TECHNICAL REPORT

COMPARING METHOD IN DOUBLE PENDULUM

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IN THE NAME OF ALLAH, THE MOST GRACIOUS, THE MOST MERCIFUL

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ABSTRACT

In this project paper, its present the motion of the curves in double pendulum by comparing the three types of method that related each other. The method that used in double pendulum are Lagrangian, Euler equation, Hamilton's and lastly Runge Kutta. This method are related each other because to derive the Euler equation, formula of Lagrangian is needed and also from Euler equation, it can derive into two types of method such as Hamilton's and Runge Kutta but Runge Kutta can also derive from Hamilton's. All this method are needed to know their motion, structure of wave, and so on. Mathematica software is needed for solving the problem of double pendulum and to get the accurate result based on graph of parametric and for animation, it is shows their movements. This software can solve all this method included Lagrangian.

1 INTRODUCTION

Pendulum that attach with another pendulum is called double pendulum. The area of dynamical system in physic and mathematics, a rich dynamic behavior of a strong sensitivity is exhibits from the double pendulum of simple physic system with initial conditions. Double pendulum have a difference types whether same mass or different mass that declare as m_1 and m_2 and same length or different length that declare as L_1 or L_2. Its also have different angles. In Stickel (2009), a diagram of a double is shown in Fig. 1.1. The

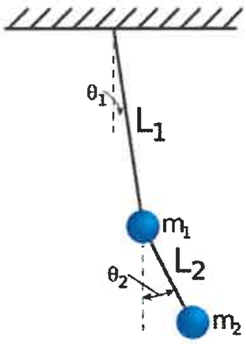

Figure 1.1: Double Pendulum

conservative system happens when double pendulum is friction-less that allows a conservation of energy, that is $Energy_{in} = Energy_{out}$. Furthermore, Stickel (2009) mentioned that double pendulum is two masses attached to rigid, mass less, rod with the base at a stationary location . In other words, the double pendulum become a linear system when angle is small and become non linear when angle is big.

To predict the behavior of double pendulum is very limited in certain regimes that is initial condition because the extreme sensitivity towards even small perturbations. In addition, Nielsen & B.T. (2013) said that the double pendulum is considered as a model system exhibiting deterministic chaotic behavior and the motion is governed by a set of coupled differential equations. This project we will use four types of methods to solve the double pendulum which are Lagrangian Equation, Range-Kutta Equation, Hamilton's Equation and lastly Euler Equation. In Stickel (2009), the Lagrangian is representation system of motion and can be used when system is conservative . Determine expressions for the kinetic energy and the potential when apply the Lagrange's equation (S.Widnall, 2009). The general equation for this method is :-

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{1}$$

Runge-Kutta equation is generally to solve differential equation numerically and its very accurate also well behaved for wide range of problems. Generally, the general solution of Runge-Kutta for double pendulum is :-

$$w_0 = \alpha \tag{2}$$

$$w_{i+1} = w_i + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \tag{3}$$

which is $w_i \approx y(t_i)$ computes an approximate solution. Hamilton's Equation is used when to solve the trajectories of double pendulum. The formula of Hamilton's is:-

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{4}$$

Lastly, Euler-Lagrange equations for θ_1 and θ_2 are :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta_1}\right) = \frac{\partial L}{\partial \theta_1} \tag{5}$$

All these methods will be discussed in details in methodology.

1.1 Definition of Terms and Concepts

The following are the definition of terms and concepts used in this project:

Analytical dynamic point	-	the relationship between motion, forces acting and the properties of bodies particularly mass and moment of inertia.
Chaotic	-	completely confused or disordered pattern.
Closed form	-	a mathematical expression that can be evaluated in a finite number of operations.
Conservative force	-	a force with the property that the work done in moving a particle between two points is independent of the taken path.
Conservative system	-	the difference between the final and initial values of an energy function.

Conversation law - any law stating that some quantity or property remains constant during and after an interaction or process, as conversation of charge or conservation of linear momentum.

Gravity force - attracts any object with mass.

Holonomic - constraints expressible as a function of the coordinates x_j and time t.

Incompressible fluid - a flow in which the material density is constant within a fluid parcel.

Oscillation - a single swing(as of an oscillating body) from one extreme limit to the other.

Pendulum - something that hanging and swinging freely from a fixed point.

Perturbations - Process from its regular or normal state or path.

Pivot point - the object rotates about an axis.

Regimes - A class of physical condition usually parameterized by some specific measures.

Rigid body fluid - a continuous mass distribution or usually thought of as a collection of point masses.

Rigid rod - one along which any disturbance travels with the fundamental speed without the dissipation of energy.

Suspension - a construction passing from a map to a flow.

1.2 Literature Review

Single Pendulum

A single pendulum has a single particle m hanging from a string of length l and fixed at a point Q. the pendulum will swing back and forth with periodic motion as shown in Figure 1.2 when displaced to an initial angle and released. As for the simple pendulum, the equation of motion for the pendulum may be obtained by applying Newton's second law for rotational systems.

$$\tau = I\alpha \Rightarrow -mg(\sin\theta)l = ml^2\frac{d^2\theta}{dt^2}$$
$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\sin\theta = 0 \qquad (6)$$

where τ and α are the force the angular acceleration . If the amplitude of angular dis-

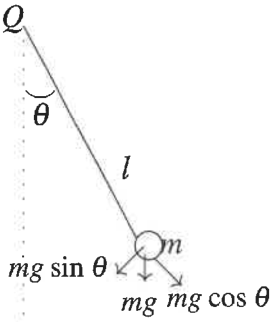

Figure 1.2: Simple pendulum

placement is small enough that $\sin\theta \approx \theta$, equation (6) becomes

$$\frac{d^2\theta}{dt^2} + \frac{g}{l}\theta = 0 \tag{7}$$

gives the simple harmonic solution

$$\theta(t) = \theta_0 \cos(\omega t + \psi) \tag{8}$$

where $\omega = \sqrt{g/l}$ is the natural frequency of the motion. According to Gonzalez (2008), the single pendulum is always at a fixed distance from point and can oscillate in (x,y) plane. Mathematically,

$$v = 0 \tag{9}$$

$$|\, r \,| = l \tag{10}$$

The angular position of the pendulum θ ; which we can use to write:

$$r = l(\sin\theta, \cos\theta, 0). \tag{11}$$

The gravity force of simple pendulum is always in the same direction and same magnitude which is 9.8 N/kg. Both its direction and its magnitude changes as the bob swings to and fro and it always towards the pivot point. Witherden (2001) also mention that the tension pointing towards the origin, along the direction of $-r$ and gravity is along y-direction with gravitational acceleration g:

$$F = T\frac{-r}{|r|} + mg = -\frac{T}{l}r + mg \tag{12}$$

Notice that there will be two unknown of two equations, T and θ. Notice also the tension force will be greater than the perpendicular component of gravity when the bob moves through this equilibrium position. Kelly (1993) stated that since the bob is in motion along a circular arc, there must be a net force at this position.

Double Pendulum

Double pendulum have a two masses m_1 and m_2 which connected by rigid weightless rod of length l_1 and l_2, subject to gravity forces. According to Gonzalez (2008), each particle moving in the (x,y) plane, and the constraints are holonomic which is they are only algebraic relationship between coordinates but not involving inequalities or derivatives and each rod having constant lengths. Since there are two generalized coordinates, the expression for r_1,r_2 in term of two angles θ_1,θ_2:

$$r_1 = l_1(\sin\theta_1, \cos\theta_2, 0) \tag{13}$$

$$r_2 = r_1 + l_2(\sin\theta_2, \cos\theta_2, 0) \tag{14}$$

The tension in the upper rod is along the direction $-r_1$, and due to the lower rod is along the direction $r_2 - r_1$ and F_1 will be;

$$F_1 = T_1\frac{-r_1}{\mid r_1 \mid} + T_2\frac{r_2 - r_1}{\mid r_2 - r_1 \mid} + m_1 g = -\frac{T_1}{l_1}r_1 + \frac{T_2}{l_2}(r_2 - r_1) + m_1 g \tag{15}$$

The tension on $m_2$is along the direction of $-(r_2 - r_1)$ will be:

$$F_2 = T_2\frac{-(r_2 - r_1)}{\mid r_2 - r_1 \mid} + m_2 g = -\frac{T_2}{l_2}(r_2 - r_1) + m_1 g \tag{16}$$

Witherden (2001) mentioned that since there exists no analytical solution for double pendulum, it must instead be done numerically which there exist several solvers. Considering several methods such as Lagrangian equation which allows for firstly verification and secondly allows for a comparison to be made between methods such as Euler, Hamilton and also Runge-Kutta.

Lagrangian Equation

The Lagrangian can be find by using the equation of the motion of the system in term of generalized coordinates :

$$L = K - P \tag{17}$$

where K represent kinetic equation and P represent potential energy.

As for the conservative system of Lagrangian, Rao & J.Srinivas (2007) stated that Lagrange's equation proposed and approach which will obtain the equation of motion in generalized coordinates of the system from the analytical dynamics points of view which can also be expressed:

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{18}$$

The results that describe the equations of motion of the system in the differential equations.

Runge-Kutta Equation

According to Stoer & Bullrsch (1980), there are many ways to evaluate f(x, y), but the higher-order error terms in a different coefficients. Adding up the right combination of these, we can eliminate the error terms order by order. That is the basic idea of the Runge-Kutta method. Lambert (1973) stated that Runge-Kutta 4^{th} order method is a numerical technique used to solve ordinary differential equation of the system.

