APPROXIMATION AND INTERPOLATION OF ENTIRE FUNCTIONS OF SEVERAL COMPLEX VARIABLES ON ARBITRARY COMPACT SETS

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Abstract

In the present paper, we study the polynomial approximation of entire functions of several complex variables. The characterizations of order and type of entire functions of several complex variables have been obtained in terms of approximation and interpolation errors.

Keywords**:** Entire function, Vandermonde determinant, Order, Type, Approximation errors, Interpolation errors.

1. Introduction

Let $g:C^N \to C, N \geq 1$, be an entire transcendental function. For $z=(z_1,z_2,...,z_N)\in C^N$, we put $S(r,g)=\sup\{|g(z)|:|z_1|^2+|z_2|^2+...+|z_N|^2=r^2\}$, $r>0$. Then we define the order $g(z)$ as

$$\rho(g)=\limsup_{r\to\infty}\frac{\log\log S(r,g)}{\log r}$$

and for $0<\rho<\infty$, the type of $g(z)$ as

$$\sigma(g)=\limsup_{r\to\infty}\frac{\log S(r,g)}{r^{\rho}}.$$

Let K be a compact set in C^N and let $\|.\|_K$ denote the sup norm on K. The function $\Phi_K(z)=\left(|p(z)|^{1/n}:\ p\text{-polynomial},\ \deg p\leq n,\ \|p\|_K\leq 1,\ n=1,2,..,z\in C^N\right)$, is called the Siciak extremal function of the compact set K (Janik, 1984a and Janik, 1984b). Given a function f defined and bounded on K, we put for $n=1,2,...$

$$E_n^1(f,K)=\|f-t_n\|_K;$$
$$E_n^2(f,K)=\|f-l_n\|_K;$$
$$E_{n+1}^3(f,K)=\|l_{n+1}-l_n\|_K;$$

where t_n denotes the n^{th} Chebyshev polynomial of the best approximation to f on K and l_n denotes the n^{th} Lagrange interpolation for f with nodes at extremal points of K (Janik, 1984a and Janik, 1984b). Let $u_1, u_2, ..., u_n \in K$, where $u_l = (u_{1l}, u_{2l}, ..., u_{Nl})$. Following (Sheinov, 1971), we define

$$V_n = \max_K \left| \prod_{i=1}^{N} V(u_{i1}, u_{i2}, ..., u_{in}) \right|,$$

where $V(u_{i1}, u_{i2}, ..., u_{in})$ is Vandermonde determinant for the i-th co-ordinates of these points , that is,

$$V(u_n) = \begin{vmatrix} 1 & u_{i1} & u_{i1}^2 & . & . & . & u_{i1}^{n-1} \\ 1 & u_{i2} & u_{i2}^2 & . & . & . & u_{i2}^{n-1} \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ 1 & u_{in} & u_{in}^2 & . & . & . & u_{in}^{n-1} \end{vmatrix}.$$

Also, let μ_n denote the smallest maximum modulus of n^{th} Chebyshev polynomial t_n on compact set K. G. M. Goluzin (Goluzin, 1966 pp. 296) obtained the relation between μ_n and $\{V_{n+1} / V_n\}$. Here we extend this result for several complex variables. Hence here we write

$$\prod_{i=1}^{N} [(z_i - u_{i1})(z_i - u_{i2})...(z_i - u_{in})] = \frac{\prod_{i=1}^{N} V(z_i, u_{i1}, u_{i2}, ..., u_{in})}{\prod_{i=1}^{N} V(u_{i1}, u_{i2}, ..., u_{in})}. \tag{1.1}$$

