vec{c}) = (\vec{a}\times\vec{b})\cdot\vec{c}$

Note 2: Three vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ <u>lie on the same plane (coplanar)</u> if $\vec{a}\cdot(\vec{b}\times\vec{c})=0$



Example 1: Given vectors $\vec{a}=\langle 1,2,-1\rangle$, $\vec{b}=\langle -2,0,3\rangle$, $\vec{c}=\langle 0,7,-4\rangle$.

(a) Find $\vec{a}\cdot(\vec{b}\times\vec{c})$.

(b) Find the volume of parallelepiped formed by $\vec{a}$, $\vec{b}$ and $\vec{c}$.

Solution:

(a) Find $\vec{a}\cdot(\vec{b}\times\vec{c})$.

$$\vec{a}\cdot(\vec{b}\times\vec{c})=\begin{vmatrix} 1 & 2 & -1 \\ -2 & 0 & 3 \\ 0 & 7 & -4 \end{vmatrix}$$

$$= 0+0+14-0-21-16$$

$$= -23$$

Use Sarrus' rule or Laplace expansion to find determinant

(b) Find the volume of parallelepiped formed by $\vec{a}$, $\vec{b}$ and $\vec{c}$.
Solution:

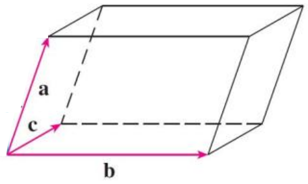
volume of parallelepiped formed by $\vec{a}$, $\vec{b}$ and $\vec{c}$

$= |\vec{a} \cdot (\vec{b} \times \vec{c})|$

$= |-23|$

$= 23$ units^3

$|-23| =$ Absolute value of -23



Parallelepiped

Example 2: Given vectors $\vec{a} = \langle 0, 1, 2 \rangle$, $\vec{b} = \langle p, 1, 0 \rangle$, $\vec{c} = \langle 3, 0, 1 \rangle$.
Find the value of p so that $\vec{a}$, $\vec{b}$ and $\vec{c}$ are coplanar.

Solution:

$$\vec{a} \cdot (\vec{b} \times \vec{c}) = \begin{vmatrix} 0 & 1 & 2 \\ p & 1 & 0 \\ 3 & 0 & 1 \end{vmatrix}$$

$$= 0 + 0 + 0 - 6 - 0 - p$$

$$= -6 - p$$

Use Sarrus' rule or Laplace expansion to find determinant

Vectors $\vec{a}$, $\vec{b}$ and $\vec{c}$ lie on the same plane (coplanar)
if $\vec{a} \cdot (\vec{b} \times \vec{c}) = 0$

$$-6 - p = 0$$

$$p = -6$$

(D) Vector Triple Product $\vec{a} \times (\vec{b} \times \vec{c})$

Note 1: $\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

Note 2: $(\vec{a} \times \vec{b}) \times \vec{c} = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{b} \cdot \vec{c})\vec{a}$

Example 1: Given vectors $\vec{a} = \langle \mathbf{2}, \mathbf{0}, -\mathbf{1} \rangle$, $\vec{b} = \langle \mathbf{1}, \mathbf{2}, \mathbf{0} \rangle$, $\vec{c} = \langle \mathbf{1}, \mathbf{2}, \mathbf{1} \rangle$.
Find $\vec{a} \times (\vec{b} \times \vec{c})$.

