Available online at
https://jcrinn.com/
https://crinn.conferencehunter.com/

Journal of Computing Research and Innovation 9(1) 2024

Journal of Computing Research and Innovation

Analysis of the Performance Parameters of Queueing Systems M/M/1 with Pre-Emptive Priority in Transient Regime

Lama Okenge Daniel[1*], Mabela Makengo Matendo Rostin[2]

[1]*Department of Mathematics and Physics, University of Kindu, Kindu, DR Congo*
[2]*Department of Mathematics, Statistics and Computer Science, University of Kinshasa, Kinshasa, DR Congo*

ARTICLE INFO

Article history:
Received: 13 January 2024
Revised: 8 February 2024
Accepted: 9 February 2024
Online first: 1 March 2024
Published 1 March 2024

Keywords:
Performance Measurement
Queuing System
Transient Regime
Absolute Priority
Non-Priority Customers

DOI:
10.24191/jcrinn.v9i1.396

ABSTRACT

In Markov chain theory, performance parameters are indicator of the proper management of a queue. In this field, an abundant literature exists, particularly in the steady state, which is not the case in the transient state. It is in this context that we can question whether it is possible to establish the equations of the performance measures in the transient regime with absolute priority given the complexity of the study of Markov chains in a transient regime. To achieve this, we used the analytical method based on the exploitation of the Laplace transform in the Kolmogorov equations, as well as the theory of convergent series in the equations resulting from the transition matrices. This analysis is supported by the descriptive technique. These tools allowed us to produce concrete results; which are the performance measures of priority and no priority customers in a transient regime M/M/1 queue. Which is a plus in the field of Markov chains. The purpose of this paper is to analyse the M/M/1 transient performance measures with absolute priority. Its originality lies in the fact that we have determined the expressions of the performance measures of non-priority customers in a transient regime. Indeed, very few publications are made in this area at this time. A numerical application was treated to illustrate the theory evoked above. This reflection could soon be carried out in a fuzzy environment.

1. INTRODUCTION

The concept of rational queuing is relevant management is essential today in several areas, among then we can cite: public transportation, service at a counter, reception in a medical office, etc. In all these cases, managers and clients need efficient services, because waiting for long has a cost. Thus, to size a service

[1*] Corresponding author. *E-mail address*: dalokenge@gmail.com
https://doi.org/10.24191/jcrinn.v9i1

center, as Agnès (2012) points out, a necessary condition is that it is capable of absorbing the average arrival and service flow rates of all categories of clients.

Initiated since the beginning of the 20th century, with the work of the Danish engineer AK Erlang on the management of Copenhague's telephone networks, the queuing phenomena were only generalized and developed in other areas after the Second World War with the remarkable contributions of Palm, Chapman, Kolmogorov, Pollaczek. Currently, much work has been carried out on the study of performance parameters with or without priority in steady state, among others (Deepak et al., 2022; Baudouin Adia et al., 2022; Y cart B, 2014), but very rare in transient regime where we can cite Alonge w'Omatete (2021). In this existing literature, nothing has been done on the performance parameters with absolute priority in the transient state of the waiting system.

The main question we want to address is whether it is possible to create the performance parameters of the M/M/1 model with absolute priority in transient conditions. The question being worrying, we believe that there would be scientific methods based in particular on state probabilities, Kolmogorov equations, Laplace transforms, transition matrices, convergent series. which would facilitate the evaluation of measurements performance of the M/M/1 queuing system with absolute priority in transient conditions. Particular emphasis is placed on non-priority customers.

Given that this thought is limited to the calculation of performance parameters in priority transient regime, our approach is structured in two sections: the first relates to generalities on state probabilities, Kolmogorov equations, Laplace transforms and transition matrix. The second, analyze the system performance measurements and the numerical example.

In this thought, we have carried out a mathematical analysis of the performance parameters which are the indicators of the good management of a queue. The case under examination is the determination of performance measures in transient state, with absolute priority. The discipline retained is FIFO, with the consequence that the service of a non-priority customer can only completely result in the absence of a priority customer in the system. The method used is based on the Laplace transform applied to the Kolmogorov equations, state probabilities and transition matrices.

