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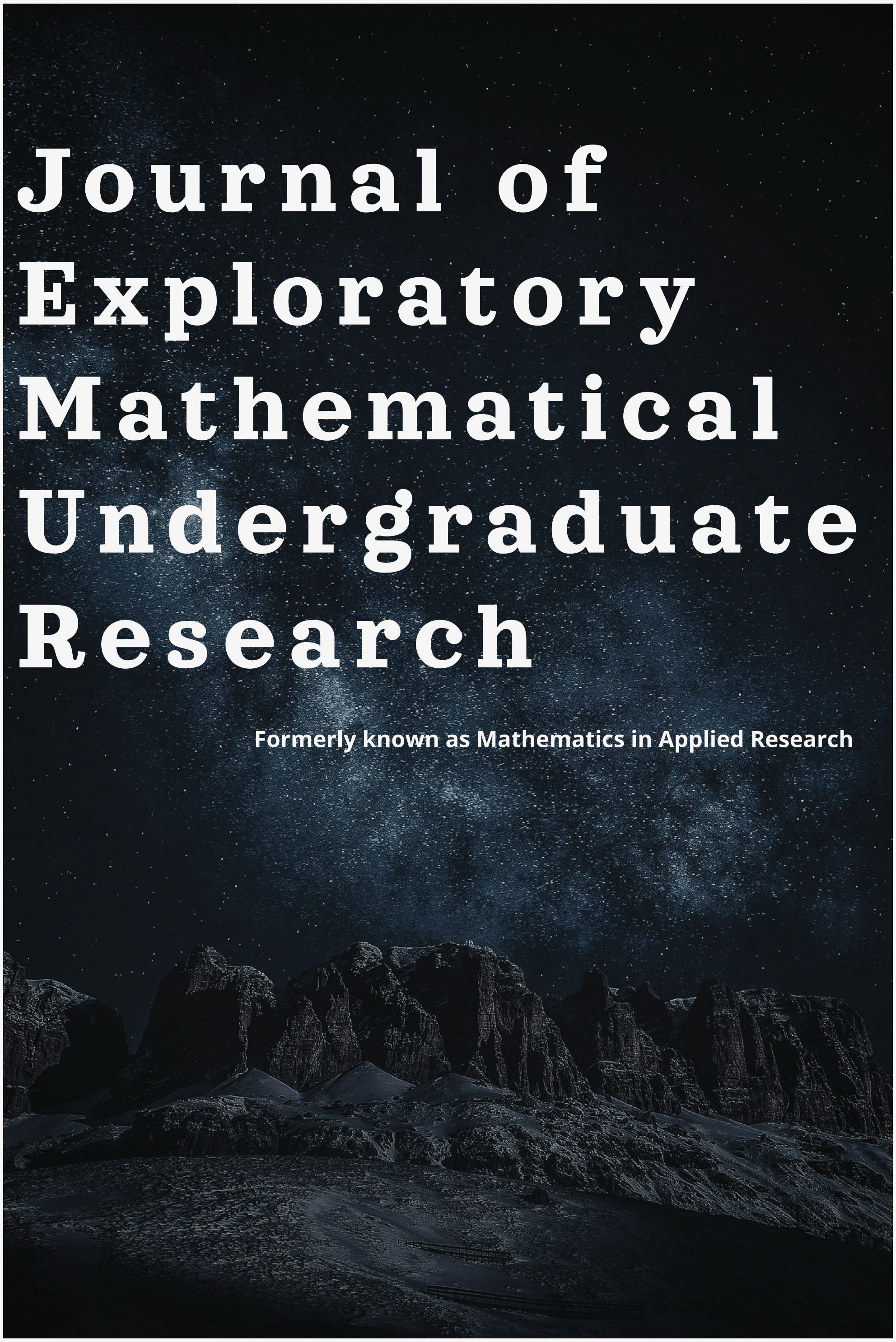

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70300 Seremban, Negeri Sembilan,
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# FEKETE-SZEGÖ INEQUALITY FOR CERTAIN CLASS OF CLOSE-TO-CONVEX FUNCTIONS ON A REAL PARAMETER

**Nonis Airina Mohd Arshad[1], Nuraliah Yasmin Md Nasir[2], Nurzahidah Zamri[3], and Abdullah Yahya[4], ***

[1,2,3,4]College of Computing, Informatics and Mathematics, UiTM Cawangan Negeri Sembilan, Kampus Seremban, Malaysia.
*abdullahyahya@uitm.edu.my

**Abstract**

In the present paper, we examine the upper bounds of the second Fekete-Szegö inequality $\left|a_3 - \mu a_2{}^2\right|$ for the $S(\beta,\delta)$ that is close-to-convex functions in the unit disk using the Toeplitz determinant.