Now if the points $z, u_1, u_2, ..., u_n \in K$ and $\left| \prod_{i=1}^{N} V(u_{i1}, u_{i2}, ..., u_{in}) \right| = V_n$ then the modulus of right hand side of (1.1) does not exceed $\{V_{n+1} / V_n\}$. Also for $z \in K$ the modulus of left hand side is not less than μ_n. So we get

$$\mu_n \le \frac{V_{n+1}}{V_n}. \tag{1.2}$$

Also we have

$$\begin{aligned} \prod_{i=1}^{N} V(u_{i1}, u_{i2}, ..., u_{i(n+1)}) \le \prod_{i=1}^{N} \{ &| u_{i1}^n || V(u_{i2}, u_{i3}, ..., u_{i(n+1)}) | \\ &+ | u_{i2}^n || V(u_{i1}, u_{i3}, ..., u_{i(n+1)}) | + \\ &....+ | u_{i(n+1)}^n || V(u_{i1}, u_{i2}, ..., u_{in}) | \}. \end{aligned} \tag{1.3}$$

Now if the points $u_1, u_2, ..., u_n, u_{n+1} \in K$ and $\left| \prod_{i=1}^{N} V(u_{i1}, u_{i2}, ..., u_{in}, u_{i(n+1)}) \right| = V_{n+1}$ then from (1.3), we get

$$V_{n+1} \le V_n \left[\prod_{i=1}^{N} \{ | u_{i1}^n | + | u_{i2}^n | + ... + | u_{i(n+1)}^n | \} \right]$$

or $$\frac{V_{n+1}}{V_n} \le \prod_{i=1}^{N} | u_{i1}^n | + \prod_{i=1}^{N} | u_{i2}^n | + ... + \prod_{i=1}^{N} | u_{i(n+1)}^n |$$

or $$\frac{V_{n+1}}{V_n} \le \mu_n (n+1) \quad . \tag{1.4}$$

Finally from (1.2) and (1.4), we get

$$\mu_n \le \frac{V_{n+1}}{V_n} \le \mu_n (n+1).$$

From the above inequalities we get

$$\lim_{n\to\infty} \left[\frac{V_{n+2}}{V_{n+1}} \right]^{1/n} = d,$$

where d is the transfinite diameter of the set K.

2. Main Results

Before proving our main results, we state and prove some lemmas.

Lemma 2.1: Let $K \subseteq C^N$ be a compact set with non-zero transfinite diameter. Let f be a continuous function on K. Then the function f can be continuously extended to an entire function $g(z)$ if and only if

$$\lim_{n\to\infty} \left[E_n^s(f,K) \frac{V_{n+1}}{V_{n+2}} \right]^{1/n} = 0 \quad ; \; s = 1,2,3.$$

Proof: Following (Janik, 1984b and Winiarski, 1973), it follows that the function f can be continuously extended to an entire function $g(z)$ if and only if

$$\lim_{n\to\infty} \left[E_n^s(f,K) \right]^{1/n} = 0 \quad ; \; s = 1,2,3.$$

Also we have

$$\lim_{n\to\infty} \left[\frac{V_{n+2}}{V_{n+1}} \right]^{1/n} = d.$$

Since transfinite diameter of K is finite, therefore, we get

$$\lim_{n\to\infty} \left[E_n^s(f,K) \frac{V_{n+1}}{V_{n+2}} \right]^{1/n} = 0.$$

Hence the Lemma 2.1 is proved.

Lemma 2.2: Let $(p_n)_{n\in N}$ be a sequence of polynomials of degree not exceeding n. If there exist $n_0 \in N$ and a positive number λ such that for all $n \ge n_0$

$$\| p_n \| \le (d_n)^n \, n^{-n/\lambda}, \text{ where } d_n = \left[\frac{V_{n+2}}{V_{n+1}} \right]^{1/n}$$

then $\sum_{n=0}^{\infty} p_n$ is an entire function and the order $\rho(\sum_{n=0}^{\infty} p_n) \le \lambda$ provided $\sum_{n=0}^{\infty} p_n$ is not a polynomial.