Solution:

$\vec{a} \cdot \vec{c} = \langle \mathbf{2}, \mathbf{0}, -\mathbf{1} \rangle \cdot \langle \mathbf{1}, \mathbf{2}, \mathbf{1} \rangle$

$= 2 + 0 - 1$

$= 1$

$\vec{a} \cdot \vec{b} = \langle \mathbf{2}, \mathbf{0}, -\mathbf{1} \rangle \cdot \langle \mathbf{1}, \mathbf{2}, \mathbf{0} \rangle$

$= 2 + 0 - 0$

$= 2$

$\vec{a} \times (\vec{b} \times \vec{c}) = (\vec{a} \cdot \vec{c})\vec{b} - (\vec{a} \cdot \vec{b})\vec{c}$

$= (1)\,\vec{b} - (2)\,\vec{c}$

$= (1)\,\langle \mathbf{1}, \mathbf{2}, \mathbf{0} \rangle - (2)\,\langle \mathbf{1}, \mathbf{2}, \mathbf{1} \rangle$

$= \langle \mathbf{1}, \mathbf{2}, \mathbf{0} \rangle - \langle \mathbf{2}, \mathbf{4}, \mathbf{2} \rangle$

$= \langle -\mathbf{1}, -\mathbf{2}, -\mathbf{2} \rangle$

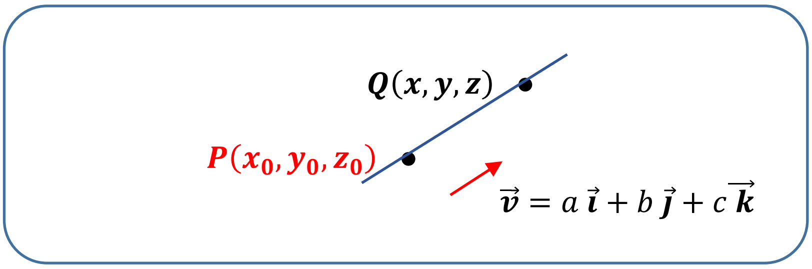
2.4 Simple Geometric Applications

Simple Geometric Applications

- **(A) Equation of line**
- **(B) Equation of plane**

(A) Equation of line

Introduction



- **If $Q(x, y, z)$ is any point on a line**
- **Given that $P(x_0, y_0, z_0)$ is a point on the line and the line is parallel to a vector $\vec{v} = a\,\vec{i} + b\,\vec{j} + c\,\vec{k}$**
- **Then, $\overrightarrow{PQ} = t\vec{v}$**

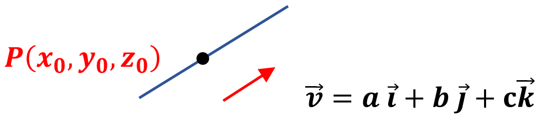
$$\overrightarrow{OQ} - \overrightarrow{OP} = t\vec{v}$$

$$\langle x, y, z\rangle - \langle x_0, y_0, z_0\rangle = t\langle a, b, c\rangle$$

$$(x - x_o)\vec{i} + (y - y_o)\,\vec{j} + (z - z_o)\vec{k} = t\,(a\,\vec{i} + b\,\vec{j} + c\vec{k})$$

This is the equation of line (vector equation)

Equation of line



Definition: <u>Equation of line</u> that passes through a point $P(x_0, y_0, z_0)$ and parallel to a vector $\vec{v} = a\,\vec{i} + b\,\vec{j} + c\vec{k}$ can be expressed in 3 forms:

<u>(a) vector equation:</u>

$$(x - x_o)\vec{i} + (y - y_o)\,\vec{j} + (z - z_o)\vec{k} = \text{t}\,(a\,\vec{i} + b\,\vec{j} + c\vec{k})$$

<u>(b) parametric equation:</u>

$$x = x_o + t\,a$$

$$y = y_o + t\,b$$

$$z = z_o + t\,c$$

<u>(c) Cartesian / symmetric equation:</u>

$$\frac{x - x_o}{a} = \frac{y - y_o}{b} = \frac{z - z_o}{c} = t$$

Example 1: Find the <u>equation of line</u> that passes through a point $P(1, 0, 4)$ and parallel to a vector $\vec{v} = 2\,\vec{i} + 4\,\vec{j} - 3\,\vec{k}$. Express the answer in