The fourth order Runge-Kutta method can be expressed as follows:

$$\begin{cases} y' & = f(t,y) \\ y(t_0) & = \alpha \end{cases} \tag{19}$$

By defining h to be the time step size and $t_i = t_0 + ih$. Then, the following formula can be expressed as:

$$w_0 = \alpha \tag{20}$$

$$m_1 = hf(t_i, w_i) \tag{21}$$

$$m_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{m_1}{2}\right) \tag{22}$$

$$m_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{m_2}{2}\right) \tag{23}$$

$$m_4 = hf(t_i + h, w_i + m_3) \tag{24}$$

$$w_{i+1} = w_i + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \tag{25}$$

which computes an approximate solution, that is $w_i \approx y(t_i)$.

According to Rice (1983), to achieve some predetermined accuracy in the solution with minimum computational effort is one of the purpose of adaptive step-size control.Thus, some related conserved quantity that can be monitored although the accuracy may be demanded not directly in the solution itself.

Hamilton's Equation

According to Ramegowda (2001), Hamiltonian or Hamiltonian formulation consists of two independent variables which are canonical coordinates and canonical momenta.These two variables come when replaced the n 2^{nd} order differential equations by $2n$ 1^{st} order differential equations for p_i and q_i.

The formula of Hamilton's Equation can be expressed as :

$$
\begin{aligned}
\dot{p}_i &= -\frac{\partial H}{\partial q_i} \\
\dot{q}_i &= \frac{\partial H}{\partial p_i} \\
-\frac{\partial L}{\partial t} &= \frac{\partial H}{\partial t}
\end{aligned}
\tag{26}
$$

Hamiltonian advantages are that it leads to powerful geometric techniques for studying the properties of dynamical system. It allows for a beautiful expression of the relation between symmetries and conversation law, and it leads to many view that can be viewed as the macroscopic "classical" (Stroup, 2004).

Euler Equation

According to Batchelor (2008), Euler's equation are the equation that written out entirely in term of the principal axes attached to the rigid body. A derivation of the Euler's equations. The torque equation in terms of the frame fixed that related to rigid body fluid is:

$$\left[I_{xx}\frac{d\omega_x}{dt}+(I_{zz}-I_{yy})\omega_y\omega_x\right]=\tau_x \tag{27}$$

$$\left[I_{yy}\frac{d\omega_y}{dt}+(I_{xx}-I_{yy})\omega_z\omega_x\right]=\tau_y \tag{28}$$

$$\left[I_{zz}\frac{d\omega_z}{dt}+(I_{yy}-I_{zz})\omega_x\omega_y\right]=\tau_z \tag{29}$$

Hunter (2004) stated that the incompressible Euler equation for the flow of inviscid, incompressible fluid, describe some of their basic mathematical features, and provide a perspective on their physical applicability.

The incompressible Euler equations are the following PDEs for (il,p) :

$$il_t+il\cdot\nabla il+\nabla p=0 \tag{30}$$

$$\nabla\cdot il=0 \tag{31}$$

The Euler-Lagrange equations for θ_1 and θ_2 are :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial\theta_1}\right)=\frac{\partial L}{\partial\theta_1} \tag{32}$$

The θ_1 equation is :

$$\ell_1\left[(m_1+m_2)\ell_1\ddot{\theta_1}+m_2\ell_2\cos(\theta_1-\theta_2)\ddot{\theta_2}+m_2\ell_2\sin(\theta_1-\theta_2)\dot{\theta_2^2}+(m_1+m_2)g\sin\theta_1\right] \tag{33}$$

and the θ_2 equation is :

$$m_2\ell_2\left[\ell_2\ddot{\theta_2}+\ell_1\cos(\theta_1-\theta_2)\ddot{\theta_1}-\ell_1\sin(\theta_1-\theta_2)\dot{\theta_1^2}+g\sin\theta_2\right]=0 \tag{34}$$

1.3 Significant of the Project

1. This project will give you the exact solution for motion force especially in double pendulum problem.
2. Double pendulum can be applied in sports. For instance, tennis, badminton and so on.
3. Its also can be applied in human biomechanics model which is hip joint control model.

1.4 Scope of the Project

1. We will study the Lagrangian equation that can use in whole dealing with scalar quantities such as the potential and the kinetic energy of the system.
2. Use the Runge-Kutta equation, Hamilton's equation and Euler equation to solve the double pendulum problem.

1.5 Problem Statement

Suppose that a motion can be described if the lower mass is given angular velocity and there are two masses hanging from a fixed point and free to rotate in the vertical plane including above pivot point. For the first initial condition, the motion is very chaotic. It produces when lower mass being forced with counter-intuitive at times to vertical at specific time and it slowly move to become normal angular motion. For this problem,the methods to solve is by using Lagrangian equation.In this study, the problem that we obtain is when the solution to the Lagrangian equation in differential equation of motion of a dynamic system cannot be obtained in closed form.

1. What is the suitable method that can solve the double pendulum problem in a closed form using numerical approach?

2. How to apply the suitable method in double pendulum problem?

1.6 Objective

The purposes of this project are

1. To derive the Lagrangian equation of double pendulum problem into Euler-Lagrangian, Hamilton and Runge-Kutta.
2. To compare three type of methods based on motion of curves which is Lagrangian Equation, Euler's Method and Runge-Kutta.
3. At the end, we want to find the best solution by comparing all the methods.

2 METHODOLOGY

2.1 Step 1:Development of Lagrangian Equation for Double Pendulum

First, the x-axis pointing along the horizontal direction and the y-axis pointing vertically upwards and fixed point O will be taken as the origin of the Cartesian coordinate system. Let θ_1 and θ_2 be the angles which the vertical direction of the first and second rods make with respectively.Now, we will consider the Lagrangian equation by :

$$L = K - P \tag{35}$$

where K represents kinetic equation and P represents potential energy.From the potential equation,$P = mga$, we can find the potential energies for the first and second pendulums by simplifying them.In order to implement in the Runge-Kutta equation, first, Lambert (1973) mentioned that we should identify the fourth-order of Runge-Kutta which can be seen as ODE integrator. In order to carry out the Runge-Kutta, we need to input the values of the independent variables on a set of n differential equation and step-size, h. In the end, the solution will be:

$$w_0 = \alpha \tag{36}$$

$$m_1 = hf(t_i, w_i) \tag{37}$$

$$m_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{m_1}{2}\right) \tag{38}$$

$$m_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{m_2}{2}\right) \tag{39}$$

$$m_4 = hf(t_i + h, w_i + m_3) \tag{40}$$

$$w_{i+1} = w_i + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \tag{41}$$

which $w_i \approx y(t_i)$ computes an approximate solution. In order to get into Runge-Kutta (RK4), we need to develop Euler-Lagrange and Hamiltonian first.By using software Mathematica, we would generate the equation to get a significant result.

2.2 STEP 2 : Euler-Lagrangian Equation Development from Lagrangian System

The Lagrangian formulation,the function $L(p_i, q_i, t)$ where p_i and q_i $(i = 1, \cdots, n)$ are n generalized coordinates (Kelly, 1993).Euler-Lagrange system is also called "Lagrange's Equation of Second Kind".The Hamilton's equation can be derive from the Lagrange equation by substitute the value of p_i and q_i. The final solution will be :

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{42}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{43}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{44}$$

These are the 2^{nd} order differential equations which require 2n initial conditions. In order to make it n generalized equation, Ramegowda (2001) stated it will be canonical momenta equation.In order to get to the Hamiltonian, we need to develop Euler-Lagrange equation. Software Mathematica could be generated in finding the result of double pendulum problem by this equation. The implementation and the result will be briefly discuss below.

2.3 STEP 3 : The Expansion of Hamiltonian into Runge-Kutta

Suppose that the upper pendulum has a massless rod of length ℓ_1 and a bob of mas m_1. The two rods provide constraints on the motion of the vertical plane which can compute into x_1, x_2, y_1 and y_2. Consider the Lagrangian equation by :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial q_i}\right) - \frac{\partial L}{\partial q_i} = 0 \tag{45}$$

From the above equation, Batchelor (2008) states that the Lagrange equation will be expand to include the Euler equation which consist of two independent generalized coordinates, θ_1 and θ_2.This two angles make the two rods going downward vertical direction repeatedly. By substituting equation of θ_1 and θ_2 in Lagrangian equation, we may have the Euler-Lagrangian equation which is :

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \theta_1}\right) = \frac{\partial L}{\partial \theta_1} \tag{46}$$

This type of formula is easily to be computed in the Mathematica software since it is only related between Euler and Lagrange formula.

3 IMPLEMENTATION

3.1 Step 1:Development of Lagrangian Equation for Double Pendulum

Based on the Lagrangian equation (35), we can derive this formula in order to implement it in Mathematica software. First, let θ_1 and θ_2 with the vertical direction be the angles which the first and second rods respectively. Hence, the position of bob is given by :

$$x_1 = \ell_1 \sin\theta_1 \tag{47}$$

$$y_1 = -\ell_1 \cos\theta_1 \tag{48}$$

$$x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2 \tag{49}$$

$$y_2 = -\ell_1 \cos\theta_1 - \ell_2 \cos\theta_2 \tag{50}$$

By differentiating, we obtain velocities of the bobs with respect to time :

$$\dot{x_1} = \ell_1 \dot{\theta}_1 \cos\theta_1 \tag{51}$$

$$\dot{y_1} = \ell_1 \dot{\theta}_1 \sin\theta_1 \tag{52}$$

$$\dot{x_2} = \ell_1 \dot{\theta}_1 \cos\theta_1 + \ell_2 \dot{\theta}_2 \cos\theta_2 \tag{53}$$

$$\dot{y_2} = \ell_1 \dot{\theta}_1 \sin\theta_1 + \ell_2 \dot{\theta}_2 \sin\theta_2 \tag{54}$$

Now, we will consider the Lagrangian equation in (35) where K represents kinetic equation and P represents potential energy.From the potential equation,$P = mga$, we can find the potential energies for the first and second pendulums by simplifying them. Hence, the kinetic energy, K is given by :

$$K=\frac{1}{2}m_1\ell_1^2\dot{\theta}_1^2+\frac{1}{2}m_2[\ell_1^2\dot{\theta}_1^2+\ell_2^2\dot{\theta}_2^2+2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)] \tag{55}$$

where above that $\cos\theta_1\cos\theta_2+\sin\theta_1\sin\theta_2=\cos(\theta_1-\theta_2)$.