***Keywords:*** Analytic and univalent functions, upper bound, Fekete-Szegö inequality, Toeplitz determinant

## 1. Introduction

The geometric function theory of complex analysis is a fascinating study area, focusing on analytic univalent functions and their geometric properties. However, this field faces significant challenges due to its complexity, involving complex mathematical concepts, proofs, and abstract logic. Researchers often struggle to select suitable methods for mathematical analysis. Coefficient inequalities are a popular topic in this field, allowing researchers to quantify and understand coefficient behavior in function power series expansions. According to Deniz & Orhan (2000), the Second Hankel Determinant problem, $\left|a_2 a_4 - a_3{}^2\right|$ and the Fekete-Szegö problem, $\left|a_3 - \mu a_2{}^2\right|$ are fundamental problems in this field, providing upper bounds for specific coefficient combinations. Thus, this study was conducted to define a new generalized class of close-to-convex functions and determine the upper bound of the Fekete-Szegö inequality, $\left|a_3 - \mu a_2{}^2\right|$.

In this paper, a new generalized class of close-to-convex functions, $S(\beta,\delta)$ defined in the unit disc, $E = \{z \in \mathbb{C} : |z| < 1\}$ which satisfied the condition

$$\operatorname{Re}\left\{e^{i\beta}\frac{zf'(z)}{g(z)}\right\} > \delta,$$

for $z \in D$, $|a| < \pi$, $\cos\beta > \delta, 0 \le \delta < 1$ and $g(z) = \dfrac{2z - z^2}{2}$ was introduced. The method used is based on Toeplitz determinant, Hankel Determinants, Pommerenke's lemma (1975), and the mathematical analysis of analytical proof originated by Rathi (2015). The central finding of this study involves the derivation of an upper bound of the Fekete-Szegö inequality and a new theorem is obtained under Fekete-Szegö determinant for $S(\beta,\delta)$. The findings contribute to a more comprehensive understanding and knowledge of the geometric function theory.

## 2. Preliminaries

Let $A$ be the class of analytic functions of the form

$$f(z) = z + a_2 z^2 + a_3 z^3 + \ldots + a_n z^n + \ldots = z + \sum_{n=2}^{\infty} a_n z^n \tag{2.1}$$

which are analytic in the unit disk, $E = \{z \in \mathbb{C} : |z| < 1\}$ and normalized by $f(0) = f'(0) - 1 = 0$. We also define subclasses of $A$ consisting of functions that are univalent, starlike, convex and close-to-convex denoted by $S$, $St$, $K$, and $C$ respectively.

Suppose that $P$ denoted the class of function of the form

$$p(z) = 1 + p_1 z + p_2 z^2 + \cdots + p_n z^n + \cdots = 1 + \sum_{n=1}^{\infty} p_n z^n \tag{2.2}$$

that are regular in $E$, where $\mathrm{Re}(p(z)) > 0$. All functions of $P$ are called functions with a positive real part of $E$. Based on Pommerenke (1975), if $P \in E$ then

$$|p_n| \le 2 \tag{2.3}$$

and

$$\left| p_2 - \frac{p_1^2}{2} \right| \le 2 - \frac{|p_1^2|}{2} \tag{2.4}$$

According to Janteng et al. (2007) the power series, $p(z)$ as in (2.2) is said to be converges in $E$ to a function in $P$ if and only if the Toeplitz determinants

$$D_n = \begin{bmatrix} 2 & p_1 & p_2 & \cdots & p_n \\ p_{-1} & 2 & p_1 & \cdots & p_{n-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ p_{-n} & p_{-n+1} & p_{-n+2} & \cdots & 2 \end{bmatrix}, n = 1, 2, 3 \ldots$$

and $p_{-4} = \overline{p}_k$, are all non-negative. The equation said to be sternly positive unless for the situation where $p(z) = \sum_{k=1}^{m} \beta_k c_0 (e^{it_k} z), \beta_k > 0$, $t_k$ real and $t_k \ne t_j$ for $k \ne j$; in this case $D_n > 0$ for $n < m - 1$ and $D_n = 0$ for $n \ge m$.

