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EXTREMAL PROPERTIES OF CERTAIN CLASS OF TILTED UNIVALENT ANALYTIC FUNCTIONS

Noor Jannah Nazri[1], Nur Adibah Abdul Razak[2], Sharifah Syazwani Syed Abdul Halim[3], and Abdullah Yahya[4,*]
[1,2,3,4]College of Computing, Informatics and Mathematics, UiTM
Cawangan Negeri Sembilan, Kampus Seremban, Malaysia.
*abdullahyahya@uitm.edu.my

Abstract

This research concerns some extremal properties of certain class of univalent analytic functions that include representation theorem and coefficient bound. Let S denote the subclass of univalent functions f in an open unit disc, $E=\{m:|m|<1\}$, given by $f(m)=m+\sum_{n=2}^{\infty}a_n m^n$. The study focuses on the generalised class of tilted univalent analytic functions, $S^*(\delta,t)$ which denoted as $\operatorname{Re}\left\{e^{i\alpha}\frac{mf'(m)}{g'(m)}\right\}>\delta, m\in E,$ where $|\alpha|\le\pi,$ $\cos\alpha>\delta,$ $0\le\delta\le1,$ $-1<t\le1$ and $g'(m)=\frac{m}{(1+tm)(1-m)}$. In this study, the new generalized class of tilted univalent analytic functions, representation theorem and the coefficient bound for the $S^*(\delta,t)$ are obtained by using Herglotz Representation Theorem.

Keywords: Extremal properties, Representation theorem, Coefficient bound

1. Introduction

Complex analysis is a part of mathematical fields that deals with analytic functions of a complex variable. The Geometric Function Theory (GFT) is subdividing of complex analysis which included geometric properties of univalent analytic function, initiated by a German Mathematicians, Ludwig Bieberbach in 1916. As stated by Goodman (1983), an analytic function $f(m)$ is a one-to-one mapping of one region to another in a complex plane.

Let $\mathbb{C}$ be an element of complex number and let $f(m)$ be a complex-valued function of the complex variable m. A function f is called univalent on a domain, $D\in\mathbb{C}$ if the function f is injective, which is for all $m_1,m_2\in D, f(m)=f(m_2)$ implies $m_1=m_2$ (Duren, 1983). Then, Darus (2002) stated that the codomain and range of one-to-one function are in real axis R which need on one axis only. In addition, at a point $m\in D$ is said to be analytic if it is differentiable at every point of some open neighborhood of m_0.

Let A denoted the class of function of the form

$$f(m)=m+a_2m^2+a_3m^3+a_4m^4+\dots+a_nm^n=m+\sum_{n=2}^{\infty}a_nm^n, \tag{1.1}$$

which are analytic in the unit disc, $E=\{m:|m|<1\}$. The function $f(m)$ is also known as normalized univalent function if it satisfy conditions of $f(0)=0$ and $f'(0)=1$ or $f(x)=f'(0)-1=0$ are fixed is denoted by S.

Based on Kaharudin (2011), if $f\in S$ is given by (1.1) and $f\in G_K(\alpha,\delta)$, then

$$|a_n|=\frac{2}{n}\left(\frac{1}{2}+A_{\alpha\delta}(n-1)\right),\quad n=2,3,4\dots$$

where the functions in this class satisfy the condition

$$\operatorname{Re}\left\{e^{i\alpha}\frac{f'(m)}{g'(m)}\right\}>\delta,\ (m\in E)$$

with $|\alpha|<\pi,\ \cos\alpha>\delta,\quad g'(m)=\dfrac{1}{1-m}$.