By developing kinetic energy, the potential energy, P is given by:

$$P=-(m_1+m_2)g\ell_1\cos\theta_1-m_2g\ell_2\cos\theta_2 \tag{56}$$

Then, the complete Lagrangian of the system is then:

$$L=\frac{1}{2}(m_1+m_2)\ell_1^2\dot{\theta}_1^2+\frac{1}{2}m_2\ell_2^2\dot{\theta}_2^2+m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)$$
$$+(m_1+m_2)g\ell_1\cos\theta_1+m_2g\ell_2\cos\theta_2 \tag{57}$$

By developing Lagrangian system, the canonical momenta can be associated with the coordinates ofθ_1 and θ_2 which can obtained directly from L :

$$p_{\theta_1}=\frac{\partial L}{\partial\dot{\theta}_1}=(m_1+m_2)\ell_1^2\dot{\theta}_1+m_2\ell_1\ell_2\dot{\theta}_2\cos(\theta_1-\theta_2) \tag{58}$$

$$p_{\theta_2}=\frac{\partial L}{\partial\dot{\theta}_2}=m_2\ell_2^2\dot{\theta}_2+m_2\ell_1\ell_2\dot{\theta}_1\cos(\theta_1-\theta_2) \tag{59}$$

In order to solve this formula, we used Mathematica software. We could get the exact curve and result from this formula. The graph is being generated from the same data which is mass of double pendulum is, $m1 = m2 = 1$ and the length of both pendulum is, $l1 = l2 = 2$. The gravity is fixed by 9.81.

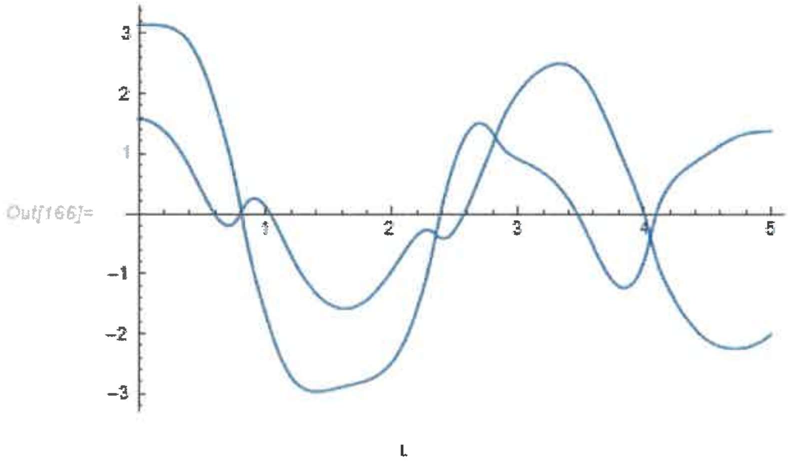

Figure 3.1: Graph of Lagrangian

From the above graph, it can be seen that the graph indicates the movement of the double pendulum at the given time,t which is 5 seconds. The graph have a gap and the curve become larger since there is less movement due to the time taken and the $(\theta 1t, \theta 2t, t)$ is $(t, 0, 5)$.The initial takes place at $\frac{Pi}{2}$ for x_1 and Pi for x_2.

3.2 STEP 2 : Euler-Lagrangian Equation Development from Lagrangian System

The Lagrangian formulation,the function $L(p_i, q_i, t)$ where p_i and q_i $(i = 1, \cdots, n)$ are n generalized coordinates (Kelly, 1993).Euler-Lagrange system is also called "Lagrange's Equation of Second Kind".The Hamilton's equation can be derive from the Lagrange equation by substitute the value of p_i and q_i. The final solution will be :

$$\dot{p}_i = -\frac{\partial H}{\partial q_i} \tag{60}$$

$$\dot{q}_i = \frac{\partial H}{\partial p_i} \tag{61}$$

$$-\frac{\partial L}{\partial t} = \frac{\partial H}{\partial t} \tag{62}$$

These are the 2^{nd} order differential equations which require 2n initial conditions. In order to make it n generalized equation, Ramegowda (2001) stated it will be canonical momenta equation.In order to get to the Hamiltonian, we need to develop Euler-Lagrange equation.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_i}\right) - \frac{\partial L}{\partial \theta_i} = 0 \Longrightarrow \frac{dp_{\theta_i}}{dt} - \frac{\partial L}{\partial \theta_i} = 0 \tag{63}$$

Then, the equation yields after dividing by ℓ_1 when $i = 1$ and $m_2\ell_2$ when $i = 2$ produce two equations:

$$\begin{aligned}&(m_1 + m_2)\ell_1\ddot{\phi}_1 + m_2\ell_2\ddot{\phi}_2\cos(\phi_1 - \phi_2) + m_2\ell_2\dot{\phi}_1^2\sin(\phi_1 - \phi_2)\\ &+ (m_1 + m_2)g\sin\phi_1 = 0\end{aligned} \tag{64}$$

$$\ell_1\ddot{\phi}_2 + \ell_1\ddot{\phi}_1\cos(\phi_1 - \phi_2) - \ell_1\dot{\phi}_1^2\sin(\phi_1 - \phi_2) + g\sin\phi_2 = 0 \tag{65}$$

Both equations coupled second order nonlinear differential equation that form a system. By dividing equation by $(m_1+m_2)\ell_1$ and by ℓ_2 and also moving all terms which do not involves $\ddot{\phi_1}$ and $\ddot{\phi_2}$ to the right hand-side, we would lastly obtain:

$$\ddot{\phi_1}+\alpha_1(\phi_1,\phi_2)\ddot{\phi_2}=f_1(\phi_1,\phi_2,\dot{\phi}_1,\dot{\phi}_2) \tag{66}$$

$$\ddot{\phi_2}+\alpha_2(\phi_1,\phi_2)\ddot{\phi_1}=f_2(\phi_1,\phi_2,\dot{\phi}_1,\dot{\phi}_2) \tag{67}$$

From the findings, f_1 does not depend on $\dot{\phi}_1$ and f_2 does not depend on $\dot{\phi}_2$. Thus, both equation can be combined into a single equation:

$$A\begin{pmatrix}\ddot{\phi_1}\\ \ddot{\phi_2}\end{pmatrix}=\begin{pmatrix}1 & \ddot{\phi_1}\\ \ddot{\phi_2} & 1\end{pmatrix}\begin{pmatrix}\ddot{\phi_1}\\ \ddot{\phi_2}\end{pmatrix}=\begin{pmatrix}f_1\\ f_2\end{pmatrix} \tag{68}$$

They have become $2x2$ matrix where matrix A depends on θ_1 and θ_2 since $alpha_1$ and $alpha_2$ depend on this variable. Hence, A can be inverted directly by:

$$A^{-1}=\frac{1}{\det(A)}\begin{pmatrix}1 & \ddot{\phi_1}\\ \ddot{\phi_2} & 1\end{pmatrix} \tag{69}$$

From (106), we obtain :

$$\begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{pmatrix} = \frac{1}{1-\alpha_1\alpha_2} \begin{pmatrix} f_1 - \alpha_1 f_2 \\ -\alpha_2 f_1 + f_2 \end{pmatrix} \tag{70}$$

Finally, by letting $\omega_1 = \phi_1$ and $\omega_2 = \phi_2$, as a system of coupled first order differential equations on the variables ϕ_1,ϕ_2,χ_1,χ_2, we can produce the equations of motion of the double pendulum which is:

$$\frac{d}{dt} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \chi_1 \\ \chi_2 \\ g_1(\phi_1, \phi_2, \omega_1, \omega_2) \\ g_2(\phi_1, \phi_2, \omega_1, \omega_2) \end{pmatrix} \tag{71}$$

Then, the above matrix can be solved to get the result.In Mathematica, we need to identify the values of each unit before being developed to get the exact solution.First, we need to make sure that we estimate the value for the length, mass and the angle for double pendulum. Since we use the same exact value from Lagrange, we just continued and extend the formula from the Lagrange earlier.

The curve is representing the movement of double pendulum at given time, t. The curve as what can be seen moving ascending between both pendulum since they are in the same time taken. The curve looks further and wavy since they oscillates lesser due to given time is 5 seconds only.The Euler-Lagrange graph is based on the development of Lagrange that has been made earlier.