Based on the Noonan & Thomas (1976), the $q^{th}$ determinant for $q \geq 1$ and $n \geq 0$ is

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q+1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q+2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix}.$$

By this definition, the Fekete-Szegö functional is clearly the Hankel determinant for $q = 2$ and $n = 1$, which yields

$$H_2(1) = \begin{vmatrix} a_1 & a_2 \\ a_2 & a_3 \end{vmatrix} = \left| a_1 a_3 - a_2{}^2 \right|.$$

## 3. Main Results

In this research we denote $S(\beta, \delta)$ to be a subclass of analytic univalent function $f$ in the open unit disk, $E = \{z \in \mathbb{C} : |z| < 1\}$, given by (2.1) and satisfy $\operatorname{Re}\left\{ e^{i\beta} \frac{zf'(z)}{g(z)} \right\} > \delta, \quad (z \in D)$ where $|a| < \pi$, $\cos\beta > \delta$ and $0 \leq \delta < 1$. According to Rathi (2015), the Fekete-Szegö problem is defined as an inequality for the coefficient of univalent functions in the unit disc which refers to the upper bound for $\left| a_3 - \mu a_2{}^2 \right|$. There are ten results of the main theorems obtained. The analytical proving has been used to find all theorems.

The first theorem is to concentrate on finding the representation theorem from the function defined before finding the inequality properties for the functions in the class $f \in S(\beta, \delta)$.

**Theorem 3.1**
*Let $f \in S$ be given by the Toeplitz determinant. Then, $S(\beta, \delta)$ if and only if*

$$\frac{e^{i\alpha} \dfrac{zf'(z)}{g(z)} - i \sin\alpha - \delta}{\cos\alpha - \delta} = P,$$

where $z \in E$.

**Theorem 3.2**
*Let $f \in S(\beta, \delta)$ be given by (2.1), then*

$$|a_2| \leq \frac{3}{4} A_{\beta\delta} - \frac{1}{4}$$

*and*

$$|a_3| \leq \frac{1}{3} A_{\beta\delta}$$

*where* $A_{\beta\delta} = \cos\beta - \delta$.

**Theorem 3.3**
*If* $f \in S(\beta,\delta)$ *is given by (2.1), then for* $A_{\beta\delta} > 0$ *have the sharp inequality*

$$\left|a_3 - \frac{2}{3(A_{\beta\delta}+1)}a_2^2\right| \le \frac{1}{3}A_{\beta\delta} - \frac{\left(3A_{\beta\delta}-1\right)^2}{24\left(A_{\beta\delta}+1\right)}.$$

**Theorem 3.4**
*If* $f \in S(\beta,\delta)$ *is given by (2.1), then for* $A_{\beta\delta} > 0$ *have the sharp inequality*

$$\left|a_3 - \mu a_2^2\right| \le \frac{1}{3}A_{\beta\delta} - \mu\left(\frac{3}{4}A_{\beta\delta} - \frac{1}{2}\right)^2$$

*for* $\mu \le \dfrac{2}{3\left(A_{\beta\delta}+1\right)}$.

**Theorem 3.5**
*If* $f \in S(\beta,\delta)$ *is given by (2.1), then for* $A_{\beta\delta} > 0$ *have the sharp inequality*

$$\left|a_3 - \mu a_2^2\right| = \frac{7}{12}A_{\beta\delta} - \frac{1}{12} + \frac{1}{36\mu}$$

*for* $\dfrac{2}{3\left(A_{\beta\delta}+1\right)} \le \mu \le \dfrac{2}{3\left(1-3A_{\beta\delta}\right)}$.