In addition, Yahya, Soh, and Mohamad (2013) stated that if $f\in S$ and $f\in G(\alpha,\delta)$ then

$$|a_n|=\frac{2}{n}\left(\frac{1}{2}+\frac{A_{\alpha\delta}(n-1)}{2}\right),\ n=2,3,4,\dots$$

where the functions in this class satisfy the condition

$$\operatorname{Re}\left\{e^{i\alpha}\frac{f'(m)}{g'(m)}\right\}>\delta,\ (m\in E)$$

with $|\alpha|<\pi,\ \cos\alpha>\delta,\quad g'(m)=\dfrac{m}{1-m^2}$.

In the present paper, we focused on the generalised class of tilted univalent analytic functions, $S^*(\delta,t)$ which denoted as S and satisfied the condition

$$\operatorname{Re}\left\{e^{i\alpha}\frac{mf'(m)}{g'(m)}\right\}>\delta, m\in E, \tag{1.2}$$

where $|\alpha|<\pi$, $\cos\alpha>\delta$, $0\le\delta<1$, $-1<t\le 1$, and $g'(m)=\dfrac{m}{(1+tm)(1-m)}$. The main objectives of this paper are to define the new generalized class of tilted univalent analytic

functions, to find representation theorem for this class of functions and to determine the coefficient bound for the $S^*(\delta, t)$ by using Herglotz Representation Theorem.

2. Preliminaries

To derive the main result, we apply Herglotz Representation Theorem to obtain Representation Theorem for the $S^*(\delta, t)$.

Theorem 2.1
Let $f \in S$ and $f \in S^*(\delta, t)$, then

$$e^{i\delta}\,\frac{f'(m)(1+tm)(1-m)-\delta-i\sin\alpha}{\cos\alpha-\delta}=p(m).$$

Proof.
Based on (1.2), let $f'(m)(1+tm)(1-m)$ be written as

$$f'(m)(1+tm)(1-m)=1+\sum_{n=1}^{\infty}p_n m^n. \tag{2.1}$$

Then, (2.1) can be written as

$$e^{i\alpha}\,f'(m)(1+tm)(1-m)-\delta=1+\sum_{n=1}^{\infty}p_n m^n.$$

By applying into the relation P, we have,

$$e^{i\alpha}\,f'(m)(1+tm)(1-m)-\delta=e^{i\alpha}\left[1+\sum_{n=1}^{\infty}p_n m^n\right]-\delta \tag{2.2}$$

Rearranging (2.2), we have,

$$e^{i\alpha}\,f'(m)(1+tm)(1-m)-\delta-i\sin\alpha=\cos\alpha-\delta+\sum_{n=1}^{\infty}\left(e^{i\alpha}p_n\right)m^n.$$

We divide the equation with $\cos\alpha-\delta$ to obtain P,

$$e^{i\alpha}\,\frac{f'(m)(1+tm)(1-m)-\delta-i\sin\alpha}{\cos\alpha-\delta}=\frac{\cos\alpha-\delta+\sum_{n=1}^{\infty}\left(e^{i\alpha}p_n\right)m^n}{\cos\alpha-\delta}.$$

Therefore,

$$e^{i\alpha}\,\frac{f'(m)(1+tm)(1-m)-\delta-i\sin\alpha}{\cos\alpha-\delta}=1+\frac{\sum_{n=1}^{\infty}\left(e^{i\alpha}p_n\right)m^n}{\cos\alpha-\delta}.$$

By applying $b_n=\dfrac{e^{i\alpha}p_n}{\cos\alpha-\delta}$, we have

$$e^{i\alpha}\frac{f'(m)(1+tm)(1-m)-\delta-i\sin\alpha}{\cos\alpha-\delta}=1+\sum_{n=1}^{\infty}b_n m^n.$$

We relate the function in P with

$$e^{i\alpha}\frac{f'(m)(1+tm)(1-m)-\delta-i\sin\alpha}{\cos\alpha-\delta}=p(m)\,. \tag{2.3}$$

So that, $f\in S^*(\delta,t)$ if and only if $p(m)\in P$. Based on (2.3) noted that $\cos\alpha-\delta$ should always be positive which brings about the condition $\cos\alpha>\delta$ in the definition of the class $S^*(\delta,t)$. In addition, by using an approach of Herglotz representation theorem for function in P give a representation function for $S^*(\delta,t)$.