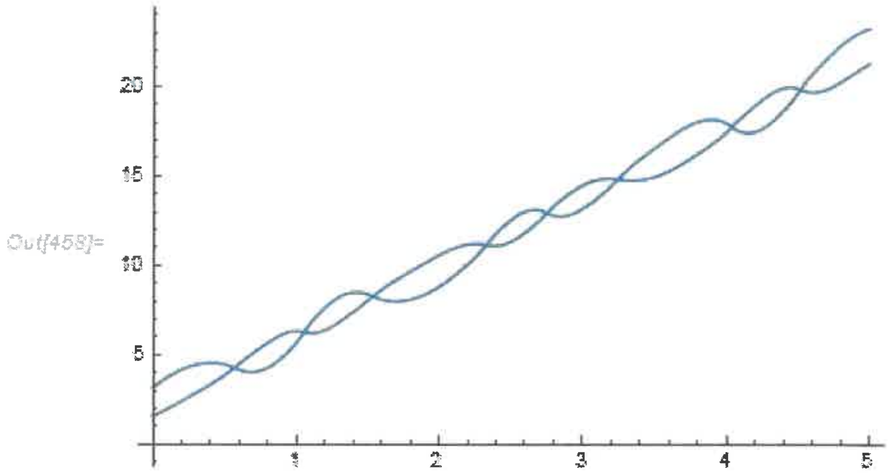

Figure 3.2: Graph of Euler-Lagrangian

3.3 STEP 3 : The Expansion of Hamiltonian into Runge-Kutta

From the Lagrangian equation that we refer on (35), we obtain :

$$L=\frac{1}{2}(m_1+m_2)\ell_1^2\dot{\theta}_1^2+\frac{1}{2}m_2\ell_2^2\dot{\theta}_2^2+m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1-\theta_2)$$
$$+(m_1+m_2)g\ell_1\cos\theta_1+m_2g\ell_2\cos\theta_2 \tag{72}$$

From the above equation, we can obtain canonical momenta of the system :

$$p_{\theta_1}=\frac{\partial L}{\partial\dot{\theta}_1}=(m_1+m_2)\ell_1^2\dot{\theta}_1+m_2\ell_1\ell_2\dot{\theta}_2\cos(\theta_1-\theta_2) \tag{73}$$
$$p_{\theta_2}=\frac{\partial L}{\partial\dot{\theta}_2}=m_2\ell_2^2\dot{\theta}_2+m_2\ell_1\ell_2\dot{\theta}_1\cos(\theta_1-\theta_2) \tag{74}$$

Then, the Hamiltonian of the system is given by :

$$H = \sum_{i=1}^{2} \dot{\theta}_i P_{\theta_i} - L \tag{75}$$

From the Hamiltonian, we can write H as a function of motion for the system that equivalent to Euler-Lagrange equations:

$$\dot{\theta}_i = \frac{\partial H}{\partial P_{\theta_i}} \quad , \quad \dot{P_{\theta_i}} = \frac{-\partial H}{\partial \theta_i} \quad for\ i = 1, 2, \ldots \tag{76}$$

From (114), we can write H as a function of the variables θ_1,θ_2,P_{θ_1}, and P_{θ_2}. Hence, from (113), we must determine $\dot{\theta}_i$ and L in terms of these variables. From what we notice, equation (111) can be written in form of matrix as shown below.

$$\begin{pmatrix} P_{\theta_1} \\ P_{\theta_1} \end{pmatrix} = B \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \tag{77}$$

where B is $2X2$ matrix entries depends on θ_1 and θ_2:

$$P \begin{pmatrix} (m_1 + m_2)\ell_1^2 & m_2\ell_1\ell_2\cos(\theta_1 - \theta_2) \\ m_2\ell_1\ell_2\cos(\theta_1 - \theta_2) & m_2\ell_2^2 \end{pmatrix} \tag{78}$$

From (115), we can obtain generalized velocities of $\dot{\theta}_i$ in terms of canonical momenta

P_{θ_i} and angles θ_i :

$$P\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix}=B^{-1}\begin{pmatrix}P_{\theta_1}\\ P_{\theta_1}\end{pmatrix} \tag{79}$$

After canceling out common factor and rearranging some terms, we get :

$$\dot{\theta}_1=\frac{\ell_2 P_{\theta_1}-\ell_1 P_{\theta_2}\cos(\theta_1-\theta_2)}{\ell_1^2\ell_2[m_1+m_2\sin^2(\theta_1-\theta_2)]} \tag{80}$$

$$\dot{\theta}_2=\frac{-m_2\ell_2 P_{\theta_1}-\ell_1 P_{\theta_2}\cos(\theta_1-\theta_2)+(m_1+m_2)\ell_1 P_{\theta_2}}{m_2\ell_1\ell_2^2[m_1+m_2\sin^2(\theta_1-\theta_2)]} \tag{81}$$

By using equation 113,118 and 119, we can get the Hamiltonian, H in terms of θ_1, θ_2, P_{θ_1} and P_{θ_1}:

$$H=\frac{m_2\ell_2^2P_{\theta_1}^2+(m_1+m_2)\ell_1^2P_{\theta_2}^2-2m_2\ell_1\ell_2P_{\theta_1}P_{\theta_2}\cos(\theta_1-\theta_2)}{2m_2\ell_1^2\ell_2^2[m_1+m_2\sin^2(\theta_1-\theta_2)]}$$
$$-(m_1+m_2)g\ell_1\cos\theta_1-m_2g\ell_2\cos\theta_2 \tag{82}$$

From equation 120, we can conclude the Hamiltonian equation of motion for double pendulum:

$$h_1 = \frac{P_{\theta_1} P_{\theta_2} \sin(\theta_1 - \theta_2)}{\ell_1 \ell_2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]} \tag{83}$$

$$h_2 = \frac{m_2 \ell_2^2 P_{\theta_1}^2 + (m_1 + m_2)\ell_1^2 P_{\theta_2}^2 - 2m_2 \ell_1 \ell_2 P_{\theta_1} P_{\theta_2} \cos(\theta_1 - \theta_2)}{2\ell_1^2 \ell_2^2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]^2} \tag{84}$$

In the process of determine the Hamiltonian in terms of canonical momenta P_{θ_i} and angles θ_i, we ended up by obtaining two Hamiltonian equation.This set of equations also can be solved numerically by using Runge-Kutta(RK4).

$$w_0 = \alpha \tag{85}$$

$$m_1 = hf(t_i, w_i) \tag{86}$$

$$m_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{m_1}{2}\right) \tag{87}$$

$$m_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{m_2}{2}\right) \tag{88}$$

$$m_4 = hf(t_i + h, w_i + m_3) \tag{89}$$

$$w_{i+1} = w_i + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \tag{90}$$

It can be implement from Hamiltonian equation to get the exact result before being develop in Runge-Kutta (RK4) formula.As for the Runge-Kutta, it needs to develop Hamiltonian to get the$h$ value before we attain the results.

From what can be seen from Fig.3.3 until Fig.3.6, the result have the same graph patterns but different oscillation made. The use of Runge-Kutta formula is to get the result of double pendulum. The graph indicates that the curves become more narrow and the curves keep repeating more often since the step-size become smaller. The smaller the step-size, the smaller the error will be made in this motion. The curves would be interestingly unique when it being done in parametric plot. It will be shown in the result section. As what have been conducted, it can be seen that the Runge-Kutta formula is the best tools to measure the error of the double pendulum curves since the graph shown smaller steps-size which will give a small error in the end.

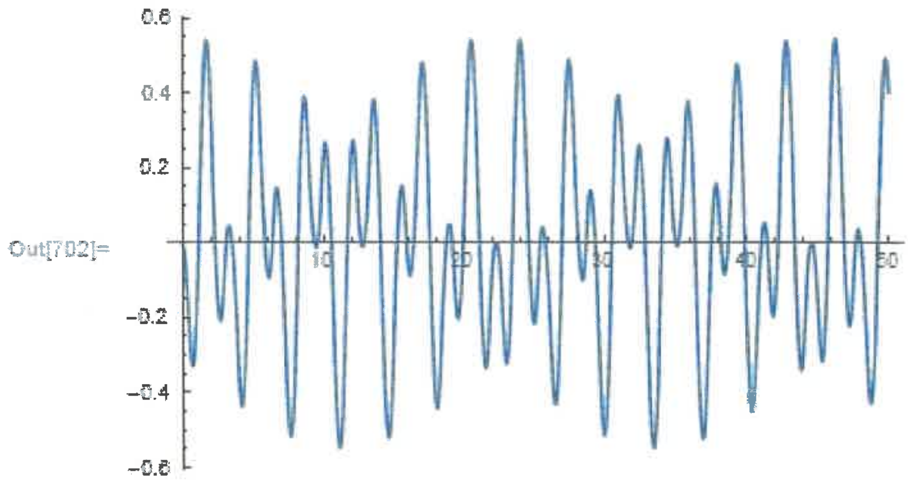

Figure 3.3: Graph of Runge-Kutta(RK1)

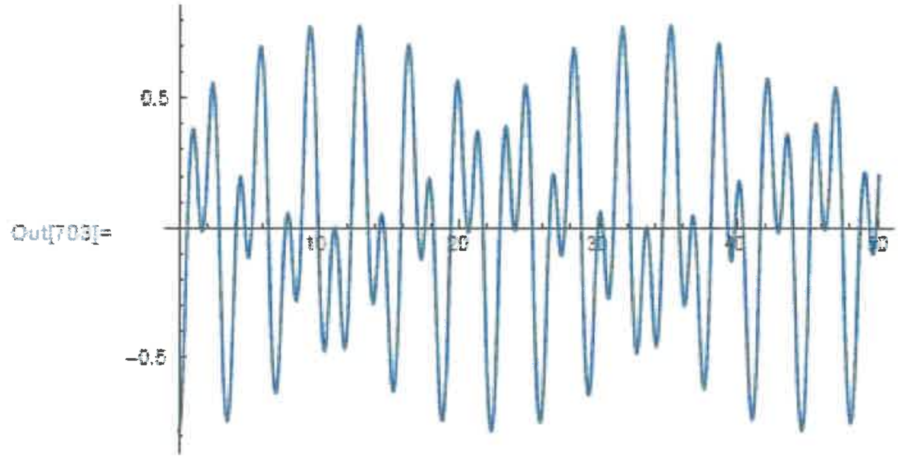

Figure 3.4: Graph of Runge-Kutta(RK2)