**Theorem 3.6**
*If* $f \in S(\beta,\delta)$ *is given by (2.1), then for* $A_{\beta\delta} > 0$ *have the sharp inequality*

$$\left|a_3 - \mu a_2^2\right| \le \frac{7}{12}A_{\beta\delta} - \frac{1}{24}.$$

**Theorem 3.7**
*If* $f \in S(\beta,\delta)$ *is given by (2.1), then for* $A_{\beta\delta} > 0$ *have the sharp inequality*

$$\left|a_3 - \mu a_2^2\right| \le \frac{7}{12}A_{\beta\delta} - \frac{1}{12} + \frac{1}{36\mu},$$

*for* $\dfrac{2}{3\left(5A_{\beta\delta}+1\right)} \le \mu \le \dfrac{2}{3\left(A_{\beta\delta}+1\right)}$.

**Theorem 3.8**
*If* $f \in S(\beta,\delta)$ *is given by (2.1), then for* $\beta \in [0,1]$ *and* $A_{\beta\delta} > 0$ *have the sharp inequality*

$$\left|a_3 - a_2^2\right| \le \frac{1}{8}A_{\beta\delta}^2 + \frac{5}{6}A_{\beta\delta} + \frac{5}{72}.$$

**Theorem 3.9**
*If $f \in S(\beta,\delta)$ is given by (2.1), then for $\beta \in [0,1]$ and $A_{\beta\delta} > 0$ have the sharp inequality.*

$$\left|a_3 - \mu a_2^{\,2}\right| \le \left(\frac{3}{8}\mu - \frac{1}{4}\right)A_{\beta\delta}^{\;2} + \left(\frac{3}{4}\mu + \frac{1}{12}\right)A_{\beta\delta} + \frac{1}{3}\mu - \frac{19}{72}$$

*for* $\frac{2}{3} \le \mu \le 1.$

**Theorem 3.10**
*If $f \in S(\beta,\delta)$ is given by (2.1), then for $\beta \in [0,1]$ and $A_{\beta\delta} > 0$ have the sharp inequality.*

$$\left|a_3 - \mu a_2^{\,2}\right| \le \frac{11}{16}A_{\beta\delta}^{\;2} + \frac{11}{24}A_{\beta\delta} + \frac{19}{144} - \mu\left(\frac{9}{16}A_{\beta\delta}^{\;2} - \frac{3}{8}A_{\beta\delta} + \frac{1}{16}\right)$$

*for* $\mu \ge 1.$

*Then the results of the upper bound for Fekete-Szegö problem are*

$$\left|a_3 - \mu a_2^{\,2}\right| = \begin{cases} \frac{1}{3}A_{\beta\delta} - \mu\left(\frac{3}{4}A_{\beta\delta} - \frac{1}{4}\right)^2, & 0 \le \mu \le \frac{2}{3\left(A_{\beta\delta}+1\right)} \\ \frac{7}{12}A_{\beta\delta} - \frac{1}{12} + \frac{1}{36\mu}, & \frac{2}{3\left(A_{\beta\delta}+1\right)} \le \mu \le \frac{2}{3\left(1-3A_{\beta\delta}\right)} \\ \left(\frac{3}{8}\mu - \frac{1}{4}\right)A_{\beta\delta}^{\;2} + \left(\frac{3}{4}\mu + \frac{1}{12}\right)A_{\beta\delta} + & \frac{2}{3} \le \mu \le 1 \\ \qquad + \frac{1}{3}\mu - \frac{19}{72}, & \\ \frac{11}{16}A_{\beta\delta}^{\;2} + \frac{11}{24}A_{\beta\delta} + \frac{19}{144} - & \mu \ge 1. \\ \qquad -\mu\left(\frac{9}{16}A_{\beta\delta}^{\;2} - \frac{3}{8}A_{\beta\delta} + \frac{1}{16}\right), & \end{cases}$$

## 4. Conclusion

In conclusion, the classes of functions designated as $S(\beta,\delta)$ have been introduced in this study. The outcome of the Fekete-Szegö inequality was shown through the Fekete-Szegö functional and upper bounds theorem justifications, which were previously demonstrated by Rathi (2015). Additionally, the upper bound of the Fekete-Szegö inequalities using the Hankel Determinant method had been successfully achieved. In this study, the crucial theories on basic geometry and the application of lemma by Pommerenke (1975) also had been fully utilized to acquire the main results. Each of the study's results not only met its goals but also made a significant contribution to the field of analytic univalent functions. As a result, a new Fekete-Szegö inequality was derived based on the class $S(\beta,\delta)$. When everything is considered, the objectives and goals of the paper have been accomplished.

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