Now, we shall prove our main result.

3. Main Result

Now, we shall focus on the coefficient bound of $S^*(\delta,t)$.

Theorem 3.1
If $f\in S$ and $f\in S^*(\delta,t)$, then

$$|a_n|\le\begin{cases}\dfrac{1}{n}\left(\dfrac{1-t^2+2A_{\alpha\delta}\left[(-1+t)+n(t+1)\right]}{(t+1)^2}\right), & n=2,4,6,\ldots\\[2ex] \dfrac{1}{n}\left(\dfrac{2A_{\alpha\delta}(n-1)+t+1}{(t+1)}\right), & n=3,5,7,\ldots\end{cases}$$

Proof.
Suppose that

$$p\in P\Leftrightarrow p(m)=\int_x\frac{1+xm}{1-xm}\,d\mu(x)\,,\quad |x|=1\,,$$

for some probability measure μ on the unit circle X. Rearranging (2.3) to make $f'(m)$ as the subject,

$$e^{i\alpha}\frac{mf'(m)}{g'(m)}-\delta-i\sin\alpha=p(m)(\cos\alpha-\delta).$$

Then,

$$e^{i\alpha}f'(m)=\frac{g'(m)\left[p(m)(\cos\alpha-\delta)+\delta+i\sin\alpha\right]}{m}\,,$$

by replacing $\cos\alpha-\delta=A_{\alpha\delta}$, we have,

$$e^{i\alpha} f'(m) = \frac{g'(m)\left[A_{\alpha\delta}\, p(m) + \delta + i\sin\alpha\right]}{m}.$$

Therefore,

$$f'(m) = e^{-i\alpha}\, \frac{g'(m)\left[A_{\alpha\delta}\, p(m) + \delta + i\sin\alpha\right]}{m}, \tag{3.1}$$

which implies $A_{\alpha\delta} > 0.$ From (3.1), we have

$$f'(m) = e^{-i\alpha}\left(\frac{g'(m)}{m}\right)\left[(\cos\alpha - \delta)\int_X \frac{1+xm}{1-xm}\, d\mu(x) + \delta + i\sin\alpha\right].$$

Then, follows that

$$f(m) = \int_0^m \frac{g'(\varphi)}{\varphi}\left[\int_X \frac{e^{-i\alpha}(\cos\alpha - \delta)(1+x\varphi) + e^{-i\alpha}(i\sin\alpha + \delta)(1-x\varphi)\, d\mu(x)}{1-x\varphi}\right] d\varphi.$$

Then,

$$f(m) = \int_0^m \left[\int_x \frac{\frac{\varphi}{(1+xt\varphi)(1-x\varphi)}}{\varphi}\left(\frac{1+x\varphi\left[\left(e^{-i\alpha}(\cos\alpha - \delta) - e^{-i\alpha}(i\sin\alpha + \delta)\right)\right] d\mu(x)}{1-x\varphi}\right)\right] d\varphi$$

$$= \int_0^m \left[\int_x \frac{1}{(1+xt\varphi)(1-x\varphi)}\left(\frac{1+x\varphi\left[e^{-i\alpha}(\cos\alpha - i\sin\alpha - 2\delta)\right] d\mu(x)}{1-x\varphi}\right)\right] d\varphi,$$

and

$$f(m) = \int_0^m \left[\int_x \frac{1}{(1+xt\phi)(1-x\phi)}\left(\frac{1+x\phi\left(e^{-i2\alpha} - 2\delta e^{-i\alpha}\right) d\mu(x)}{1-x\phi}\right)\right] d\phi$$

$$= \int_X \left[\int_0^m \frac{1+x\phi\left(e^{-i2\alpha} - 2\delta e^{-i\alpha}\right) d\phi}{(1+xt\phi)(1-x\phi)^2}\right] d\mu(x).$$