<u>(a) vector equation:</u>

$$(x - 1)\vec{i} + (y - 0)\,\vec{j} + (z - 4)\vec{k} = \text{t}\,(2\,\vec{i} + 4\,\vec{j} - 3\,\vec{k})$$

$$\boxed{\langle x - 1, y, z - 4\rangle = \text{t}\,\langle 2, 4, -3\rangle}$$

<u>(b) parametric equation:</u>

$$x = 1 + t\,2$$

$$y = 0 + t\,4$$

$$z = 4 + t\,(-3)$$

$$\boxed{\begin{aligned} x &= 1 + 2\,t \\ y &= \;4\,t \\ z &= 4 - 3\,t \end{aligned}}$$

<u>(c) Cartesian / symmetric equation:</u>

$$\frac{x - 1}{2} = \frac{y - 0}{4} = \frac{z - 4}{-3} = t$$

$$\boxed{\frac{x - 1}{2} = \frac{y}{4} = \frac{z - 4}{-3}}$$

Example 2:

(a) Find the parametric equation of line that passes through points A (0, 2, 1) and B (2, 3, 4).

(b) Find the intersection point where the line intersects the xz-plane.

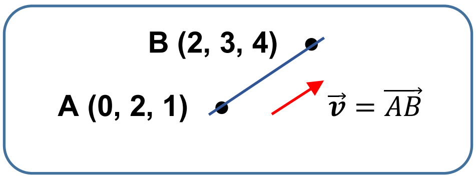
Solution:

(a) Find the parametric equation of line that passes through points A (0, 2, 1) and B (2, 3, 4).

$$\vec{v} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \langle \mathbf{2}, \mathbf{3}, \mathbf{4} \rangle - \langle \mathbf{0}, \mathbf{2}, \mathbf{1} \rangle$$

$$= \langle \mathbf{2}, \mathbf{1}, \mathbf{3} \rangle$$



Using point A (0, 2, 1), parametric equation of line:

$x = x_o + t\,a$

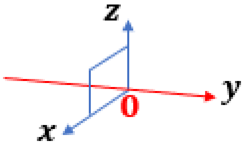
$y = y_o + t\,b$

$z = z_o + t\,c$

$$\boxed{\begin{aligned} x &= 0 + 2\,t \\ y &= 2 + 1\,t \\ z &= 1 + 3\,t \end{aligned}} \qquad \boxed{\begin{aligned} x &= 2\,t \\ y &= 2 + \ t \\ z &= 1 + 3\,t \end{aligned}}$$

(b) Find the intersection point where the line intersects the xz-plane.

Solution:



$\boxed{\textbf{Theory: On xz-plane, } \ y = 0}$

$$\boxed{\begin{aligned} x &= 2\,t \\ y &= 2 + \ t \\ z &= 1 + 3\,t \end{aligned}} \longrightarrow \boxed{\begin{aligned} y = 2 + \ t &= 0 \\ t &= -2 \end{aligned}} \longrightarrow \boxed{\begin{aligned} &\textbf{Substitute } t = -2\textbf{:} \\ &x = 2\,t = 2(-2) = -4 \\ &y = 2 + \ t = 2 - 2 = 0 \\ &z = 1 + 3\,t = 1 + 3(-2) = -5 \end{aligned}}$$

Intersection point $= (\mathbf{-4}, \mathbf{0}, \mathbf{-5})$

**** Tips: On** xy **- plane,** $\ z = \mathbf{0}$

On yz **- plane,** $\ x = \mathbf{0}$

Example 3:
Given two lines:

L_1: $x = -3 - 2\,t, \qquad y = -2 + 2\,t, \qquad z = 6 + 4\,t$

L_2: $x = -1 + 4\,\mathrm{s}, \qquad y = -4 - 2\,s, \qquad z = 2 + 3\,\mathrm{s}$

(a) Find the Cartesian equation of line that passes through point $A(2,4,0)$ and parallel to line L_1 .
(b) Find the intersection point of lines L_1 and L_2 .
(c) Find the angle between lines L_1 and L_2 .