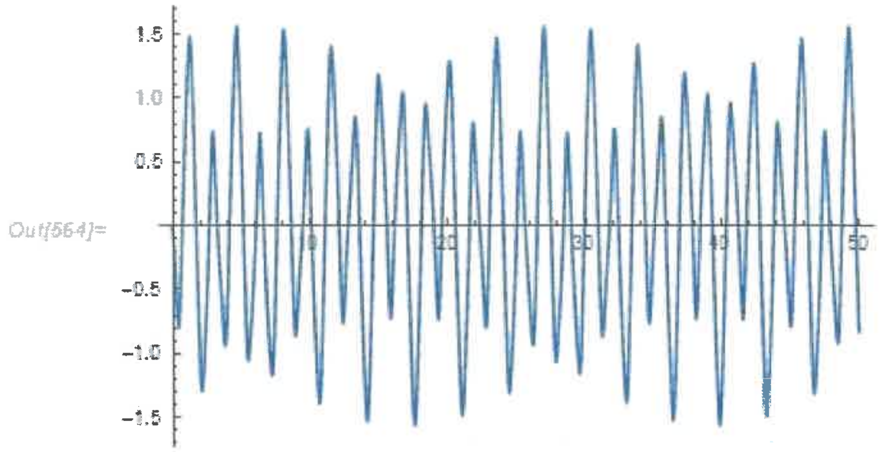

Figure 3.5: Graph of Runge-Kutta(RK3)

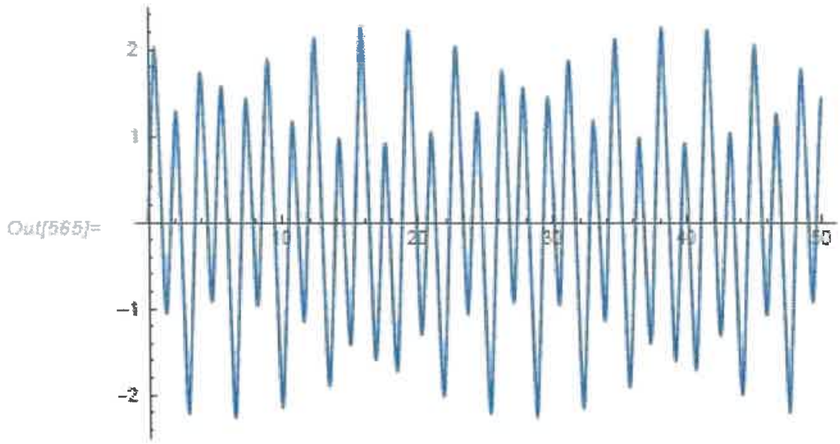

Figure 3.6: Graph of Runge-Kutta(RK4)

4 RESULTS AND DISCUSSION

4.1 Lagrange

The results of motion of curves by comparing three type of methods in double pendulum when using the fixed data such as m_1=1kg and m_2 =1kg, ℓ_1=2m and ℓ_2=2m, g=9.81n. According to mathematica result, the Figure 4.1 shows the result of Lagrangian equation. The graph in Figure 4.1 shows that the double pendulum move extremely starting when

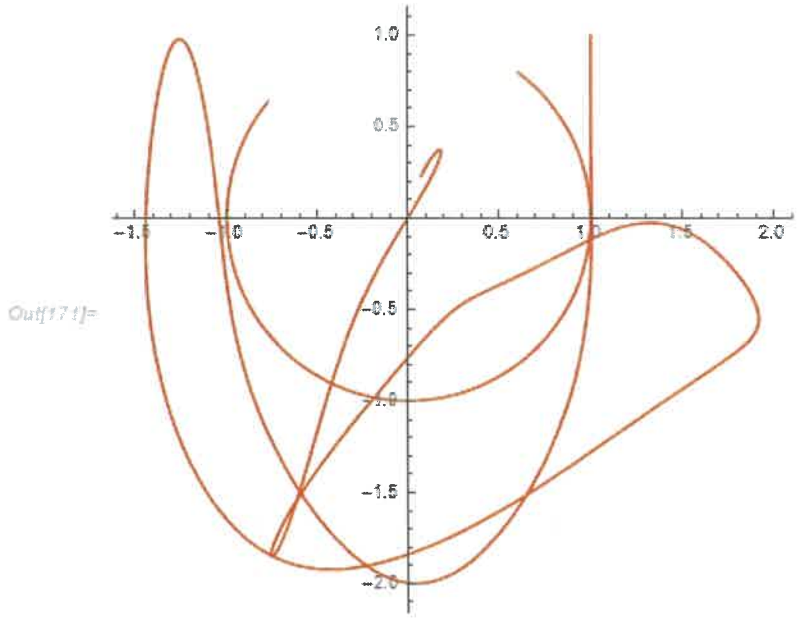

Figure 4.1: Graph of Lagrangian

t=o until the t_max=5sec and its show the motion of curves are dramatically.

4.2 Euler-Lagrange

For the second method, Euler equation, the Figure 4.2 present the motion of curves in double pendulum are more straight upward and shows some curves starting when $t = 0$. The graph also shows, the Euler's method unsuitable for long set time because the t_max=5sec.

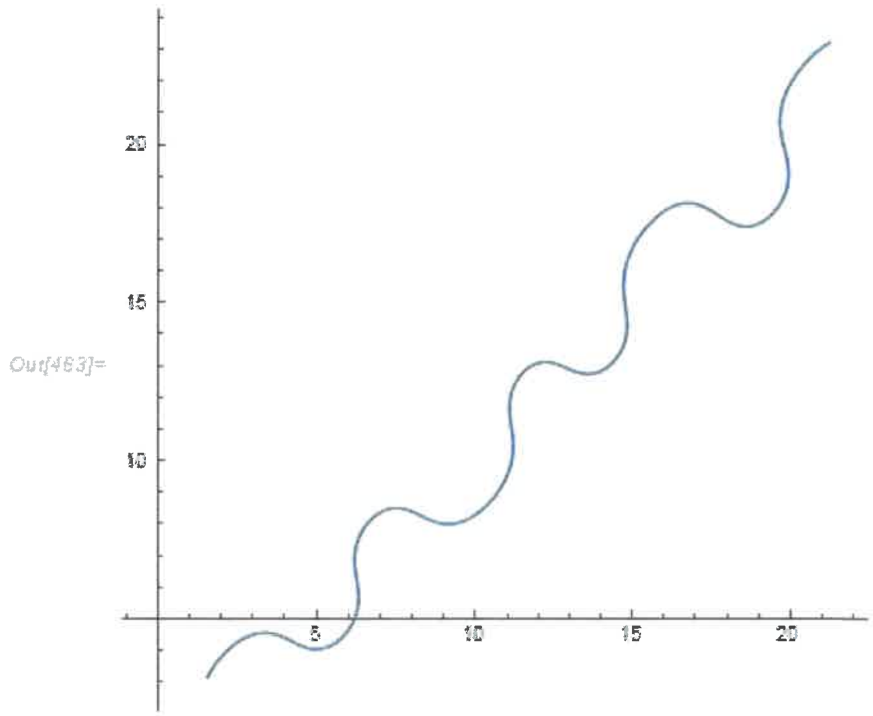

Figure 4.2: Graph of Euler-Lagrangian

4.3 Runge-Kutta

Figure 4.3, Figure 4.4 and Figure 4.5 represent the result by applying Runge Kutta. The result show the smooth line motion and the curve of the graph can generate the other form. From all this above result for three type of methods it can concluded the Runge Kutta are best method for applying double pendulum.

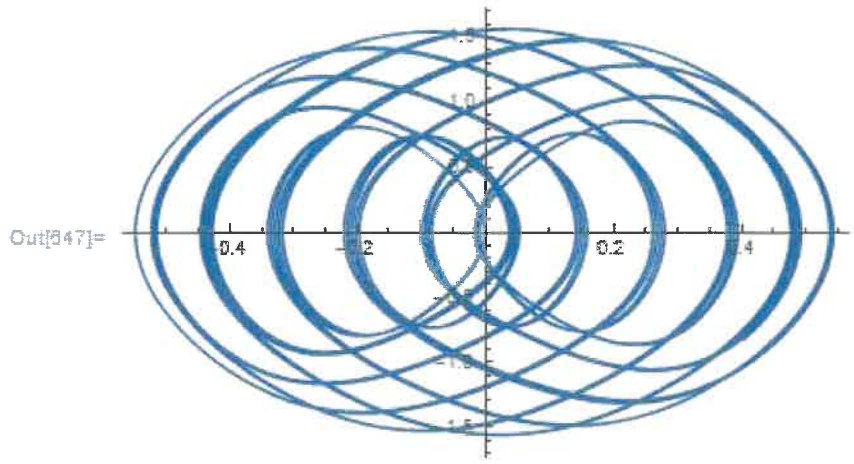

Figure 4.3: Graph of Runge-Kutta(RK1)