Let $e^{-i2\alpha} - 2\delta e^{-ia} = m,$

$$f(m) = \int_X \left[\int_0^m \frac{1+x\phi m}{(1-x\phi)^2(1+xt\phi)}\, d\phi\right] d\mu(x). \tag{3.2}$$

Rearranging the equation (3.2),

$$f(m) = \int_X \left[\int_0^m \frac{-m + x\varphi m + 1 + m}{(1-x\varphi)^2(1+xt\varphi)}\, d\varphi\right] d\mu(x)$$

$$= \int_X \int_0^m \left[\frac{-m}{(1-x\varphi)(1+xt\varphi)} + \frac{1+m}{(1-x\varphi)^2(1+xt\varphi)}\right] d\varphi\, d\mu(x),$$

Therefore,

$$f(m)=\int_X\int_0^m\left[\frac{-\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)}{(1-x\phi)(1+xt\phi)}+\frac{1+e^{-i2\alpha}-2\delta e^{-i\alpha}}{(1-x\phi)^2(1+xt\phi)}\right]d\phi d\mu(x).$$

Next, separate the variable

$$f(m)=\int_x\left[\int_0^m\left[\left(-e^{-i2\alpha}+2\delta e^{-i\alpha}\right)\left(\frac{1}{(1-x\phi)(1+xt\phi)}\right)+\left(1+e^{-i2\alpha}-2\delta e^{-i\alpha}\right)\left(\frac{1}{(1-x\phi)^2(1+xt\phi)}\right)\right]d\phi\right]d\mu(x).$$

By using partial fraction, we get

$$f(m)=\int_X\left[\int_0^m\left[\left(-e^{-i2\alpha}+2\delta e^{-i\alpha}\right)\left(\frac{1}{(t+1)(1-x\phi)}+\frac{t}{(t+1)(1+xt\phi)}\right)+\left(1+e^{-i2\alpha}-2\delta e^{-i\alpha}\right)\left(\frac{t}{(t+1)^2(1-x\phi)}+\frac{1}{(t+1)(1-x\phi)^2}\frac{t^2}{(t+1)^2(1+xt\phi)}\right)\right]d\phi\right]d\mu(x),$$

by replacing,

$$1+e^{-2i\alpha}-2\delta e^{-i\alpha}=2A_{\alpha\delta}(\cos\alpha-i\sin\alpha)$$

and rearrange the equation,

$$f(m)=\int_X\left[\int_0^m\left[\left(\frac{-e^{-i2\alpha}+2\delta e^{-i\alpha}}{(t+1)}\right)\left(\frac{1}{(1-x\phi)}+\frac{t}{(1+xt\phi)}\right)+\left(\frac{2A_{\alpha\delta}e^{-i\alpha}}{(t+1)^2}\right)\left(\frac{t}{(1-x\phi)}+\frac{1+t}{(1-x\phi)^2}+\frac{t^2}{(1+xt\phi)}\right)\right]d\phi\right]d\mu(x). \quad (3.3)$$

From (3.3), we have

$$f(m)=\frac{1}{t+1}\int_0^m\left[\int_X\left[\left(-e^{-i2\alpha}+2\delta e^{-i\alpha}\right)\left(\frac{1}{(1-x\phi)}+\frac{t}{(1+xt\phi)}\right)+\left(\frac{2A_{\alpha\delta}e^{-i\alpha}}{t+1}\right)\left(\frac{t-tx\phi}{(1-x\phi)^2}+\frac{1+t}{(1-x\phi)^2}+\frac{t^2}{(1+xt\phi)}\right)\right]d\mu(x)\right]d\phi,$$