Solution:

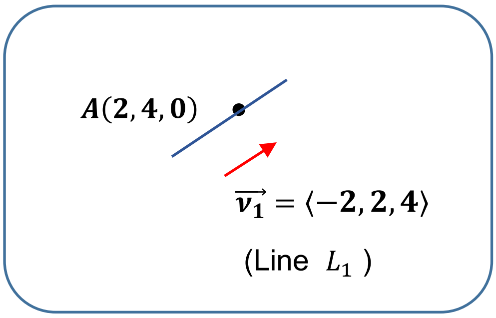
(a) Find the Cartesian equation of line that passes through point $A(2,4,0)$ and parallel to line L_1 .

Cartesian equation of line:

$$\frac{x - x_o}{a} = \frac{y - y_o}{b} = \frac{z - z_o}{c}$$

$$\frac{x - 2}{-2} = \frac{y - 4}{2} = \frac{z - 0}{4}$$

$$\boxed{\frac{x - 2}{-2} = \frac{y - 4}{2} = \frac{z}{4}}$$



(b) Find the intersection point of lines L_1 and L_2 .

Solution:

$-3-2t=-1+4s$ --- ①

$-2+2t=-4-2s$ --- ②

$6+4t=2+3s$ --- ③

$L_1: x=-3-2t, \quad y=-2+2t, \quad z=6+4t$

x = x　　y = y　　z = z

$L_2: x=-1+4s, \quad y=-4-2s, \quad z=2+3s$

Solve the above simultaneous equations:

①+ ②: $-5=-5+2s$ → $s=0$

Substitute $s=0$ into ③: $6+4t=2+3s$

$$6+4t=2+3(0)$$

→ $t=-1$

To find intersection point, substitute $s=0$ into

$L_2: x=-1+4s, \quad y=-4-2s, \quad z=2+3s$

$x=-1+4(0)=-1$

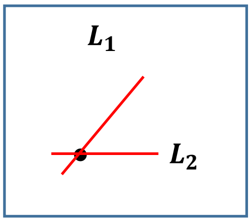
$y=-4-2(0)=-4$

$z=2+3(0)=2$

→

Intersection point:

$(\mathbf{-1,-4,2})$



(c) Find the angle between lines L_1 and L_2 .

Solution:

Line L_1:

$$\overrightarrow{v_1} = \langle -2, 2, 4\rangle$$

Line L_2:

$$\overrightarrow{v_2} = \langle 4, -2, 3\rangle$$

$L_1: x = -3 - 2\,t, \quad y = -2 + 2\,t, \quad z = 6 + 4\,t$

$L_2: x = -1 + 4\,s, \quad y = -4 - 2\,s, \quad z = 2 + 3\,s$

To find the angle between lines L_1 and L_2 :

$$\overrightarrow{v_1} \cdot \overrightarrow{v_2} = |\overrightarrow{v_1}|\,|\overrightarrow{v_2}|\,\cos\theta$$

$$0 = \sqrt{24}\,\sqrt{29}\,\cos\theta$$

$$\cos\theta = \frac{0}{\sqrt{24}\,\sqrt{29}}$$

$$\cos\theta = 0$$

$$\theta = \cos^{-1}(0) = 90^{\circ}$$

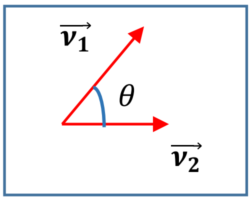
$$\overrightarrow{v_1} \cdot \overrightarrow{v_2} = \langle -2, 2, 4\rangle \cdot \langle 4, -2, 3\rangle$$