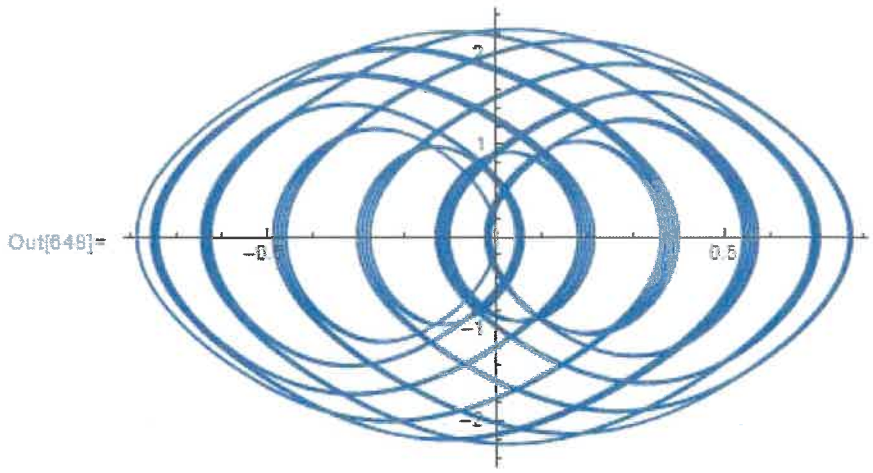

Figure 4.4: Graph of Runge-Kutta(RK2)

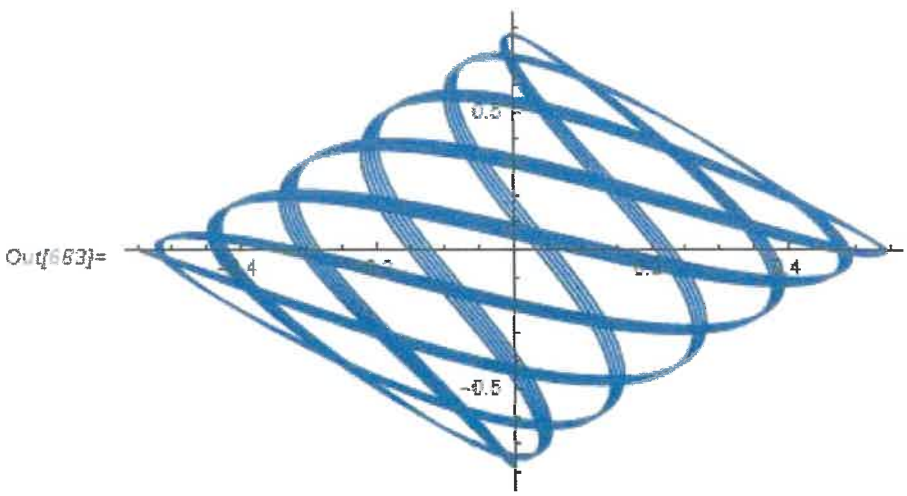

Figure 4.5: Graph of Runge-Kutta(RK3)

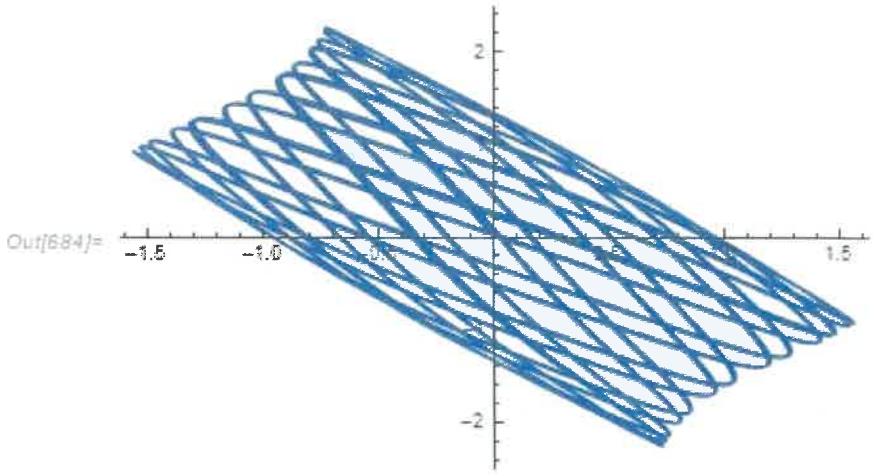

Figure 4.6: Graph of Runge-Kutta(RK4)

5 CONCLUSIONS AND RECOMMENDATIONS

The objective are to comparing three type of methods based on their motion of curves. Motion of curves represents the initial condition that sensitive depends and how much chaotic motion. Each of method are related to each other. To knows which one are the best method, Mathematica software has been used to solve this problem. By using the same data and different method which is m_1=1kg and m_2=1kg, ℓ_1=2m and ℓ_2=2m, g=9.81n, the result show that Runge Kutta are the best method other than Lagrangian Equation and Euler's Method because the line of curves are smooth rather than line of curves for other difference method. In addition, the result also shows that Euler's method are not suitable for long time-step. In conclusion, Runge Kutta are the best method to solve double pendulum whether in long time-step or short time-step. Lastly, for the recommendation to get the better result the findings can take a long time when doing an experiment for double pendulum but if other researchers want to get the better result for Euler's method, it can take a short time-step because Euler's method only suitable for short time-step. Furthermore, it also can use difference mass which is $m_1 < m_2$ or $m_1 > m_2$.

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APPENDIX A

1. Developing Lagrangian Equation

The position of bob is given by :

$$x_1 = \ell_1 \sin\theta_1$$

$$y_1 = -\ell_1 \cos\theta_1$$

$$x_2 = \ell_1 \sin\theta_1 + \ell_2 \sin\theta_2$$

$$y_2 = -\ell_1 \cos\theta_1 - \ell_2 \cos\theta_2$$

By differentiating with respect to time, we obtain velocities of the bobs :

$$\dot{x_1} = \ell_1 \dot{\theta}_1 \cos\theta_1$$

$$\dot{y_1} = \ell_1 \dot{\theta}_1 \sin\theta_1$$

$$\dot{x_2} = \ell_1 \dot{\theta}_1 \cos\theta_1 + \ell_2 \dot{\theta}_2 \cos\theta_2$$

$$\dot{y_2} = \ell_1 \dot{\theta}_1 \sin\theta_1 + \ell_2 \dot{\theta}_2 \sin\theta_2$$

From the Lagrangian formula, $L = K - P$, the kinetic energy, K is given by :

$$
\begin{aligned}
K &= \frac{1}{2}m_1v_1^2 + \frac{1}{2}m_2v_2^2 \\
&= \frac{1}{2}m_1(\dot{x}_1^2 + \dot{y}_1^2) + \frac{1}{2}m_2(\dot{x}_2^2 + \dot{y}_2^2) \\
&= \frac{1}{2}m_1\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2[\ell_1^2\dot{\theta}_1^2 + \ell_2^2\dot{\theta}_2^2 + 2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)] \qquad (91)
\end{aligned}
$$

where above that $\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 = \cos(\theta_1 - \theta_2)$.

By developing kinetic energy, the potential energy, P is given by:

$$
\begin{aligned}
P &= m_1gy_1 + m_2gy_2 \\
&= -m_1g\ell_1\cos\theta_1 - m_2g(\ell_1\cos\theta_1 + \ell_2\cos\theta_2) \\
&= -(m_1 + m_2)g\ell_1\cos\theta_1 - m_2g\ell_2\cos\theta_2 \qquad (92)
\end{aligned}
$$

Then, the Lagrangian of the system is then:

$$
\begin{aligned}
L &= \frac{1}{2}(m_1 + m_2)\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\ell_2^2\dot{\theta}_2^2 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2) \\
&\quad + (m_1 + m_2)g\ell_1\cos\theta_1 + m_2g\ell_2\cos\theta_2 \qquad (93)
\end{aligned}
$$

By developing Lagrangian system, the canonical momenta can be associated with the coordinates ofθ_1 and θ_2 which can obtained directly from L :

$$p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2)\ell_1^2\dot{\theta}_1 + m_2\ell_1\ell_2\dot{\theta}_2\cos(\theta_1 - \theta_2) \tag{94}$$

$$p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2} = m_2\ell_2^2\dot{\theta}_2 + m_2\ell_1\ell_2\dot{\theta}_1\cos(\theta_1 - \theta_2) \tag{95}$$

2. Euler-Lagrangian Equation Development from Lagrangian System

First, we need to develop Euler-Lagrange equation.

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\phi}_i}\right) - \frac{\partial L}{\partial \phi_i} = 0 \Longrightarrow \frac{dp_{\phi_i}}{dt} - \frac{\partial L}{\partial \phi_i} = 0 \quad for\, i = 1, 2, \ldots \tag{96}$$

Since :

$$\frac{dp\phi_1}{dt} = (m_1 + m_2)\ell_1^2\ddot{\phi}_1 + m_2\ell_1\ell_2\ddot{\phi}_2\cos(\phi_1 - \phi_2) - m_2\ell_1\ell_2\dot{\phi}_2\dot{\phi}_1\sin(\phi_1 - \phi_2)$$
$$+ m_2\ell_1\ell_2\dot{\phi}_2^2\sin(\phi_1 - \phi_2)$$
$$\frac{dp\phi_2}{dt} = m_2\ell_2^2\ddot{\phi}_2 + m_2\ell_1\ell_2\ddot{\phi}_1\cos(\phi_1 - \phi_2) - m_2\ell_1\ell_2\dot{\phi}_1^2\dot{\phi}_1\sin(\phi_1 - \phi_2)$$
$$+ m_2\ell_1\ell_2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1 - \phi_2)$$
$$\frac{dL}{d\phi_1} = -m_2\ell_1\ell_2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1 - \phi_2) - (m_1 + m_2)g\ell_1\sin\phi_1$$
$$\frac{dL}{d\phi_2} = m_2\ell_1\ell_2\dot{\phi}_1\dot{\phi}_2\sin(\phi_1 - \phi_2) - m_2 g\ell_2\sin\phi_2$$