$$=\frac{1}{t+1}\int_0^m\left[\int_X\left[\left(-e^{-i2\alpha}+2\delta e^{-i\alpha}\right)\left(\frac{1}{(1-x\phi)}+\frac{t}{(1+xt\phi)}\right)+\left(\frac{2A_{\alpha\delta}e^{-i\alpha}}{t+1}\right)\left(\frac{-tx\phi}{(1-x\phi)^2}+\frac{1+2t}{(1-x\phi)^2}+\frac{t^2}{(1+xt\phi)}\right)\right]d\mu(x)\right]d\phi,$$

and

$$
\begin{aligned}
f(m)=\frac{1}{t+1}\int_0^m \Big[& -\left(e^{-i2\alpha}-2\delta e^{-i\alpha}\right)\int_X \sum_{n=0}^{\infty}(x)^n\, d\mu(x)(\phi)^n \\
& +\left(-te^{-i2\alpha}+2t\delta e^{-i\alpha}+\frac{2t^2 A_{\alpha\delta}e^{-i\alpha}}{(t+1)}\right)\int_X \sum_{n=0}^{\infty}(-1)^n(t)^n(x)^n\, d\mu(x)(\phi)^n \\
& -\frac{2tA_{\alpha\delta}e^{-i\alpha}}{(t+1)}\int_X \sum_{n=0}^{\infty}(n)(x)^n\, d\mu(x)(\phi)^n \\
& +\frac{2A_{\alpha\delta}e^{-i\alpha}(1+2t)}{(t+1)}\int_X \sum_{n=0}^{\infty}(n+1)(x)^n\, d\mu(x)(\phi)^n \Big] d\phi. \qquad (3.4)
\end{aligned}
$$

From (3.4), substitute $n=0$, then

$$
f(m)=\frac{1}{(t+1)^2}\left(\left[\left(-e^{-i2\alpha}+2\delta e^{-i\alpha}-te^{-i2\alpha}+2t\delta e^{-i\alpha}\right)(t+1)\right]+2t^2A_{\alpha\delta}e^{-i\alpha}+2A_{\alpha\delta}e^{-i\alpha}(1+2t)\right),
$$

and

$$
f(m)=\frac{1}{(t+1)^2}\left(\left(-1-2t-t^2\right)e^{-i2\alpha}+\left(2t^2+4t+2\right)\delta e^{-i\alpha}+\left(2t^2+4t+2\right)A_{\alpha\delta}e^{-i\alpha}\right).
$$

Substitute $A_{\alpha\delta}=\cos\alpha-\delta$ and $\cos\alpha=\dfrac{e^{i\alpha}+e^{-i\alpha}}{2}$, thus

$$
\begin{aligned}
f(m)&=\frac{1}{(t+1)^2}\left(\left(-1-2t-t^2+t^2+2t+1\right)e^{-i2\alpha}+\left(2t^2+4t+2-2t^2-4t-2\right)\delta e^{-i\alpha}+\left(t^2+2t+1\right)\right) \\
&=1
\end{aligned}
$$

Then, rewrite the equation (3.4) will yield to

$$
\begin{aligned}
f(m)=\int_0^m \Bigg[& 1-\left(\frac{e^{-i2\alpha}-2\delta e^{-i\alpha}}{t+1}\right)\int_X \sum_{n=1}^{\infty}(x)^n\, d\mu(x)(\phi)^n \\
& -\left(\frac{te^{-i2\alpha}(t+1)-2t\delta e^{-i\alpha}(t+1)-2t^2A_{\alpha\delta}e^{-i\alpha}}{(t+1)^2}\right)\int_X \sum_{n=1}^{\infty}(-t)^n(x)^n\, d\mu(x)(\phi)^n \\
& -\frac{2tA_{\alpha\delta}e^{-i\alpha}}{(t+1)^2}\int_X \sum_{n=1}^{\infty}(n)(x)^n\, d\mu(x)(\phi)^n \\
& +\frac{2A_{\alpha\delta}e^{-i\alpha}(1+2t)}{(t+1)^2}\int_X \sum_{n=1}^{\infty}(n+1)(x)^n\, d\mu(x)(\phi)^n \Bigg] d\phi.
\end{aligned}
$$