1.1 State probability

Consider a stochastic process with time space T and state space $E = \{X(\Omega),\ t \in T\}$. According to (Mabela et al., 2021), the conditional probability for the system to be in the state $e_k$at time t_k, knowing that it was in the state $e_j$at time $t_j$and in the state $e_i$at time $t_i$is written:

$$p_{e_i e_j e_k}(t_i, t_j, t_k) = \mathbb{P}\left[\, X(t_k) = e_k \mid X(t_j) = e_j, X(t_i) = e_i \,\right] \quad (1)$$

With $e_i, e_j, e_k \in E$, or simply,

$$p_{i,j,k}(t_i, t_j, t_k) = \mathbb{P}\left[\, X(t_k) = k \mid X(t_j) = j, X(t_i) = i \,\right]$$

Let us note by,

$$P(t) = \left(p_{i,j}(t)\right) \text{ où } p_{i,j}(t) = \mathbb{P}([\, X_{t'+t} = j \mid X_{t'} = i \,]) ,$$

So,

https://doi.org/10.24191/jcrinn.v9i1

$$P(t) = e^{Lt} = \sum_{k=0}^{+\infty} \frac{t^k}{k!} L^k \tag{2}$$

1.2 Kolmogorov equations

Or $\vec{\pi}(t) = \vec{\pi}(0)P(t)$, its derivative (Dassa Meriyam, 2019) :

$$\overrightarrow{\pi'}(t) = \vec{\pi}(0)P'(t) = \vec{\pi}(t)L$$

Where L is the infinitesimal generator of the stochastic process defined by,

$$L(i,j) = \begin{cases} p_{i,i} = -\lambda \\ p_{i,i+1} = \lambda \\ p_{i,j} \quad = 0 \end{cases} \tag{3}$$

The life and death processes can be used to model the evolution of a population over time, the number of customers in a queue, in the system, etc. They are continuous Markov processes ($T = \mathbb{R}_+$)with values in $E = \mathbb{N}$such that the only possible non-negligible transitions from $N$are to $N + 1 \;\; ou \; N - 1$.

We will ask $a_{i,i+1} = \lambda_i \;\; et \;\; a_{i,i-1} = \mu_i$where $i \geq 1 \;\; et \;\; a_{0,0} = \lambda_0$represents $\lambda_i$the birth rate from the state $i$and $\mu_i$the death rate from the state i.

The following system is called "Kolmogorov equations",

$$\begin{cases} p'_0(t) = -\lambda p_0(t) + \mu_1 p_1(t) \\ p'_n(t) = \lambda_{n-1}p_{n-1}(t) - (\lambda_n + \mu_n)p_n(t) + \mu_{n+1}p_{n+1}(t) \end{cases} \tag{4}$$

1.3 Laplace transforms

We define a causal function as any function defined on $\mathbb{R}$, zero on $]-\infty, 0[$and continuous piecewise on $[0, +\infty[$ (Alonge w'Omatete, 2021). If $f$is a causal function, the Laplace transform of $f$is defined by (Norbert Verdier et al., 2022):

$$F(p) = \mathcal{L}(f)(p) = \int_0^{+\infty} e^{-pt} f(t)dt = \lim_{x \to +\infty} \int_0^{x} e^{-pt} f(t)dt, \tag{5}$$

for the values of $p$for which this integral converges.

1.4 Properties

- *The Laplace transform is linear:*

$$\mathcal{L}(af + bg) = a\mathcal{L}(f) + b\mathcal{L}(g). \tag{6}$$

- *Effect of a translation:*

either $a > 0$ and $g(t) = f(t-a) \;\; alors \;\; \mathcal{L}(g)(p) = e^{-ap}\mathcal{L}(f)(p)$. (7)

- *Effect of multiplication by an exponential:*

https://doi.org/10.24191/jcrinn.v9i1

If $g(t) = e^{at}f(t),\ \ avec\ a \in \mathbb{R}\ \ alors\ \ \ \mathcal{L}(g)(p) = \mathcal{L}(f)(p-a).$ (8)

- *Derivation.*

Let be fa causal function differentiable on $]0,+\infty[$. So, for everything $p$for which both members have meaning,