$$= -8 - 4 + 12$$

$$= 0$$

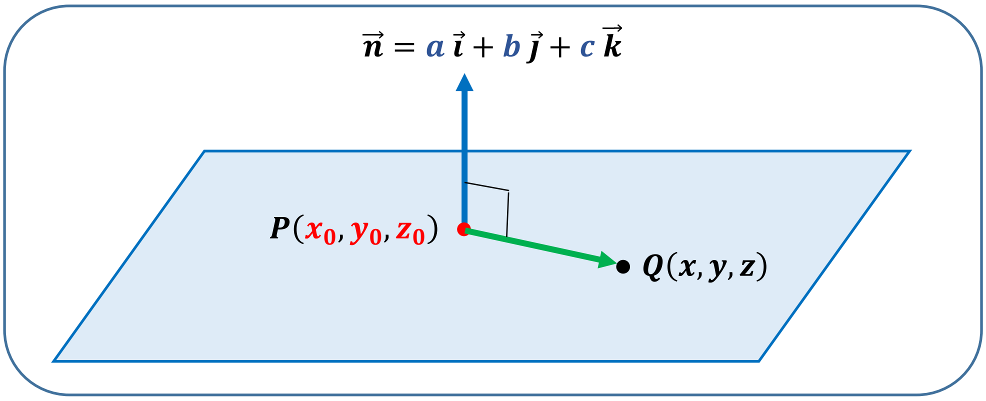
$$|\overrightarrow{v_1}| = \sqrt{(-2)^2 + 2^2 + 4^2} = \sqrt{24}$$

$$|\overrightarrow{v_2}| = \sqrt{4^2 + (-2)^2 + 3^2} = \sqrt{29}$$



(B) Equation of Plane

Introduction



- If $Q(x, y, z)$ is any point on a plane
- Given that $P(x_0, y_0, z_0)$ is a point on the plane and the plane is perpendicular to a vector $\vec{n} = a\,\vec{i} + b\,\vec{j} + c\,\vec{k}$
- Since $\overrightarrow{PQ}$ and $\vec{n}$ are perpendicular,

  then $\overrightarrow{PQ} \cdot \vec{n} = 0$

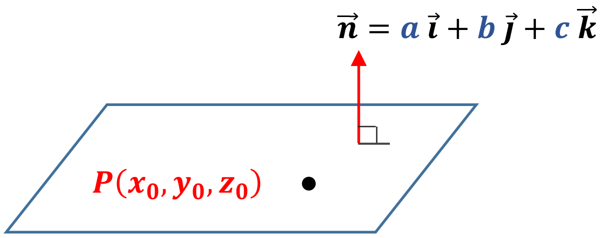
  $(\,\overrightarrow{OQ} - \overrightarrow{OP}\,) \cdot \vec{n} = 0$

  $[\,\langle x, y, \mathrm{z}\rangle - \langle x_0, y_0, z_0\rangle\,] \cdot \langle a, b, c\rangle = 0$

$\langle x - x_o, y - y_o, \mathrm{z} - z_o\rangle \cdot \langle a, b, c\rangle = 0$

This is the equation of plane

Equation of plane



Definition: **<u>Equation of plane</u>** that passes through a point $P(x_0, y_0, z_0)$ and perpendicular (orthogonal) to a vector $\vec{n} = a\,\vec{i} + b\,\vec{j} + c\,\vec{k}$ can be expressed as:

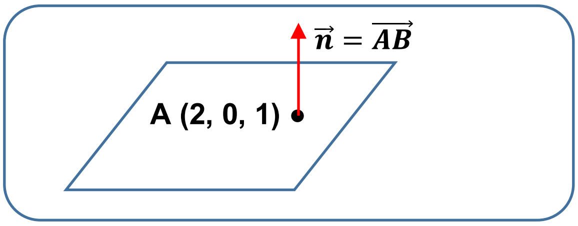
$$\langle x - x_o, y - y_o, z - z_o\rangle \cdot \langle a, b, c\rangle = 0$$

$$ax + by + cz = ax_o + by_o + cz_o$$

$$\boxed{ax + by + cz = D}$$

Example 1: **Given points $A(2,0,1)$ and $B(4,1,0)$.**
Find the equation of plane that passes through point A and perpendicular to $\overrightarrow{AB}$.

