SECOND ORDER DIFFERENTIAL EQUATION

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Second Order Differential Equation

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PREFACE

This eBook was created to offer a clear, intuitive, and structured introduction to the core concepts of calculus: limits, derivatives, integrals, and their applications. Whether you're a student encountering calculus for the first time, a self-learner brushing up on fundamentals, or an instructor seeking a supplemental resource, this book is designed to guide you step-by-step through the subject with clarity and purpose. Our goal is not only to help you succeed in calculus but to foster an appreciation for its power, elegance, and relevance.

We hope this eBook becomes a valuable companion on your mathematical journey !

SECOND ORDER ODE

Homogeneous Second Order

Steps in solving second order differential equation :

1. Find characteristic equation by substituting $y'' = m^2$, $y' = m$ and $y = 1$. Therefore, characteristic equation for $ay'' + by' + cy = 0$ is $am^2 + bm + c = 0$.

2.Solve the characteristic equation - either by calculator or using formula.

3.When m_1 and m_2 are known as the roots of the characteristic equation, then the solution can be expressed as the following forms :

a) $y = c_1e^{m1x} + c_2e^{m2x}$

b) $y = c_1^{mx} + c_2xe^{mx}$

c) $y = e^{\infty x}(c_1 \cos Bx + c_2 \sin Bx)$

Nature Of Roots	Roots	Solution
1. Real and Distinct	m_1 and m_2	$y = c_1e^{m1x} + c_2e^{m2x}$
2. Real and Equal	m	$y = c_1^{mx} + c_2xe^{mx}$
3. Complex and Distinct	$\alpha \pm \beta i$	$y = e^{\infty x}$ $(c_1 \cos Bx + c_2 \sin Bx)$

Solve the following second order differential equation :

a) $y'' - 15y' + 56y = 0$

$$y'' - 15y' + 56y = 0$$

$$m^2 - 15m + 56 = 0$$

$$(m - 7)(m - 8) = 0$$

$$m - 7 = 0, m - 8 = 0$$

$$m = 7, m = 8$$

$$\therefore y = c_1e^{7x} + c_2e^{8x}$$

b) $y'' + 6y' + 9y = 0$

$$y'' + 6y' + 9y = 0$$

$$m^2 + 6m + 9 = 0$$

$$(m + 3)(m + 3) = 0$$

$$m + 3 = 0, \quad m + 3 = 0$$

$$m = -3, m = -3$$

$$\therefore y = c_1^{-3x} + c_2xe^{-3x}$$

c) $y'' + 2y' + 10y = 0$

$$y'' + 2y' + 10y = 0$$

$$m^2 + 2m + 10 = 0$$

$$a = 1,\ b = 2,\ c = 10$$

$$-b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$-2 \pm \frac{\sqrt{(2)^2 - 4(1)(10)}}{2(1)}$$

$$-2 \pm \frac{\sqrt{-36}}{2}$$

$$\frac{-2 \pm 6i}{2}$$

$$-1 \pm 3i$$

$$\therefore y = e^{-1x}(\ c_1 \cos 3x + c_2 \sin 3x\ )$$

Non-Homogeneous Second Order

Function of R(x)	Example of R(x)	Trial Solution, yp
a) Polynomial	$R(x) = 8$	$yp(x) = c$
	$R(x) = 5x+2$	$yp(x) = ax+b$
	$R(x) = 3x^2+2x+7$	$yp(x) = ax^2+bx+c$
b) Exponential	$R(x) = 6e^{5x}$	$yp(x) = ce^{5x}$
	$R(x) = e^{4x}$	$yp(x) = ce^{4x}$
c) Trigonometry	$R(x) = 4\sin(3x)$	$yp(x) = A\cos(3x)+B\sin(3x)$
	$R(x) = \cos(3x)$	$yp(x) = A\cos(3x)+B\sin(3x)$
	$R(x) = \sin(3x)-4\cos(3x)$	$yp(x) = A\cos(3x)+B\sin(3x)$
d) Addition or Multiplication	$R(x) = x^2+8e^{3x}$	$yp(x) = (ax^2+bx+c)+de^{3x}$
	$R(x) = 7xe^x$	$yp(x) = (ax+b)e^x$
	$R(x) = xe^x+8e^{7x}$	$yp(x) = (ax+b)e^x+ce^{7x}$
	$R(x) = e^{6x}\cos(9x)$	$yp(x) = e^{6x}(A\cos(9x)+B\sin(9x))$
	$R(x) = 5x\cos(6x)$	$yp(x) = (ax+b)(A\cos(6x)+B\sin(6x))$