Proof: By assumption of lemma, we have

$$\| p_n \| \left[\frac{r}{d_n}\right]^n \le r^n\, n^{-n/\lambda} \quad , \; n \ge n_0 \quad , r > 0.$$

Let us consider the function

$$\phi_1(x) = r^x\, x^{-x/\lambda}.$$

The maximum of $\phi_1(x)$ is attained at $x = \exp[\lambda \log r - 1]$ and is equal to $\exp\left[\frac{1}{\lambda}\exp(\lambda \log r - 1)\right]$. Hence we get

$$\| p_n \| \left[\frac{r}{d_n}\right]^n \le \exp\left[\frac{1}{\lambda}\exp(\lambda \log r - 1)\right]. \tag{2.1}$$

Let us write $K_r = \{z \in C^N : \Phi_k(z) < r, r > 1\}$, then for every polynomial p of degree $\le n$, we have (Janik, 1984b)

$$| p_n(z) | \le \| p_n \|_K\, \Phi_K^n(z) \quad , \; z \in C^N. \tag{2.2}$$

So the series $\sum_{n=0}^{\infty} p_n$ is convergent in every set K_r, $r > 1$, whence $\sum_{n=0}^{\infty} p_n$ an entire function. Put

$$M^*(r) = \sup\left\{\| p_n \|_K \left[\frac{r}{d_n}\right]^n : n \in N, r > 0\right\}.$$

On account of (2.1), for every $r > 0$, there exists a positive integer $\nu(r)$ such that

$$M^*(r) = \| p_{\nu(r)} \|_K \left[\frac{r}{d_{\nu(r)}}\right]^{\nu(r)}$$

and

$$M^*(r) > \| p_n \|_K \left[\frac{r}{d_n}\right]^n \quad , \quad n > \nu(r).$$

It is evident that $\nu(r)$ increases with r. First suppose that $\nu(r) \to \infty$ as $r \to \infty$. Then putting $n = \nu(r)$ in (2.1) we get for sufficiently large r

$$M^*(r) \le \exp\left[\frac{1}{\lambda}\exp(\lambda \log r - 1)\right]. \tag{2.3}$$

Put

$$F_r = \{z \in C^N : \Phi_K(z) = r\} \quad , \quad r > 1$$

and

$$M(r) = \sup\{| \sum_{n=0}^{\infty} p_n(z) | : z \in F_r\} \quad , \quad r > 1.$$

Now following (Janik, 1984b) for some positive constant k, we have

$$S\left(r,\sum_{n=0}^{\infty}p_n\right)\le M(kr)\le 2M^*(2kr). \tag{2.4}$$

Combining (2.2), (2.3) and (2.4), we get

$$S\left(r,\sum_{n=0}^{\infty}p_n\right)\le 2\exp\left[\frac{1}{\lambda}\exp\{\lambda\log(2kr)-1\}\right]$$

or

$$\frac{\log\log S\left(r,\sum_{n=0}^{\infty}p_n\right)}{\log r}\le\frac{\log\log 2}{\log r}+\frac{\log(1/\lambda)}{\log r}+\lambda\left[\frac{\log(2k)}{\log r}+1\right]-\frac{1}{\log r}.$$

Now proceeding to limits as $r\to\infty$, we get

$$\rho\left(\sum_{n=0}^{\infty}p_n\right)\le\lambda.$$

In the case when $\nu(r)$ is bounded then $M^*(r)$ is also bounded, whence $\sum_{n=0}^{\infty}p_n$ reduces to a polynomial. Hence the Lemma 2.2 is proved.

Lemma 2.3: Let $(p_n)_{n\in N}$ be a sequence of polynomials of degree not exceeding n. Let there exist positive numbers λ and μ such that

(i) for every $n\ge n_0$

$$\|p_n\|\le(d_n)^n\left[\frac{\mu}{n}\right]^{n/\lambda},\ \text{where } d_n=\left[\frac{V_{n+2}}{V_{n+1}}\right]^{1/n}.$$

(ii) $\rho\left(\sum_{n=0}^{\infty}p_n\right)=\lambda.$

Then $\sum_{n=0}^{\infty}p_n$ is an entire function and the type $\sigma\left(\sum_{n=0}^{\infty}p_n\right)\le\frac{\mu}{e\lambda}$ provided $\sum_{n=0}^{\infty}p_n$ is not a polynomial.