Solution:



$$\vec{n} = \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \langle 4,1,0\rangle - \langle 2,0,1\rangle$$

$$= \langle 2,1,-1\rangle$$

Using point $A(2,0,1)$, the equation of plane:

$$\langle x - x_o, y - y_o, \mathrm{z} - z_o\rangle \cdot \langle a,b,c\rangle = 0$$

$$\langle x - 2, y - 0, \mathrm{z} - 1\rangle \cdot \langle 2,1,-1\rangle = 0$$

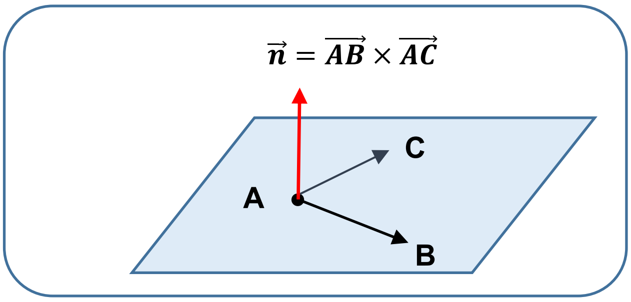
$$2x - 4 + y - z + 1 = 0$$

$$\boxed{2x + y - z = 3}$$

Please take note that the coefficients are taken from $\vec{n} = \langle 2,1,-1\rangle$

Example 2: Given points $A(4,-1,0)$, $B(2,4,1)$ and $C(0,2,0)$.
Find the equation of plane containing the points A, B and C.

Solution:



$$\vec{n} = \overrightarrow{AB} \times \overrightarrow{AC}$$

$$= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -2 & 5 & 1 \\ -4 & 3 & 0 \end{vmatrix}$$

$$= \vec{i}\begin{vmatrix} 5 & 1 \\ 3 & 0 \end{vmatrix} \textcircled{-} \vec{j}\begin{vmatrix} -2 & 1 \\ -4 & 0 \end{vmatrix} + \vec{k}\begin{vmatrix} -2 & 5 \\ -4 & 3 \end{vmatrix}$$

$$= -3\,\vec{i} - 4\,\vec{j} + 14\,\vec{k}$$

$$\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$$

$$= \langle 2,4,1\rangle - \langle 4,-1,0\rangle$$

$$= \langle -2,5,1\rangle$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$= \langle 0,2,0\rangle - \langle 4,-1,0\rangle$$

$$= \langle -4,3,0\rangle$$

Using point $A(4,-1,0)$, the equation of plane:

$$\langle x - x_o, y - y_o, z - z_o\rangle \cdot \langle a,b,c\rangle = 0$$

$$\langle x - 4, y - (-1), z - 0\rangle \cdot \langle -3,-4,14\rangle = 0$$

$$\langle x - 4, y + 1, z\rangle \cdot \langle -3,-4,14\rangle = 0$$

$$-3x + 12 - 4y - 4 + 14z = 0$$

$$\boxed{-3x - 4y + 14z = -8}$$

Please take note that the coefficients are taken from $\vec{n} = \langle -3,-4,14\rangle$

Example 3: Given a line L_1 and a plane P_1 .

$\boldsymbol{L_1}$: $\boldsymbol{x = 3\,t,} \qquad \boldsymbol{y = 2 + 2\,t,} \qquad \boldsymbol{z = 4 + t}$

$\boldsymbol{P_1}$: $\boldsymbol{2x - y + z = 22}$

Find the intersection point of line L_1 and plane P_1.