$\mathcal{L}(f')(p) = p\mathcal{L}(f)(p) - f(0^+).$ (9)

- *The reciprocal of the Laplace transform*

If $F(p) =\ \mathcal{L}\big(f(t)\big)\quad so\ \ f(t) = \mathcal{L}^{-1}\big(F(p)\big)$ (10)

Consequences

$if\ f(t) = 1\ \ so\ \ F(p) = \frac{1}{p}.$ (11)

$if\ f(t) = \sin(at)\ so\ \ \ F(p) = \frac{a}{p^2+a^2}.$ (12)

$for\ ft) = \cos(at)\ we\ have\ \ \ F(p) = \frac{p}{p^2+a^2}.$ (13)

$for\ f(t) = e^{-\alpha t}\ \ we\ have\ \ \ F(p) = \frac{1}{p+\alpha}.$ (14)

$si\ f(t) = t^n\quad so\quad F(p) = \frac{n!}{p^{n+1}}.$ (15)

Definition. A Markovian queue system is in a transient state, when the performance parameters are a function of time.

1.5 Transition matrix and graph of a CMTD

The advantage of this approach is that we are interested in the state of the system only at particular moments $t_n$in the evolution of the process, by means of state and transition probabilities.

Let us denote by P, the one-step matrix $P^{(1)}$and by$P_{x,y} = P_{x,y}^{(1)}$.

For everything $t, t' \in \mathbb{N}$, we have:

$P(t+t') = P(t).P(t')$

Especially,

$P^{(n+1)} = P^{(n)}.P^{(1)} = P^{(n)}P\ \ and\ \ \ P^{(n)} = P^n.$

The one-step transition probabilities are given by:

$p_{i,j} = \mathbb{P}[X_t = j \mid X_{t-1} = i]\ \forall i,j \in E\ \ such\ as\ \sum_{j\in E} p_{i,j} = 1.$ (16)

The transition probabilities with $n$steps (steps) denoted by $P_{i,j}^{(n)}$are given by:

https://doi.org/10.24191/jcrinn.v9i1

$$p_{i,j}^{(n)} = \mathbb{P}[\, X_{t+t'} = j \mid X_t = i \,]. \tag{17}$$

The transition matrix Pcharacterizing the chain is often useful for creating a graph of the chain, where the states are represented by points and the transition probability $P_{x,y} > 0$by an arc oriented above which the value of is noted $p_{x,y}$.

2. RESULTS

2.1 Performance parameters of priority clients in system M/M/1 transient state

Note that this category of customers is served as in a standard queue (Baudouin Adia et al., 2022).

Starting from the state probabilities described by the Kolmogorov equations, according to (Mabela, M., 2023) and (Ritha. W and Rajeswari, 2021).

For $\lambda_n = \lambda > 0,\ \mu_n = \mu > 0$, system (4) is written:

$$\begin{cases} p'_x(t) = \lambda p_{x-1}(t) - (\lambda + \mu) p_x(t) + \mu p_{x+1}(t)\ ; \quad x \geq 1 \\ p'_0(t) = -\lambda p_0(t) + \mu p_1(t) \end{cases} \tag{18}$$

The Laplace transforms (5) and (9) applied to $p'_x(t)\ and\ p_x(t)$are written:

$$\begin{cases} \mathcal{L}\left(p'_x(t)\right) = s p_x^*(s) - p_x(0) \\ \mathcal{L}\left(p_x(t)\right) = p_s^*(s) \end{cases} \tag{19}$$

Substituting (19) into (18) we obtain:

$$\begin{cases} s p_x^*(s) - p_x(s) = \lambda p_{x-1}^*(s) - (\lambda + \mu) p_x^*(s) + \mu p_{x+1}^*(s) \\ s p_x^*(s) - p_0(0) = -\lambda p_0^*(s) + \mu p_1^*(s) \end{cases} \tag{20}$$

Returning to initial conditions where $p_x(0) = 0, if\ x \neq 0\ and\ p_0(x) = 1, if\ x = 0$, (20) becomes:

$$\begin{cases} s p_x^*(s) = \lambda p_{x-1}^*(s) - (\lambda + \mu) p_x^*(s) + \mu p_{x+1}^*(s) & (21) \\ \ s p_x^*(s) - 1 = -\lambda p_0^*(s) + \mu p_1^*(s) & (22) \end{cases}$$