Integrating with respect to ϕ gives us,

$$f(m) = m - \left(\frac{e^{-i2\alpha} - 2\delta e^{-i\alpha}}{t+1} \right) \int_X \sum_{n=2}^{\infty} (x)^{n-1} d\mu(x) \left(\frac{m^n}{n} \right)$$
$$- \left(\frac{te^{-i2\alpha}(t+1) - 2t\delta e^{-i\alpha}(t+1) - 2t^2 A_{\alpha\delta} e^{-i\alpha}}{(t+1)^2} \right) \int_X \sum_{n=2}^{\infty} (-t)^{n-1} (x)^{n-1} d\mu(x) \left(\frac{m^n}{n} \right)$$
$$- \frac{2tA_{\alpha\delta} e^{-i\alpha}}{(t+1)^2} \int_X \sum_{n=2}^{\infty} (n-1)(x)^{n-1} d\mu(x) \left(\frac{m^n}{n} \right)$$
$$+ \frac{2A_{\alpha\delta} e^{-i\alpha}(1+2t)}{(t+1)^2} \int_X \sum_{n=2}^{\infty} (n)(x)^{n-1} d\mu(x) \left(\frac{m^n}{n} \right) \Bigg].$$

Rearranging the equation, we have

$$f(m) = m + \sum_{n=2}^{\infty} \left(- \frac{\left(e^{-i2\alpha} - 2\delta e^{-i\alpha}\right)}{(t+1)n} \right) \left[\int_X (x)^{n-1} d\mu(x) \right]$$
$$- \left(\frac{te^{-i2\alpha}(t+1) - 2t\delta e^{-i\alpha}(t+1) - 2t^2 A_{\alpha\delta} e^{-i\alpha}}{(t+1)^2 n} \right) \left[\int_X (-t)^{n-1} (x)^{n-1} d\mu(x) \right]$$
$$- \frac{2tA_{\alpha\delta} e^{-i\alpha}}{(t+1)^2 n} \left[\int_X (n-1)(x)^{n-1} d\mu(x) \right]$$
$$+ \frac{2A_{\alpha\delta} e^{-i\alpha}(1+2t)}{(t+1)^2 n} \left[\int_X (n)(x)^{n-1} d\mu(x) \right] \Bigg) m^n, \tag{3.5}$$

and from (1.1), we have that $f(m) = m + \sum_{n=2}^{\infty} a_n m^n$. By comparing (3.5) with (1.1), we have found that

$$a_n = \left(- \frac{\left(e^{-i2\alpha} - 2\delta e^{-i\alpha}\right)}{(t+1)n} \right) \left[\int_X (x)^{n-1} d\mu(x) \right]$$
$$- \left(\frac{te^{-i2\alpha}(t+1) - 2t\delta e^{-i\alpha}(t+1) - 2t^2 A_{\alpha\delta} e^{-i\alpha}}{(t+1)^2 n} \right) \left[\int_X (-t)^{n-1} (x)^{n-1} d\mu(x) \right]$$
$$- \frac{2tA_{\alpha\delta} e^{-i\alpha}}{(t+1)^2 n} \left[\int_X (n-1)(x)^{n-1} d\mu(x) \right]$$
$$+ \frac{2A_{\alpha\delta} e^{-i\alpha}(1+2t)}{(t+1)^2 n} \left[\int_X (n)(x)^{n-1} d\mu(x) \right] \Bigg) m^n.$$