Then, the equation yields after dividing by ℓ_1 when $i = 1$ and $m_2\ell_2$ when $i = 2$ produce two equations:

$$(m_1+m_2)\ell_1\ddot{\phi_1}+m_2\ell_2\ddot{\phi_2}\cos(\phi_1-\phi_2)+m_2\ell_2\dot{\phi}_1^2\sin(\phi_1-\phi_2)$$
$$+(m_1+m_2)g\sin\phi_1=0 \tag{97}$$

$$\ell_1\ddot{\phi_2}+\ell_1\ddot{\phi_1}\cos(\phi_1-\phi_2)-\ell_1\dot{\phi}_1^2\sin(\phi_1-\phi_2)+g\sin\phi_2=0 \tag{98}$$

By dividing equation by $(m_1+m_2)\ell_1$ and by ℓ_2 and also moving all terms which do not involves $\ddot{\phi_1}$ and $\ddot{\phi_2}$ to the right hand-side, we would lastly obtain:

$$\ddot{\phi_1}+\alpha_1(\phi_1,\phi_2)\ddot{\phi_2}=f_1(\phi_1,\phi_2,\dot{\phi_1},\dot{\phi_2}) \tag{99}$$
$$\ddot{\phi_2}+\alpha_2(\phi_1,\phi_2)\ddot{\phi_1}=f_2(\phi_1,\phi_2,\dot{\phi_1},\dot{\phi_2}) \tag{100}$$

where;

$$\alpha_1(\phi_1,\phi_2):\frac{\ell_2}{\ell_1}\left(\frac{m_2}{m_1+m_2}\right)\cos(\phi_1-\phi_2) \tag{101}$$
$$\alpha_2(\phi_1,\phi_2):\frac{\ell_1}{\ell_2}\cos(\phi_1-\phi_2) \tag{102}$$

and also;

$$f_1(\phi_1,\phi_2,\dot{\phi_1},\dot{\phi_2})=-\frac{\ell_2}{\ell_1}\left(\frac{m_2}{m_1+m_2}\right)\dot{\phi}_2^2\sin(\phi_1-\phi_2)-\frac{g}{\ell_1}\sin\phi_1 \tag{103}$$
$$f_2(\phi_1,\phi_2,\dot{\phi_1},\dot{\phi_2})=\frac{\ell_1}{\ell_2}\dot{\phi}_1^2\sin(\phi_1-\phi_2)-\frac{g}{\ell_2}\sin\phi_2 \tag{104}$$

Then, f_1 does not depend on $\dot{\phi}_1$ and f_2 does not depend on $\dot{\phi}_2$. Thus, both equation can be combined into a single equation:

$$A\begin{pmatrix}\ddot{\phi}_1\\ \ddot{\phi}_2\end{pmatrix}=\begin{pmatrix}1 & \ddot{\phi}_1\\ \ddot{\phi}_2 & 1\end{pmatrix}\begin{pmatrix}\ddot{\phi}_1\\ \ddot{\phi}_2\end{pmatrix}=\begin{pmatrix}f_1\\ f_2\end{pmatrix} \tag{105}$$

Hence, A can be inverted directly by:

$$\begin{aligned}A^{-1}&=\frac{1}{\det(A)}\begin{pmatrix}1 & \ddot{\phi}_1\\ \ddot{\phi}_2 & 1\end{pmatrix}\\ &=\frac{1}{1-\alpha_1\alpha_2}\begin{pmatrix}1 & -\ddot{\phi}_1\\ -\ddot{\phi}_2 & 1\end{pmatrix}\end{aligned} \tag{106}$$

From (106), we notice A is invertible since :

$$\begin{aligned}det(A)&=1-\alpha_1\alpha_2\\ &=1-\left(\frac{m_2}{m_1+m_2}\right)\cos^2(\phi_1-\phi_2)>0\end{aligned} \tag{107}$$

This is because :

$$\frac{m_2}{m_1+m_2}<1 \; and \; \cos^2(x)\leq 1 \; for \; all \; real \; values \; of \; x.$$

From there, we obtain :

$$
\begin{pmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \end{pmatrix} = A^{-1} \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}
$$

$$
= \frac{1}{1-\alpha_1\alpha_2} \begin{pmatrix} f_1 - \alpha_1 f_2 \\ -\alpha_2 f_1 + f_2 \end{pmatrix} \tag{108}
$$

Finally, by letting $\omega_1 = \phi_1$ and $\omega_2 = \phi_2$, we can produce the equations of motion of the double pendulum as a system of coupled first order differential equations on the variables ϕ_1,ϕ_2,ω_1,ω_2 :

$$
\frac{d}{dt} \begin{pmatrix} \phi_1 \\ \phi_2 \\ \omega_1 \\ \omega_2 \end{pmatrix} = \begin{pmatrix} \omega_1 \\ \omega_2 \\ g_1(\phi_1, \phi_2, \omega_1, \omega_2) \\ g_2(\phi_1, \phi_2, \omega_1, \omega_2) \end{pmatrix} \tag{109}
$$

where :

$$
g_1 = \frac{f_1 - \alpha_1 f_2}{1-\alpha_1\alpha_2}
$$

$$
g_2 = \frac{-\alpha_2 f_1 + f_2}{1-\alpha_1\alpha_2}
$$

and also :

$$\alpha_i = (\phi_1, \phi_2)$$

$$f_1 = f_i(\phi_1, \phi_2, \omega_1, \omega_2)$$

for $i = 1, 2, \ldots$ given in the equation. The equation above can be solved numerically by using (RK4) method.

3. The Expansion of Hamiltonian into Runge-Kutta

From the Lagrangian equation that we refer on (35), we obtain :

$$L = \frac{1}{2}(m_1 + m_2)\ell_1^2\dot{\theta}_1^2 + \frac{1}{2}m_2\ell_2^2\dot{\theta}_2^2 + m_2\ell_1\ell_2\dot{\theta}_1\dot{\theta}_2\cos(\theta_1 - \theta_2)$$
$$+ (m_1 + m_2)g\ell_1\cos\theta_1 + m_2 g\ell_2\cos\theta_2 \qquad (110)$$

From the above equation, we can obtain canonical momenta of the system :

$$p_{\theta_1} = \frac{\partial L}{\partial \dot{\theta}_1} = (m_1 + m_2)\ell_1^2\dot{\theta}_1 + m_2\ell_1\ell_2\dot{\theta}_2\cos(\theta_1 - \theta_2) \qquad (111)$$

$$p_{\theta_2} = \frac{\partial L}{\partial \dot{\theta}_2} = m_2\ell_2^2\dot{\theta}_2 + m_2\ell_1\ell_2\dot{\theta}_1\cos(\theta_1 - \theta_2) \qquad (112)$$

Then, the Hamiltonian of the system is given by :

$$H = \sum_{i=1}^{2} \dot{\theta}_i P_{\theta_i} - L \tag{113}$$

From the Hamiltonian, we can write H as a function of motion for the system that equivalent to Euler-Lagrange equations:

$$\dot{\theta}_i = \frac{\partial H}{\partial P_{\theta_i}} \quad , \quad \dot{P}_{\theta_i} = \frac{-\partial H}{\partial \theta_i} \quad for\ i = 1,2,\ldots \tag{114}$$

From what we notice, the equation can be written in form of matrix as shown below.

$$\begin{pmatrix} P_{\theta_1} \\ P_{\theta_1} \end{pmatrix} = B \begin{pmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{pmatrix} \tag{115}$$

where B is $2X2$ matrix entries depends on θ_1 and θ_2:

$$B = \begin{pmatrix} (m_1 + m_2)\ell_1^2 & m_2\ell_1\ell_2\cos(\theta_1 - \theta_2) \\ m_2\ell_1\ell_2\cos(\theta_1 - \theta_2) & m_2\ell_2^2 \end{pmatrix} \tag{116}$$

From (115), we can obtain generalized velocities of $\dot{\theta}_i$ in terms of canonical momenta P_{θ_i} and angles θ_i :

$$P\begin{pmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{pmatrix}=B^{-1}\begin{pmatrix}P_{\theta_1}\\ P_{\theta_1}\end{pmatrix} \tag{117}$$

Matrix B is indeed invertible for all valuesof θ_1 and θ_2 since:

$$\begin{aligned} det(B) &= m_1m_2\ell_1^2\ell_2^2+m_2^2\ell_1^2\ell_2^2[1-\cos^2(\theta_1-\theta_2)] \\ &= m_1m_2\ell_1^2\ell_2^2+m_2^2\ell_1^2\ell_2^2\sin^2(\theta_1-\theta_2) \\ &\geq m_1m_2\ell_1^2\ell_2^2 \end{aligned}$$

B can be inverted directly since it is $2x2$ matrix by :

$$B^{-1}=\frac{1}{det(B)}\begin{pmatrix} m_2\ell_2^2 & -m_2\ell_1\ell_2\cos(\theta_1-\theta_2) \\ -m_2\ell_1\ell_2\cos(\theta_1-\theta_2) & (m_1+m_2)\ell_1^2 \end{pmatrix}$$

After canceling out common factor and rearranging some terms, we get :

$$\dot{\theta}_1=\frac{\ell_2P_{\theta_1}-\ell_1P_{\theta_2}\cos(\theta_1-\theta_2)}{\ell_1^2\ell_2[m_1+m_2\sin^2(\theta_1-\theta_2)]} \tag{118}$$

$$\dot{\theta}_2 = \frac{-m_2\ell_2 P_{\theta_1} - \ell_1 P_{\theta_2}\cos(\theta_1-\theta_2) + (m_1+m_2)\ell_1 P_{\theta_2}}{m_2\ell_1\ell_2^2[m_1+m_2\sin^2(\theta_1-\theta_2)]} \tag{119}$$