According to (Y cart B, 2014), the $p_x^*(s)$are solutions of equation (21). By setting $k(s) = \frac{p_{x+i}^*(s)}{p_{x-(i+1)}^*(s)};\quad (i = 0,1)$, we obtain the characteristic equation associated with (21),

$$\mu k^2(s) - (s + \lambda + \mu) k(s) + \lambda = 0 \tag{23}$$

Whose roots are given by:

$$\alpha(s) = \frac{1}{2\mu}\left[s + \lambda + \mu + ((s + \lambda + \mu)^2 - 4\mu\lambda)^{\frac{1}{2}}\right] \tag{24}$$

$$\beta(s) = \frac{1}{2\mu}\left[s + \lambda + \mu - ((s + \lambda + \mu)^2 - 4\mu\lambda)^{\frac{1}{2}}\right] \tag{25}$$

https://doi.org/10.24191/jcrinn.v9i1

The solution to (21) is written according to (Ycart B., 2014):

$$p_x^*(s) = A(s)\big(\alpha(s)\big)^x + B(s)\big(\beta(s)\big)^x \tag{26}$$

With $0 < \beta(s) < 1 < \alpha(s) \quad for \;\; s > 0$,

The root $\alpha(s)$being rejected because $(s) \notin [0,1]$, the linearity of the Laplace transforms leads to the following:

$$\sum_{x=1}^{n} p_x^*(s) = \frac{1}{s},\; (s > 0) \tag{27}$$

The series $\sum_{x=1}^{n} p_x^*(s)$being convergent implies that

$$\sum_{x=1}^{\infty}\left[B(s)\big(\beta(s)\big)^x\right] = \frac{B(s)}{1-\beta(s)} \tag{28}$$

The relations (27) and (28) give:

$$\frac{1}{s} = \frac{B(s)}{1-\beta(s)} \quad \Longrightarrow \qquad B(s) = \frac{1}{s}\big(1-\beta(s)\big) \tag{29}$$

And (26) becomes:

$$p_x^*(s) = \frac{1}{s}\big(1-\beta(s)\big)\big(\beta(s)\big)^x = \frac{1}{s}\Big(\big(\beta(s)\big)^x - \big(\beta(s)\big)^{x+1}\Big) \tag{30}$$

By deriving the relation (25) we obtain:

$$\beta'(s) = \frac{1}{2\mu}\left(1 - \frac{s+\lambda+\mu}{\left((s+\lambda+\mu)^2 - 4\mu\lambda\right)^{\frac{1}{2}}}\right) \tag{31}$$

The relations (25) and (31) at the point $s = 0$give:

$$\beta(0) = \begin{cases} 1 & if\; \lambda \geq \mu \\ \frac{\lambda}{\mu} & if\;\; \lambda < \mu \end{cases} \tag{32}$$

$$\beta'(0) = \begin{cases} -\frac{1}{\lambda-\mu} & if\;\; \lambda \geq \mu \\ \frac{\lambda}{\mu}\left(\frac{1}{\lambda-\mu}\right) & if\;\; \lambda < \mu \end{cases} \tag{33}$$

The case $\lambda < \mu$ interests us more compared to the other, because the opposite leads to explosion.

Developments in entire series of $\big(\beta(s)\big)^x \;\; if \;\; \big(\beta(s)\big)^{x+1}$; according to Mac-Laurin. By putting $f(x) = \big(\beta(s)\big)^x \quad et \quad g(x) = \big(\beta(s)\big)^{x+1}$we obtain:

$$f(x) = \sum_{n=0}^{\infty} x!\left(\frac{\lambda}{\mu}\right)^x \left(\frac{1}{\lambda-\mu}\right)^x \frac{s^n}{n!} = \left(\frac{\lambda}{\mu}\right)^x \left(\frac{\lambda-\mu}{(\lambda-\mu)-s}\right) \;\; ; \qquad \left|\frac{\lambda}{\lambda-\mu}\right| < 1 \tag{34}$$

https://doi.org/10.24191/jcrinn.v9i1

$$g(x) = \sum_{n=0}^{\infty}(x+1)!\left(\frac{\lambda}{\mu}\right)^{x+1}\left(\frac{1}{\lambda-\mu}\right)^{x+1}\frac{s^{n+1}}{(n+1)!} = \left(\frac{\lambda}{\mu}\right)^{x+1}\left(\frac{\lambda-\mu}{(\lambda-\mu)-s}\right) \;;\quad \left|\frac{s}{\lambda-\mu}\right| < 1 \tag{35}$$