Proof: By assumption of lemma, we have

$$\|p_n\|\left[\frac{r}{d_n}\right]^n\le r^n\left[\frac{\mu}{n}\right]^{n/\lambda},\quad n\ge n_0,\ r>0.$$

Let us consider the function

$$\phi_2(x)=r^x\left[\frac{\mu}{x}\right]^{x/\lambda}.$$

The maximum of $\phi_2(x)$ is attained at $x=\mu\exp[\lambda\log r-1]$ and is equal to $\exp\left[\frac{\mu}{\lambda}\exp(\lambda\log r-1)\right]$. Now as in Lemma 2.2, here we have

$$S\left(r,\sum_{n=0}^{\infty}p_n\right)\le 2\exp\left[\frac{\mu}{\lambda}\exp\{\lambda\log(2kr)-1\}\right]$$

or $$\log S\left(r,\sum_{n=0}^{\infty}p_n\right) \le \log 2+\frac{\mu\,(2kr)^{\lambda}}{\lambda e}$$

or $$\log S\left(r,\sum_{n=0}^{\infty}p_n\right) \le \log 2+\frac{\mu\, r^{\lambda+2k\delta}}{\lambda e}\ ,$$ where $\delta>0$ is suitably small,

or $$\frac{\log S\left(r,\sum_{n=0}^{\infty}p_n\right)}{r^{\lambda+2k\delta}} \le \frac{\log 2}{r^{\lambda+2k\delta}}+\frac{\mu}{\lambda e}\,.$$

Now proceeding to limits as $r\to\infty$, since $\delta>0$ is arbitrarily small, we get

$$\sigma(\sum_{n=0}^{\infty}p_n)\le\frac{\mu}{e\lambda}.$$

In the case when $\nu(r)$ is bounded then $M^*(r)$ is also bounded, whence $\sum_{n=0}^{\infty}p_n$ reduces to a polynomial. Hence the Lemma 2.3 is proved.

Now we prove

Theorem 2.1: Let $K\subseteq C^N$ be a compact set with non-zero transfinite diameter such that Φ_K is locally bounded in C^N. Then the function f, defined and bounded on K, is a restriction to K of an entire function g of order $\rho(g)\left(0<\rho(g)<\infty\right)$ if and only if

$$\rho(g)=\limsup_{n\to\infty}\frac{n\log n}{-\log\left\{E_n^s(f,K)\dfrac{V_{n+1}}{V_{n+2}}\right\}}\quad ;\ s=1,2,3.$$

Proof: Let g be an entire transcendental function. Write $\rho=\rho(g)$ and

$$\theta_s=\limsup_{n\to\infty}\frac{n\log n}{-\log\left\{E_n^s\dfrac{V_{n+1}}{V_{n+2}}\right\}}\ ;\ s=1,2,3.$$

Here E_n^s stands for $E_n^s\left(g\,|_K,K\right)$, $s=1,2,3.$ We claim that $\rho=\theta_s$, $s=1,2,3.$ It is known (Winiarski, 1973) that

$$E_n^1\ \le\ E_n^2\ \le\ (n_*+2)E_n^1\quad ,\ n\ge 0 \tag{2.5}$$

and

$$E_n^3\ \le\ 2(n_*+2)\,E_{n-1}^1\quad ,\ n\ge 1, \tag{2.6}$$

where $n_*=\binom{n+N}{n}$. Using Stirling formula for the approximate value of

$$n!\approx e^{-n}n^{n+1/2}\sqrt{2\pi},$$

we get $n_*\approx\dfrac{n^N}{N!}$ for all large values of n. Hence for all large values of n, we have

$$E_n^1 \;\le\; E_n^2 \;\le\; \frac{n^N}{N!}[1+o(1)]E_n^1$$

and

$$E_n^3 \;\le\; 2\frac{n^N}{N!}[1+o(1)]\,E_n^1 .$$

Thus $\theta_3 \;\le\; \theta_2 \;=\; \theta_1$ and it suffices to prove that $\theta_1 \;\le\; \rho \;\le\; \theta_3$. First we prove that $\theta_1 \le \rho$. Using the definition of order, for $\varepsilon > 0$ and $r > r_0(\varepsilon)$, we have

$$S(r,g) \le \exp\left(r^{\bar{\rho}}\right) \quad ,$$

where $\overline{\rho} = \rho + \varepsilon$, provided r is sufficiently large. Without loss of generality, we may suppose that

$$K \subset B = \{z \in C^N : |z_1|^2 + |z_2|^2 + ... + |z_N|^2 \le 1\}.$$

Then $\quad E_n^1 \dfrac{V_{n+1}}{V_{n+2}} \;\le\; E_n^1(g,B)\dfrac{V_{n+1}}{V_{n+2}}$.