Upon simplification, we have

$$|a_n| = \left|\left(-\frac{\left(e^{-i2\alpha} - 2\delta e^{-i\alpha}\right)}{(t+1)n} - \frac{te^{-i2\alpha}(t+1) + 2t\delta e^{-i\alpha}(t+1) + 2t^2 A_{\alpha\delta} e^{-i\alpha}}{(t+1)^2 n}(-t)^{n-1} - \frac{2t(n-1)A_{\alpha\delta}e^{-i\alpha}}{(t+1)^2 n} + \frac{2n(1+2t)A_{\alpha\delta}e^{-i\alpha}}{(t+1)^2 n}\right)\left[\int_X (x)^{n-1} d\mu(x)\right]\right|.$$

Rearranging the equation,

$$|a_n| = \frac{1}{(t+1)n}\left|\left(-e^{-i2\alpha} + 2\delta e^{-i\alpha} - \left(te^{-i2\alpha} - 2t\delta e^{-i\alpha} - \frac{2t^2 A_{\alpha\delta}e^{-i\alpha}}{(t+1)}\right)(-t)^{n-1} - \frac{2ntA_{\alpha\delta}e^{-i\alpha}}{(t+1)} + \frac{2tA_{\alpha\delta}e^{-i\alpha}}{(t+1)} + \frac{2nA_{\alpha\delta}e^{-i\alpha}}{(t+1)} + \frac{4ntA_{\alpha\delta}e^{-i\alpha}}{(t+1)}\right)\right|\left|\int_X (x)^{n-1} d\mu(x)\right|,$$

we have,

$$|a_n| = \frac{1}{(t+1)n}\left|\left(-e^{-i2\alpha} + 2\delta e^{-i\alpha} - \left(-e^{-i2\alpha} + 2\delta e^{-i\alpha} + \frac{2tA_{\alpha\delta}e^{-i\alpha}}{(t+1)}\right)(-t)^{n} + \frac{2ntA_{\alpha\delta}e^{-i\alpha}}{(t+1)} + \frac{2tA_{\alpha\delta}e^{-i\alpha}}{(t+1)} + \frac{2nA_{\alpha\delta}e^{-i\alpha}}{(t+1)}\right)\right|\left|\int_X (x)^{n-1} d\mu(x)\right|. \tag{3.6}$$

Based on (3.6), we will obtain two equations of a_n. One of the equations of a_n when n is an even number starting with $n = 2,4,6,...$

$$|a_n| = \frac{1}{(t+1)n}\left|\left(-e^{-i2\alpha} + 2\delta e^{-i\alpha} + te^{-i2\alpha} - 2t\delta e^{-i\alpha} - \frac{2t^2 A_{\alpha\delta}e^{-i\alpha}}{(t+1)} + \frac{2ntA_{\alpha\delta}e^{-i\alpha}}{(t+1)} + \frac{2tA_{\alpha\delta}e^{-i\alpha}}{(t+1)} + \frac{2nA_{\alpha\delta}e^{-i\alpha}}{(t+1)}\right)\right|\left|\int_X (x)^{n-1} d\mu(x)\right|,$$

and

$$|a_n| = \frac{1}{(t+1)n}\left|\frac{(-1+t)\left[(t+1)\left(e^{-i2\alpha} - 2\delta e^{-i\alpha}\right) - 2tA_{\alpha\delta}e^{-i\alpha}\right]}{(t+1)} + 2nA_{\alpha\delta}e^{-i\alpha}\right|\left|\int_X (x)^{n-1} d\mu(x)\right|,$$

then again since $e^{-i2\alpha} - 2\delta e^{-i\alpha} = 2A_{\alpha\delta}e^{-i\alpha} - 1$ and $\left|e^{-i\alpha}\right| = 1$,

$$|a_n| = \frac{1}{(t+1)n}\left|\frac{(-1+t)\left[-t + 2A_{\alpha\delta}e^{-i\alpha} - 1\right]}{(t+1)} + 2nA_{\alpha\delta}e^{-i\alpha}\right|\int_X \left|(x)^{n-1}\right| d\mu(x),$$