Substituting (34) and (35) into (30) we obtain:

$$p_x^*(s) = \frac{1}{s}\left[\left(\frac{\lambda}{\mu}\right)^{x}\left(\frac{\mu-\lambda}{s-(\lambda-\mu)}\right) + \left(\frac{\lambda}{\mu}\right)^{x+1}\left(\frac{\mu-\lambda}{s-(\lambda-\mu)}\right)\right] \tag{36}$$

(8) allows us to write (with ci $.p_x(0) = 0, x \neq 0)$:

$$p_x'(t) = \mathcal{L}^{-1}\big(sp_x^*(s)\big) = \mathcal{L}^{-1}\left[\left(\frac{\lambda}{\mu}\right)^{x}\left(\frac{\mu-\lambda}{s-(\lambda-\mu)}\right) + \left(\frac{\lambda}{\mu}\right)^{x+1}\left(\frac{\mu-\lambda}{s-(\lambda-\mu)}\right)\right] \tag{37}$$

(6) and (14) applied to (37) gives the following:

$$p_x'(t) = -\left(\frac{\lambda}{\mu}\right)^{x}(\lambda-\mu)\, e^{-(\mu-\lambda)t} + \left(\frac{\lambda}{\mu}\right)^{x+1}(\lambda-\mu)e^{-(\lambda-\mu)t} \tag{38}$$

The integration of (38) in $[0, t]$gives:

$$p_x(t) = \left(\frac{\lambda}{\mu}\right)^{x} e^{-(\mu-\lambda)t} - \left(\frac{\lambda}{\mu}\right)^{x+1} e^{-(\mu-\lambda)t} + c \tag{39}$$

$$p_x(0) = 0,\;\; C = \left(\frac{\lambda}{\mu}\right)^{x}\left(1-\frac{\lambda}{\mu}\right) \tag{40}$$

Hence the general solution is

$$p_x(t) = \left(\frac{\lambda}{\mu}\right)^{x}\left(1-\frac{\lambda}{\mu}\right)\left(1-e^{-(\mu-\lambda)t}\right) \tag{41}$$

For $x = 0$,

$$p_0(t) = \mathbb{P}[x(t) = 0] = \left(1-\frac{\lambda}{\mu}\right)\left(1-e^{-(\mu-\lambda)t}\right) \tag{42}$$

Knowing that $\rho_1 = \frac{\lambda}{\mu}$, and posing: $\alpha = \mu - \lambda$; (41) and (42) become:

$$p_x(t) = {\rho_1}^{x}(1-\rho_1)(1-e^{-\alpha t}) \tag{43}$$

$$p_0(t) = \mathbb{P}[x(t) = 0] = (1-\rho_1)(1-e^{-\alpha t}) \tag{44}$$

Therefore, the performance parameters are such as:

- *Server utilization rate at a time $\boldsymbol{t}$.*

$$\mathrm{u}(t) = 1 - p_0(t) = 1 - (1-\rho_1)(1-e^{-\alpha t}) = \rho_1 + (1-\rho_1)e^{-\alpha t}. \tag{45}$$

- *System throughput on a date $\boldsymbol{t}$:*

https://doi.org/10.24191/jcrinn.v9i1

$$d(t) = \left(1 - p_0(t)\right)\mu = \mu\left(1 - (1-\rho_1)(1 - e^{-\alpha t})\right) = \lambda + \alpha e^{-\alpha t}. \tag{46}$$

- *Average number of customers in system at the moment* $\boldsymbol{t}$:

$$\bar{N}_s(t) = \sum_{x=0}^{\infty} x\, p_x(t) = \sum_{x=0}^{\infty} x \rho^x (1-\rho_1)(1-e^{-\alpha t}) = \frac{\rho_1(1-e^{-\alpha t})}{1-\rho_1}, \tag{47}$$