Now following Janik[1984b, p.324], we get

$$E_n^1(g,B)\frac{V_{n+1}}{V_{n+2}} \le r^{-n}\,S(r,g) \quad , \; r \ge 2 \quad , \; n \ge 0$$

or
$$E_n^1\frac{V_{n+1}}{V_{n+2}} \;\le\; r^{-n}\exp\left(r^{\bar{\rho}}\right).$$

Putting $r = \left(n/\overline{\rho}\right)^{1/\bar{\rho}}$ in the above inequality, we get

$$E_n^1\frac{V_{n+1}}{V_{n+2}} \;\le\; \left(n/\overline{\rho}\right)^{-n/\bar{\rho}}\exp\left(n/\overline{\rho}\right)$$

or
$$-\log\left[E_n^1\frac{V_{n+1}}{V_{n+2}}\right] \;\ge\; \frac{n\log n}{\overline{\rho}}\left[1-\frac{\log\overline{\rho}}{\log n}-\frac{1}{\log n}\right]$$

or
$$\limsup_{n\to\infty}\frac{n\log n}{-\log\left\{E_n^1\dfrac{V_{n+1}}{V_{n+2}}\right\}} \;\le\; \overline{\rho}$$

or
$$\theta_1 \;\le\; \overline{\rho}.$$

Since $\varepsilon > 0$ is arbitrarily small therefore finally we get

$$\theta_1 \;\le\; \rho. \tag{2.7}$$

Now we will prove that $\rho \;\le\; \theta_3$. If $\theta_3 = \infty$, then there is nothing to prove. So let us assume that $0 \le \theta_3 < \infty$. Therefore for a given $\varepsilon > 0$ there exists $n_0 \in N$ such that for all $n > n_0$, we have

$$0 \le \frac{n\log n}{-\log\left\{E_n^3\dfrac{V_{n+1}}{V_{n+2}}\right\}} \le \theta_3 + \varepsilon = \overline{\theta_3}$$

or
$$E_n^3\frac{V_{n+1}}{V_{n+2}} \le n^{-n/\overline{\theta_3}}.$$

Now from the property of maximum modulus, we have

$$S(r,g) \le \sum_{n=0}^{\infty} E_n^3 \frac{V_{n+1}}{V_{n+2}} r^n \le \sum_{n=0}^{n_0} E_n^3 \frac{V_{n+1}}{V_{n+2}} r^n + \sum_{n=n_0+1}^{\infty} r^n\, n^{-n/\overline{\theta}_3}.$$

Now for $r > 1$, we have

$$S(r,g) \le A_1 r^{n_0} + \sum_{n=n_0+1}^{\infty} r^n\, n^{-n/\overline{\theta}_3} \quad , \tag{2.8}$$

where A_1 is a positive real constant. We take

$$N(r) = \left[(N+1)r\right]^{\overline{\theta}_3}. \tag{2.9}$$

Now if r is sufficiently large, then from (2.8) and (2.9) we have

$$S(r,g) \le A_1 r^{n_0} + r^{N(r)} \sum_{n_0+1 \le n \le N(r)} n^{-n/\overline{\theta}_3} + \sum_{n > N(r)} r^n\, n^{-n/\overline{\theta}_3}$$

or $$S(r,g) \le A_1 r^{n_0} + r^{N(r)} \sum_{n=1}^{\infty} n^{-n/\overline{\theta}_3} + \sum_{n > N(r)} r^n\, n^{-n/\overline{\theta}_3}. \tag{2.10}$$

Now we have

$$\limsup_{n\to\infty} \left(n^{-n/\overline{\theta}_3}\right)^{1/n} = 0.$$

Hence the first series in (2.10) converges to a positive real number A_2. So from (2.10) we get

$$S(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n > N(r)} r^n \left[(N+1)r\right]^{-n}$$

or $$S(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n > N(r)} \left(\frac{1}{N+1}\right)^n$$

or $$S(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + \sum_{n=1}^{\infty} \left(\frac{1}{N+1}\right)^n. \tag{2.11}$$