Steps to solve second order differential equation : $ay'' + by' + cy = R(x)$

Step 1 :

Solve $ay'' + by' + cy = 0$

Step 2 :

Solve $ay'' + by' + cy = R(x)$

a) Find y_p

b) Find y'_p

c) Find y''_p

d) Substitute y_p, y'_p, y''_p into $ay'' + by' + cy = R(x)$

e) Find values of constant in $R(x)$

Step 3 :

$y = y_c + y_p$

Find the general solution for the following :

a) $y'' - 6y' + 8y = 8x + 2$

LHS :

$$y'' - 6y' + 8y = 0$$

$$m^2 - 6m + 8 = 0$$

$$(m - 4)(m - 2) = 0$$

$$m - 4 = 0, \quad m - 2 = 0$$

$$m = 4, \quad m = 2$$

$$\therefore y_c = c_1e^{4x} + c_2e^{2x}$$

RHS :

$$R(x) = 8x + 2$$

$$y_p = ax + b$$

$$y'_p = a$$

$$y''_p = 0$$

Substitute :

$$y'' - 6y' + 8y = 8x + 2$$

$$(0) - 6(a) + 8\,(ax + b) = 8x + 2$$

$$-6a + 8ax + 8b = 8x + 2$$

Compare :

$$8ax = 8x$$
$$8a = 8$$
$$a = 1$$

$$-6a + 8b = 2$$
$$-6(1) + 8b = 2$$
$$-6 + 8b = 2$$
$$8b = 8$$
$$b = 1$$

$$y_p = ax + b$$
$$\therefore y_p = 1x + 1$$

$$y = y_c + y_p$$

$$y_c = c_1e^{4x} + c_2e^{2x},\ y_p = 1x + 1$$

$$y = (c_1e^{4x} + c_2e^{2x}) + (1x + 1)$$

$$\therefore\ y = c_1e^{4x} + c_2e^{2x} + x + 1$$

$$b)\ y'' - 4y' + 4y = 5e^{-x}$$

LHS :

$$y'' - 4y' + 4y = 0$$

$$m^2 - 4m + 4 = 0$$

$$(m - 2)(m - 2) = 0$$

$$m - 2 = 0,\ m - 2 = 0$$

$$m = 2,\ m = 2$$

$$\therefore yc = c_1^{2x} + c_2xe^{2x}$$

RHS :

$$R(x) = 5e^{-x}$$

$$y_p = ce^{-x}$$

$$y'_p = -ce^{-x}$$

$$y''_p = ce^{-x}$$

Substitute :

$$y'' - 4y' + 4y = 5e^{-x}$$

$$(ce^{-x}) - 4(-ce^{-x}) + 4(ce^{-x}) = 5e^{-x}$$

$$ce^{-x} + 4ce^{-x} + 4ce^{-x} = 5e^{-x}$$

$$9ce^{-x} = 5e^{-x}$$

Compare :

$$9ce^{-x} = 5e^{-x}$$

$$9c = 5$$

$$c = 5/9$$

$$yp = ce^{-x}$$

$$\therefore yp = 5/9e^{-x}$$

$$y = yc + yp$$

$$yc = c_1^{2x} + c_2xe^{2x},\ yp = 5/9e^{-x}$$

$$y = (c_1^{2x} + c_2xe^{2x}) + (5/9e^{-x})$$

$$\therefore\ y = c_1^{2x} + c_2xe^{2x} + 5/9e^{-x}$$

c) $y'' + 4y' + 5y = 30x^2 + 12$

LHS :