- *Average number of customers in queue on a date* $\boldsymbol{t}$:

$$\bar{N}_f(t) = \sum_{x=0}^{\infty}(x-1)\, p_x(t) = \sum_{x=0}^{\infty} x\, p_x(t) - \sum_{x=1}^{\infty} p_x(t) - p_0(t) = \frac{{\rho_1}^2 - ({\rho_1}^2 - \rho_1 + 1)e^{-\alpha t}}{1-\rho_1} \tag{48}$$

- *Average stay time of a customer in the system during a period*$\boldsymbol{t}$

$$\bar{T}_s(t) = \frac{\bar{N}_s(t)}{d(t)} = \frac{\rho_1(1-e^{-\alpha t})}{(1-\rho_1)(\lambda+\alpha e^{-\alpha t})}. \tag{49}$$

- *Average time a customer waits in line for an instant* $\boldsymbol{t}$**:**

$$\bar{T}_f(t) = \frac{\bar{N}_f(t)}{d(t)} = \frac{{\rho_1}^2 - ({\rho_1}^2 - \rho_1 + 1)e^{-\alpha t}}{(1-\rho_1)(\lambda+\alpha e^{-\alpha t})}. \tag{50}$$

2.2 Non-Priority Customers Performance Settings

Note that for this category of customers, the task is not so easy because their services are strongly disrupted by the arrivals and services of priority customers in the system. Service to a non-priority customer can only result in the total absence of a priority customer in the system (Babu et al., 2020; Yin et al., 2023).

Let $X(t)\ et\ Y(t)$, be the numbers of respectively priority and non-priority customers at the moment $\boldsymbol{t}$. As $\left(X(t), Y(t)\right)$it is a discrete state, the continuous-time Markov chain and its techniques will allow us to describe the limit distribution, when it exists, from a diagram.

Consider $\lambda_1(\theta)\ et\ \lambda_2(\theta)$, the arrival rates of priority and non-priority customers respectively in the system during the period $[0,\theta]$and by $\gamma(\theta)\ et\ \delta(\theta)$, the rates of priority and non-priority customers respectively served during the period $[0,\theta]$.

- The arrival rate in the system is the sum $\lambda = \lambda_1 + \lambda_2$;
- Or $p = \frac{\lambda_1}{\lambda}\ et\ q = \frac{\lambda_2}{\lambda}$ the respective proportions of priority and non-priority customers in the system;
- The average times of services in the system are given by their weighted averages $\frac{1}{\gamma}\ et\ \frac{1}{\delta}$ respectively for priority and non-priority customers;
- The average service rate in the system is given by:

$$\frac{1}{\mu} = p\left(\frac{1}{\gamma}\right) + q\left(\frac{1}{\delta}\right) = \frac{1}{\lambda}\left(\frac{\lambda_1}{\gamma} + \frac{\lambda_2}{\delta}\right) \tag{51}$$

- The traffic intensity in the system is given by:

https://doi.org/10.24191/jcrinn.v9i1

$$\rho = \frac{\lambda}{\mu} = \frac{\lambda_1}{\gamma} + \frac{\lambda_2}{\delta} = \rho_1 + \rho_2. \tag{52}$$

Consider the following transition diagram (Fig. 1):