Now we have

$$\limsup_{n\to\infty} \left[\left(\frac{1}{N+1}\right)^n\right]^{1/n} = \frac{1}{N+1} < 1.$$

Hence the series in (2.11) converges to a positive real constant A_3. Therefore from (2.11), we get

$$S(r,g) \le A_1 r^{n_0} + A_2 r^{N(r)} + A_3$$

or $$S(r,g) \le A_1 r^{n_0} + A_2 r^{[(N+1)r]^{\overline{\theta}_3}} + A_3$$

or $$S(r,g) \le r^{r^{\overline{\theta}_3 + (N+1)\delta_1}}$$, where $\delta_1 > 0$ is suitably small,

or $$\log\log S(r,g) \le [\overline{\theta}_3 + (N+1)\delta_1]\log r + \log\log r$$

or $$\frac{\log\log S(r,g)}{\log r} \le [\overline{\theta}_3 + (N+1)\delta_1] + \frac{\log\log r}{\log r}\ .$$

Now proceeding to limits as $r \to \infty$, since $\delta_1 > 0$ is arbitrarily small, we get $\rho \le \overline{\theta}_3$.

Finally, since $\varepsilon > 0$ is arbitrarily small, we get

$$\rho \le \theta_3. \tag{2.12}$$

Now let f be a function defined and bounded on K and such that for $s = 1, 2, 3$

$$\theta_s = \limsup_{n\to\infty} \frac{n\log n}{-\log\left\{E_n^s \dfrac{V_{n+1}}{V_{n+2}}\right\}}.$$

So for every $\lambda_1 > \theta_s$ and for sufficiently large n, we have

$$\frac{n\log n}{-\log\left\{E_n^s \dfrac{V_{n+1}}{V_{n+2}}\right\}} \le \lambda_1$$

or $$\left[E_n^s \frac{V_{n+1}}{V_{n+2}}\right]^{1/n} \le n^{-1/\lambda_1}.$$

Proceeding to limits as $n\to\infty$, we get

$$\lim_{n\to\infty}\left[E_n^s \frac{V_{n+1}}{V_{n+2}}\right]^{1/n} \le 0.$$

Also it is obvious that

$$\lim_{n\to\infty}\left[E_n^s \frac{V_{n+1}}{V_{n+2}}\right]^{1/n} \ge 0.$$

Hence finally we get

$$\lim_{n\to\infty}\left[E_n^s \frac{V_{n+1}}{V_{n+2}}\right]^{1/n} = 0.$$

So by Lemma 2.1 we can say that function f can be continuously extended to an entire function g. Let us put

$$g = l_0 + \sum_{n=1}^{\infty}(l_n - l_{n-1}) \ ,$$

where $\{l_n\}$ is the sequence of Lagrange interpolation polynomials of f as defined earlier. Now we claim that g is the required continuation of f and $\rho(g) = \theta_s$.

For every $\lambda_1 > \theta_3$ and for sufficiently large n, we have

$$E_n^3 \frac{V_{n+1}}{V_{n+2}} \le n^{-n/\lambda_1}$$

or $$\| l_n - l_{n-1} \| \le (d_n)^n\, n^{-n/\lambda_1}.$$

So using Lemma 2.2, we get

$$\rho(g) \le \lambda_1.$$

Since $\lambda_1 > \theta_3$ is arbitrary, so finally we get

$$\rho(g) \le \theta_3.$$

Now using (2.5), (2.6) and the proof of first part given above, we have $\rho(g) = \theta_s$, as claimed. This completes the proof of Theorem 2.1.