$y'' + 4y' + 5y = 0$

$m^2 + 4m + 5 = 0$

$a = 1, b = 4, c = 5$

$$-b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$-4 \pm \frac{\sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$

$$-4 \pm \frac{\sqrt{-4}}{2}$$

$$\frac{-4 \pm 2i}{2}$$

$$-2 \pm i$$

$\therefore yc = e^{2x}(c_1 \cos x + c_2 \sin x)$

LHS :

$$-b \pm \frac{\sqrt{b^2 - 4ac}}{2a}$$

$$-4 \pm \frac{\sqrt{(-4)^2 - 4(1)(5)}}{2(1)}$$

$$-4 \pm \frac{\sqrt{-4}}{2}$$

$$\frac{-4 \pm 2i}{2}$$

$$-2 \pm i$$

$$\therefore yc = e^{2x}(\, c_1 \cos x + c_2 \sin x \,)$$

RHS :

$$R(x) = 30x^2 + 12$$

$$yp = ax^2 + bx + c$$

$$y'p = 2ax + b$$

$$y''p = 2a$$

Substitute :

$$y'' + 4y' + 5y = 30x^2 + 12$$

$$(2a) + 4(2ax + b) + 5(ax^2 + bx + c) = 30x^2 + 12$$

$$2a + 8ax + 4b + 5ax^2 + 5bx + 5c = 30x^2 + 12$$

Compare :

$$5ax^2 = 30x^2$$
$$5a = 30$$
$$a = 30/5$$
$$a = 6$$

$$8ax + 5bx = 0x$$
$$8a + 5b = 0$$
$$8(6) + 5b = 0$$
$$48 + 5b = 0$$
$$5b = -48$$
$$b = -48/5$$

$$2a + 4b + 5c = 12$$

$$2(6) + 4(-48/5) + 5c = 12$$

$$12 - 192/5 + 5c = 12$$

$$-192/5 + 5c = 0$$

$$-192/5 = -5c$$

$$-192 = -25c$$

$$-192/-25 = c$$

$$192/25 = c$$

$$\therefore yp = 6x^2 - 48/5x + 192/25$$

$$y = y_c + y_p$$

$$y_c = e^{2x}(\ c_1 \cos x + c_2 \sin x\ ),$$

$$y_p = 6x^2 - 48/5x + 192/25$$

$$y = (e^{2x}(\ c_1 \cos x + c_2 \sin x\ )) + (6x^2 - 48/5x + 192/25)$$

$$\therefore\ y = e^{2x}(\ c_1 \cos x + c_2 \sin x\ ) + 6x^2 - 48/5x + 192/25$$

Modified Version :

d) $y'' + 3y' = 3e^{-3x}$

LHS :

$$y'' + 3y' = 0$$

$$m^2 + 3m = 0$$

$$m(m+3) = 0$$

$$m = 0,\ m = -3$$

$$\therefore y_c = c_1e^{0x} + c_2e^{-3x}$$

RHS :

$$R(x) = 3e^{-3x}$$

$$yp = cxe^{-3x}$$

$$y'p = -3cxe^{-3x} + ce^{-3x}$$

$$y''p = 9cxe^{-3x} - 3ce^{-3x} - 3ce^{-3x}$$

$$= 9cxe^{-3x} - 6ce^{-3x}$$

$$u = cx, v = e^{-3x}$$

$$u' = c, v' = -3e^{-3x}$$

$$uv' + vu' = (cx)(-3e^{-3x}) + (e^{-3x})(c)$$

$$= -3cxe^{-3x} + ce^{-3x}$$

$$u = -3cx, v = e^{-3x}$$

$$u' = -3c, v' = -3e^{-3x}$$

$$uv' + vu' = (-3cx)(-3e^{-3x}) + (e^{-3x})(-3c)$$

$$= 9cxe^{-3x} - 3ce^{-3x}$$

Substitute :

$$y'' + 3y' = 3e^{-3x}$$

$$(9cxe^{-3x} - 6ce^{-3x}) + 3(-3cxe^{-3x} + ce^{-3x}) = 3e^{-3x}$$

$$9cxe^{-3x} - 6ce^{-3x} - 9cxe^{-3x} + 3ce^{-3x} = 3e^{-3x}$$

$$-3ce^{-3x} = 3e^{-3x}$$

$$-3c = 3$$

$$c = -1$$

$$\therefore yp = -1xe^{-3x}$$

$$y = y_c + y_p$$

$$y_c = c_1e^{0x} + c_2e^{-3x},\ y_p = -1xe^{-3x}$$

$$y = (c_1e^{0x} + c_2e^{-3x}) + (-1xe^{-3x})$$

$$\therefore\ y = c_1e^{0x} + c_2e^{-3x} - xe^{-3x}$$

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