By using equation 113,118 and 119, we can get the Hamiltonian, H in terms of θ_1, θ_2, P_{θ_1} and P_{θ_1}:

$$H = \frac{m_2\ell_2^2 P_{\theta_1}^2 + (m_1+m_2)\ell_1^2 P_{\theta_2}^2 - 2m_2\ell_1\ell_2 P_{\theta_1} P_{\theta_2}\cos(\theta_1-\theta_2)}{2m_2\ell_1^2\ell_2^2[m_1+m_2\sin^2(\theta_1-\theta_2)]}$$
$$-(m_1+m_2)g\ell_1\cos\theta_1 - m_2 g\ell_2\cos\theta_2 \tag{120}$$

From equation 120, we can conclude the Hamiltonian equation of motion for double pendulum:

$$\begin{aligned}
\dot{\theta}_1 &= \frac{\partial H}{\partial P_{\theta_1}} = \frac{\ell_2 P_{\theta_1} - \ell_1 P_{\theta_2}\cos(\theta_1-\theta_2)}{\ell_1^2\ell_2[m_1+m_2\sin^2(\theta_1-\theta_2)]} \\
\dot{\theta}_2 &= \frac{\partial H}{\partial P_{\theta_2}} = \frac{-m_2\ell_2 P_{\theta_1}\cos(\theta_1-\theta_2) + (m_1+m_2)\ell_1 P_{\theta_2}}{m_2\ell_1\ell_2^2[m_1+m_2\sin^2(\theta_1-\theta_2)]} \\
\dot{P_{\theta_1}} &= \frac{-\partial H}{\partial \theta_1} = -(m_1+m_2)g\ell_1\sin\theta_1 - h_1 + h_2\sin[2(\theta_1 = \theta_2)] \\
\dot{P_{\theta_2}} &= \frac{-\partial H}{\partial \theta_2} = -m_2 g\ell_2\sin\theta_2 + h_1 - h_2\sin[2(\theta_1 = \theta_2)]
\end{aligned}$$

where :

$$h_1 = \frac{P_{\theta_1} P_{\theta_2} \sin(\theta_1 - \theta_2)}{\ell_1 \ell_2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]} \tag{121}$$

$$h_2 = \frac{m_2 \ell_2^2 P_{\theta_1}^2 + (m_1 + m_2) \ell_1^2 P_{\theta_2}^2 - 2 m_2 \ell_1 \ell_2 P_{\theta_1} P_{\theta_2} \cos(\theta_1 - \theta_2)}{2 \ell_1^2 \ell_2^2 [m_1 + m_2 \sin^2(\theta_1 - \theta_2)]^2} \tag{122}$$

This is a form of set of coupled first order differential equation on variables θ_1,θ_2,P_{θ_1} and P_{θ_2}. This set of equations also can be solved numerically by using RK4.

$$w_0 = \alpha \tag{123}$$

$$m_1 = hf(t_i, w_i) \tag{124}$$

$$m_2 = hf\left(t_i + \frac{h}{2}, w_i + \frac{m_1}{2}\right) \tag{125}$$

$$m_3 = hf\left(t_i + \frac{h}{2}, w_i + \frac{m_2}{2}\right) \tag{126}$$

$$m_4 = hf(t_i + h, w_i + m_3) \tag{127}$$

$$w_{i+1} = w_i + \frac{1}{6}(m_1 + 2m_2 + 2m_3 + m_4) \tag{128}$$

APPENDIX B

```
In[165]:= sol1 = NDSolve[{
      2 * theta1''[t] + theta2''[t] * Cos[theta1[t] - theta2[t]] +
        theta2'[t] ^ (2) * Sin[theta1[t] - theta2[t]] + 2 * 9.81 * Sin[theta1[t]] ==
      theta2''[t] + theta1''[t] * Cos[theta1[t] - theta2[t]] -
        theta1'[t] ^ (2) * Sin[theta1[t] - theta2[t]] + 9.81 * Sin[theta2[t]] == 0,
      theta1[0] == Pi / 2, theta2[0] == Pi, theta1'[0] == 0, theta2'[0] == 0},
     {theta1[t], theta2[t], t}, {t, 0, 5}]

Plot[{theta1[t], theta2[t]} /. sol1, {t, 0, 5}]

x1[t] := Evaluate[Sin[theta1[t]] /. sol1]
y1[t] := Evaluate[-Cos[theta1[t]] /. sol1]
x2[t] := Evaluate[Sin[theta1[t]] + Sin[theta2[t]] /. sol1]
y2[t] := Evaluate[-(Cos[theta1[t]] + Cos[theta2[t]]) /. sol1]

ParametricPlot[Evaluate[{{x1[t], y1[t]}, {x2[t], y2[t]}} /. sol1], {t, 0, 5}]
```

Figure 1: Coding of Lagrangian by using Mathematica

```
In[583]:= Needs["VariationalMethods`"]
Clear[s1, s2, phi1, phi2, t, g, m1, m2];
variables = {phi1[t], phi2[t]};
r1 = s1 {Sin[phi1[t]], -Cos[phi1[t]]};
r2 = r1 + s2 {Sin[phi2[t]], -Cos[phi2[t]]};

lagrangian =
  m1 / 2 D[r1, t] . D[r1, t] + m2 / 2 D[r2, t] . D[r2, t] -
   g {0, 1} . (m1 r1 + m2 r2);
eqs = EulerEquations[lagrangian, variables, t];

s1 = 2;
s2 = 2;
m1 = 1;
m2 = 1;
g = 9.81;
tMax = 5;
initial = {phi1[0] == Pi / 2, phi2[0] == Pi,
       phi1'[0] == Pi, phi2'[0] == 2Pi};
sol = First[NDSolve[Join[eqs, initial], variables, {t, 0, tMax}]];
Plot[{phi1[t], phi2[t]} /. sol, {t, 0, tMax}]

x1[t] := Evaluate[Sin[phi1[t]] /. sol2]
y1[t] := Evaluate[-Cos[phi1[t]] /. sol2]
x2[t] := Evaluate[Sin[phi1[t]] + Sin[phi2[t]] /. sol2]
y2[t] := Evaluate[-(Cos[phi1[t]] + Cos[phi2[t]]) /. sol2]
ParametricPlot[{phi1[t], phi2[t]} /. sol, {t, 0, tMax}]
```

Figure 2: Coding of Euler-Lagrangian by using Mathematica

```
In[855]:= ClearAll[x, u, y, v, f1, f2, f3, f4, pts1, pt2, pt3, pt4, pt5,
    pt6, pt7, pt8]
  g = 9.81; mas1 = 1.; mas2 = 1.; le1 = 2; le2 = 2;
  f1[t_, x_, u_, y_, v_] := y;
  f2[t_, x_, u_, y_, v_] := v;
  f3[t_, x_, u_, y_,
    v_] := (-g (2 mas1 + mas1) Sin[x] - mas2 g Sin[x - 2 u] -
    2 Sin[x - u] mas2 (v ^ 2 le2 +
      y ^ 2 le1 Cos[x - u] )) / (le1 (2 mas1 + mas2 -
      mas2 Cos[2 x - 2 u] ));
  f4[t_, x_, u_, y_,
    v_] := (2 Sin[
      x - u] (y ^ 2 le1 (mas1 + mas2) + g (mas1 + mas2) Cos[x] +
      v ^ 2 le2 mas2 Cos[x - u] )) / (le2 (2 mas1 + mas2 -
      mas2 Cos[2 x - 2 u] ));
  x = 0.0 Pi; u = -0.25 Pi; y = 0; v = 0; t = 0; h = 0.01; n = 5000;

  pt1 = {{x, y}}; pt2 = {{u, v}}; pt3 = {{x, u}}; pt4 = {{y,
      v}}; pt5 = {{t, x}}; pt6 = {{t, u}}; pt7 = {{t, y}}; pt8 = {{t,
      v}};
  Do[
    {j1, k1, l1, m1} = Map[#[t, x, u, y, v] &, {f1, f2, f3, f4}];
   {j2, k2, l2, m2} = Map[#[t + h / 2, x + h / 2 * j1, u + h / 2 * k1, y + h / 2 * l1, v + h / 2 * m1] &, {f1, f2, f3, f4}];
   {j3, k3, l3, m3} = Map[#[t + h / 2, x + h / 2 * j2, u + h / 2 * k2, y + h / 2 * l2, v + h / 2 * m2] &, {f1, f2, f3, f4}];
   {j4, k4, l4, m4} = Map[#[t + h, x + h * j3, u + h * k3, y + h * l3, v + h * m3] &, {f1, f2, f3, f4}];
```

Figure 3: Coding of Runge-Kutta by using Mathematica

```
x =x +h*(j1 +2*j2 +2*j3 +j4)/6;
u =u +h*(k1 +2*k2 +2*k3 +k4)/6;
y =y +h*(l1 +2*l2 +2*l3 +l4)/6;
v =v +h*(m1 +2*m2 +2*m3 +m4)/6;
t =t +h;
{AppendTo[pt1, {x,y}];AppendTo[pt2, {u,v}],
 AppendTo[pt3, {x,u}],AppendTo[pt4, {y,v}],
 AppendTo[pt5, {t,x}],
 AppendTo[pt6, {t,u}]AppendTo[pt7, {t,y}],
 AppendTo[pt8, {t,v}]},
{i,1,n}]

ListPlot[pt1, Joined → True]
ListPlot[pt2, Joined → True]
ListPlot[pt3, Joined → True]
ListPlot[pt4, Joined → True]
ListPlot[pt5, Joined → True]
ListPlot[pt6, Joined → True]
ListPlot[pt7, Joined → True]
ListPlot[pt8, Joined → True]
```

Figure 4: Coding of Runge-Kutta by using Mathematica(continuition)