Next we prove

Theorem 2.2: Let $K \subseteq C^N$ be a compact set with non-zero transfinite diameter such that Φ_K is locally bounded in C^N. Then the function f, defined and bounded on K, is restriction to K of an entire function g of type $\sigma(g)\left(0<\sigma(g)<\infty\right)$ if and only if

$$\sigma(g)=\limsup_{n\to\infty}\frac{n}{e\rho}\left\{E_n^s(f,K)\frac{V_{n+1}}{V_{n+2}}\right\}^{\rho/n} \quad ; \; s=1,2,3,$$

where $0<\rho<\infty$.

Proof: Let g be an entire transcendental function. Write $\sigma=\sigma(g)$ and

$$\eta_s=\limsup_{n\to\infty} n\left\{E_n^s\frac{V_{n+1}}{V_{n+2}}\right\}^{\rho/n} \quad ; \; s=1,2,3.$$

Here E_n^s stands for $E_n^s\left(g\,|_K,K\right)$, $s=1,2,3.$ We claim that $\eta_s=\sigma e\rho$, $s=1,2,3.$ Now as in previous theorem, here we prove that $\eta_1\le\sigma e\rho\le\eta_3$. First we prove that $\eta_1\le\sigma e\rho.$

Using the definition of the type, for $\varepsilon>0$ and $r>r_0(\varepsilon)$, we have

$$S(r,g)\le\exp\left(\overline{\sigma}\, r^\rho\right),$$

where $\overline{\sigma}=\sigma+\varepsilon$, provided r is sufficiently large. Again from [3, p.324], we have

$$E_n^1\frac{V_{n+1}}{V_{n+2}} \;\le\; r^{-n}S(r,g) \;\le\; r^{-n}\exp\left(\overline{\sigma}\, r^\rho\right).$$

Putting $r=\left[\dfrac{n}{\rho\overline{\sigma}}\right]^{1/\rho}$ in the above inequality, we get for all sufficiently large values of n

$$E_n^1\frac{V_{n+1}}{V_{n+2}} \;\le\; \left[\frac{n}{\rho\overline{\sigma}}\right]^{-n/\rho}\exp\left(\frac{n}{\rho}\right)$$

or $$\limsup_{n\to\infty} n\left\{E_n^1\frac{V_{n+1}}{V_{n+2}}\right\}^{\rho/n} \le \overline{\sigma}e\rho.$$

Since $\varepsilon>0$ is arbitrarily small we finally get

$$\eta_1\le\sigma e\rho \quad . \tag{2.13}$$

Now we will prove that $\sigma e\rho \le \eta_3$. If $\eta_3=\infty$, then there is nothing to prove. So let us assume that $0\le\eta_3<\infty$. For a given $\varepsilon>0$, there exists positive integer n_0 such that for all $n>n_0$, we have

$$0\le n\left\{E_n^3\frac{V_{n+1}}{V_{n+2}}\right\}^{\rho/n}\le\eta_3+\varepsilon=\overline{\eta_3}$$

or $$E_n^3\frac{V_{n+1}}{V_{n+2}} \;\le\; \left[\frac{\overline{\eta_3}}{n}\right]^{n/\rho}.$$

Now from the property of maximum modulus, we have

$$S(r,g)\le\sum_{n=0}^{\infty}E_n^3\frac{V_{n+1}}{V_{n+2}}r^n$$

or $$S(r,g) \le \sum_{n=0}^{n_0} E_n^3 \frac{V_{n+1}}{V_{n+2}} r^n + \sum_{n=n_0+1}^{\infty} r^n \left[\frac{\overline{\eta_3}}{n}\right]^{n/\rho}.$$

Now for $r > 1$, we have

$$S(r,g) \le B_1 r^{n_0} + \sum_{n=n_0+1}^{\infty} r^n \left[\frac{\overline{\eta_3}}{n}\right]^{n/\rho}, \tag{2.14}$$

where B_1 is a positive real constant. We take

$$N(r) = \overline{\eta_3}(N+1)^{\rho} r^{\rho}. \tag{2.15}$$

Now if r is sufficiently large, then from (2.14) and (2.15) we have

$$S(r,g) \le B_1 r^{n_0} + \sum_{n_0+1 \le n \le N(r)} \left[\frac{\overline{\eta_3} r^{\rho}}{n}\right]^{n/\rho} + \sum_{n > N(r)} r^n \left[\frac{\overline{\eta_3}}{n}\right]^{n/\rho}. \tag{2.16}$$