Solution:

Substitute L_1 into P_1:

L_1: $x = 3\,t$

$y = 2 + 2\,t$

$z = 4 + t$

P_1: $2x - y + z = 22$

$2(3t) - (2 + 2t) + (4 + t) = 22$

$t = 4$

To find intersection point, substitute $t = 4$ into

L_1: $x = 3\,t, \qquad y = 2 + 2\,t, \qquad z = 4 + t$

$x = 3\,(4) = 12$

$y = 2 + 2\,(4) = 10$

$z = 4 + (4) = 8$

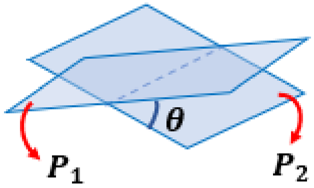
Intersection point:

(12, 10, 8)

Example 4: Find the angle of intersection between plane P_1 and P_2 .

$\boldsymbol{P_1}$: $\boldsymbol{3x - 6y - 2z = 15}$

$\boldsymbol{P_2}$: $\boldsymbol{2x + y - 2z = 5}$



Solution:

Plane P_1:

$\overrightarrow{n_1} = \langle 3, -6, -2 \rangle$

Plane P_2:

$\overrightarrow{n_2} = \langle 2, 1, -2 \rangle$

P_1: $3x - 6y - 2z = 15$

P_2: $2x + y - 2z = 5$

Please take note that $\vec{n}$ is taken from the coefficients

To find the angle:

$$\overrightarrow{n_1} \cdot \overrightarrow{n_2} = |\overrightarrow{n_1}|\,|\overrightarrow{n_2}|\, \boldsymbol{\cos\theta}$$

$$4 = (7)(3)\, cos\theta$$

$$cos\theta = \frac{4}{(7)(3)}$$

$$cos\theta = 0.1905$$

$$\theta = cos^{-1}(0.1905) = \mathbf{79.02}^{\circ}$$

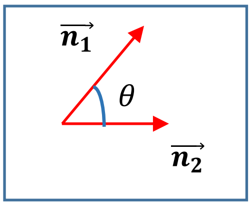
$$\overrightarrow{n_1} \cdot \overrightarrow{n_2} = \langle 3, -6, -2 \rangle \cdot \langle 2, 1, -2 \rangle$$

$$= 6 - 6 + 4$$

$$= 4$$

$$|\overrightarrow{n_1}| = \sqrt{3^2 + (-6)^2 + (-2)^2} = 7$$

$$|\overrightarrow{n_2}| = \sqrt{2^2 + 1^2 + (-2)^2} = 3$$



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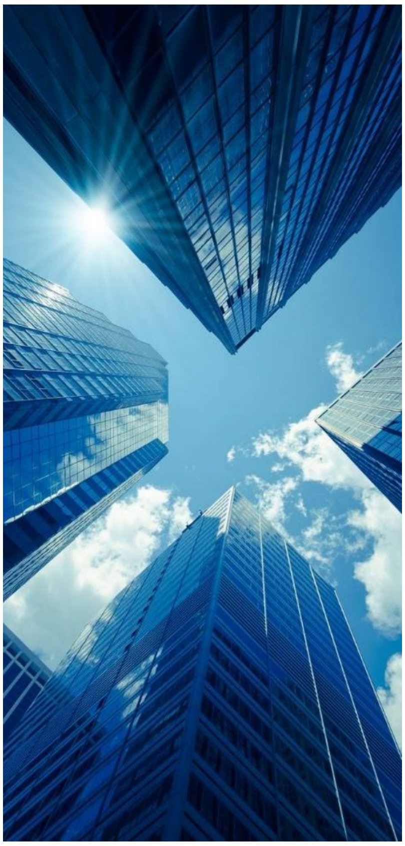

MATRIX ALGEBRA AND VECTOR ALGEBRA

The book *Matrix Algebra and Vector Algebra* covers a series of topics in mathematics such as operations on matrices, determinants, inverse matrices, vector in two- and three-dimensional space. With the exposure of these basic concepts in matrix and vector algebra, students would be able to understand and apply the knowledge in solving system of linear equations using matrix and solving simple geometric applications.

The concept of each topic is explained in the beginning of each chapter. Examples are given to illustrate the concept. Detailed steps are shown on how to solve the problems. This module is suitable to be used during lecture session to support teaching and learning processes.

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