and

$$|a_n| = \frac{1}{(t+1)n}\left|\frac{t+1-t^2-t+2(-1+t)A_{\alpha\delta}e^{-i\alpha}+2n(t+1)A_{\alpha\delta}e^{-i\alpha}}{(t+1)}\right|\int_X\left|(x)^{n-1}\right|d\mu(x),$$

thus

$$|a_n| = \frac{1}{(t+1)n}\left|\frac{1-t^2+2A_{\alpha\delta}e^{-i\alpha}\left[(-1+t)+n(t+1)\right]}{(t+1)}\right|\int_X\left|(x)^{n-1}\right|d\mu(x)$$
$$\leq \frac{1}{(t+1)n}\left(\frac{1-t^2+2A_{\alpha\delta}\left[(-1+t)+n(t+1)\right]}{(t+1)}\right)\int_X\left|(x)^{n-1}\right|d\mu(x),$$

and

$$|a_n| = \frac{1}{n}\left(\frac{1-t^2+2A_{\alpha\delta}\left[(-1+t)+n(t+1)\right]}{(t+1)^2}\right), \quad n = 2,4,6,\ldots$$

Based on (3.6), another equation of a_n when n is an odd number starting with $n = 3,5,7,\ldots$

$$|a_n| = \frac{1}{(t+1)n}\left|\left(-e^{-i2\alpha}+2\delta e^{-i\alpha}-\left(te^{-i2\alpha}-2t\delta e^{-i\alpha}-\frac{2t^2A_{\alpha\delta}e^{-i\alpha}}{(t+1)}\right)\right.\right.$$
$$\left.\left.+\frac{2ntA_{\alpha\delta}e^{-i\alpha}}{(t+1)}+\frac{2tA_{\alpha\delta}e^{-i\alpha}}{(t+1)}+\frac{2nA_{\alpha\delta}e^{-i\alpha}}{(t+1)}\right)\right|\left|\int_X(x)^{n-1}\,d\mu(x)\right|,$$

and

$$|a_n| = \frac{1}{(t+1)n}\left|\left((t+1)\left(2\delta e^{-i\alpha}-e^{-i2\alpha}\right)+2nA_{\alpha\delta}e^{-i\alpha}+2tA_{\alpha\delta}e^{-i\alpha}\right)\right|\left|\int_X(x)^{n-1}\,d\mu(x)\right|,$$

then again since $e^{-i2\alpha}-2\delta e^{-i\alpha} = 2A_{\alpha\delta}e^{-i\alpha}-1$ and $\left|e^{-i\alpha}\right| = 1$,

$$|a_n| = \frac{1}{(t+1)n}\left|\left(-2tA_{\alpha\delta}e^{-i\alpha}+t-2A_{\alpha\delta}e^{-i\alpha}+1+2nA_{\alpha\delta}e^{-i\alpha}+2tA_{\alpha\delta}e^{-i\alpha}\right)\right|\int_X\left|(x)^{n-1}\right|d\mu(x),$$

thus

$$|a_n| = \frac{1}{(t+1)n}\left|\left(2A_{\alpha\delta}(n-1)+t+1\right)\right|\int_X\left|(x)^{n-1}\right|d\mu(x)$$
$$\leq \frac{1}{(t+1)n}\left(2A_{\alpha\delta}(n-1)+t+1\right)\int_X\left|(x)^{n-1}\right|d\mu(x),$$

and

$$|a_n| = \frac{1}{n}\left(\frac{2A_{\alpha\delta}(n-1)+t+1}{(t+1)}\right), \quad n = 3,5,7,\ldots$$

as required.

4. Conclusion

In conclusion, there are three purposes of this paper, which are to to define the new generalized class of tilted univalent analytic functions, to find representation theorem for this class of functions and to determine the coefficient bound for the $S^{*}(\delta,t)$. We believe that we have achieved all the objectives that we highlighted.

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