Now we choose r such that $\overline{\eta_3}\, r^{\rho} = e\, n$. Then we have

$$\sum_{n_0+1 \le n \le N(r)} \left[\frac{\overline{\eta_3} r^{\rho}}{n}\right]^{n/\rho} \le \overline{\eta_3}(N+1)^{\rho} r^{\rho} \exp\left\{\frac{\overline{\eta_3} r^{\rho}}{e\rho}\right\}.$$

Therefore we have

$$S(r,g) \le B_1 r^{n_0} + \overline{\eta_3}(N+1)^{\rho} r^{\rho} \exp\left\{\frac{\overline{\eta_3} r^{\rho}}{e\rho}\right\} + \sum_{n > N(r)} r^n \left[\frac{\overline{\eta_3}}{\overline{\eta_3}(N+1)^{\rho} r^{\rho}}\right]^{n/\rho}$$

or $$S(r,g) \le B_1 r^{n_0} + \overline{\eta_3}(N+1)^{\rho} r^{\rho} \exp\left\{\frac{\overline{\eta_3} r^{\rho}}{e\rho}\right\} + \sum_{n > N(r)} \left(\frac{1}{N+1}\right)^{n}$$

or $$S(r,g) \le B_1 r^{n_0} + \overline{\eta_3}(N+1)^{\rho} r^{\rho} \exp\left\{\frac{\overline{\eta_3} r^{\rho}}{e\rho}\right\} + \sum_{n=1}^{\infty} \left(\frac{1}{N+1}\right)^{n}. \tag{2.17}$$

Now as in the proof of previous theorem, we can say that series in (2.17) converges to a positive real number B_2. Hence

$$S(r,g) \le B_1 r^{n_0} + \overline{\eta_3}(N+1)^{\rho} r^{\rho} \exp\left\{\frac{\overline{\eta_3} r^{\rho}}{e\rho}\right\} + B_2$$

or $$S(r,g) \le \exp\left\{\frac{(\overline{\eta_3} + \delta_2) r^{\rho}}{e\rho}\right\},$$ where $\delta_2 > 0$ is suitably small,

or $$\frac{\log S(r,g)}{r^{\rho}} \le \frac{(\overline{\eta_3} + \delta_2)}{e\rho}.$$

Now proceeding to limits as $r \to \infty$, since δ_2>0 is arbitrary, we get

$$\sigma \le \frac{\overline{\eta_3}}{e\rho}.$$

Since $\varepsilon > 0$ is arbitrarily small we get

$$\sigma e \rho \le \eta_3. \tag{2.18}$$

Now let f be a function defined and bounded on K and such that for $s = 1, 2, 3$

$$\eta_s = \lim_{n\to\infty}\sup n\left\{E_n^s \frac{V_{n+1}}{V_{n+2}}\right\}^{\rho/n}.$$

Now as in Theorem 2.1 we can easily prove that function f can be continuously extended to an entire function g. Let us put

$$g = l_0 + \sum_{n=1}^{\infty}(l_n - l_{n-1}) \ ,$$

where $\{l_n\}$ is the sequence of Lagrange interpolation polynomials of f as defined earlier. Now we claim that g is required continuation of f and $\rho\ e\ \sigma(g) = \eta_s$. For every $\mu_1 > \eta_3$ and for sufficiently large n, we have

$$E_n^3 \frac{V_{n+1}}{V_{n+2}} \le \left[\frac{\mu_1}{n}\right]^{n/\rho}$$

or $$\| l_n - l_{n-1} \| \le (d_n)^n \left[\frac{\mu_1}{n}\right]^{n/\rho}.$$

So using Lemma 2.3, we get

$$\rho\ e\ \sigma(g) \le \mu_1.$$

Since $\mu_1 > \eta_3$ is arbitrary, so finally we get

$$\rho\ e\ \sigma(g) \le \eta_3.$$

Now using (2.5), (2.6) and the proof of first part given above, we have $\rho\ e\ \sigma(g) = \eta_s$, as claimed. This completes the proof of Theorem 2.2.

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