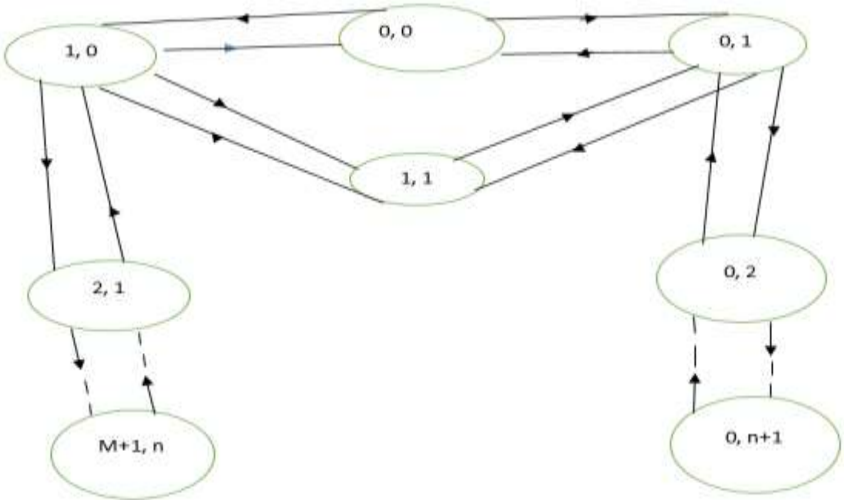

Fig 1. Transition diagram

Description of the transition diagram:

- Arrival of a priority client, the system goes from state $(m, n)$to state $(m + 1, n)$with a transition rate λ_1.
- The arrival of a non-priority customer, the system goes from state (m, n)) to the state $(m, n + 1)$with a transition rate λ_2.
- The service of a non-priority customer is over; the system transitions from state $(0, n)$to state $(0, n - 1)$, $(n \geq 1)$, with a transition rate δ.
- The service of a priority customer is completed; the system transitions from state $(m, n)$to state $(m - 1, n)$with a transition rate γ.

From this description follow the following balance equations:

$$(\lambda_1 + \lambda_2)p_{(0,0)}(t) = \gamma p_{1,0}(t) + \delta p_{0,1}(t)(1 - e^{-\alpha t}) \tag{53}$$

$$(\lambda_1 + \lambda_2 + \gamma)p_{m,0}(t) = \gamma p_{m+1,0}(t) + \lambda_1 p_{m-1,0}(t)(1 - e^{-\alpha t}) \tag{54}$$

$$(\lambda_1 + \lambda_2 + \delta)p_{0,n}(t) = \gamma p_{1,n}(t) + \left(\delta p_{0,n+1}(t) + \lambda_2 p_{0,n-1}(t)\right)(1 - e^{-\alpha t}), \tag{55}$$

$$(\lambda_1 + \lambda_2 + \delta)p_{0,n}(t) = \gamma p_{m+1,n}(t) + \lambda_1 p_{m-1,n}(t) + \lambda_2 p_{m,n-1}(t)(1 - e^{-\alpha t}) \tag{56}$$

https://doi.org/10.24191/jcrinn.v9i1

Equation (43) applied to $X(t)$and $Y(t)$gives us the following expressions:

$$p_m(t) = \mathbb{P}([X(t) = m]) = \sum_{n=0}^{\infty} p_{m,n}(t) = (1 - \rho_1)\,\rho_1^m(1 + e^{-\alpha t}) \tag{57}$$

$$p_n(t) = \mathbb{P}([X(t) = n]) = \sum_{m=0}^{\infty} p_{m,n}(t) = (1 - \rho_1)\,\rho_2^n(1 + e^{-\alpha t}) \tag{58}$$

By summing and adding member to member on m, equations (53) and (54) we obtain:

$$(\lambda_1 + \lambda_2)p_{0,0}(t) + \gamma \sum_{m=1}^{\infty} p_{0,0}(t) = \gamma \sum_{m=1}^{\infty} p_{m,0}(t) + \lambda_1 p_{0,0}(t) + \delta p_{0,1}(t)(1 - e^{-\alpha t})$$

Let,

$$\lambda_2 p_{0,0}(t) = \delta p_{0,1}(t)(1 - e^{-\alpha t}) \qquad \forall t \geq 0 \tag{59}$$

The member-to-member sum over m(55) and (56) gives us:

$$(\lambda_1 + \lambda_2 + \delta)p_{0,n}(t) + (\lambda_1 + \lambda_2 + \gamma) \sum_{m=0}^{\infty} p_{m,n}(t) = \gamma \sum_{m=1}^{\infty} p_{1,n}(t) + \Big(\delta p_{0,n+1}(t) + \lambda_2 p_{0,n-1}(t)\Big)(1 - e^{-\alpha t}) + \gamma \sum_{m=0}^{\infty} p_{m+1}(t) + \lambda_1 \sum_{m=0}^{\infty} p_{m-1,n}(t) + \lambda_2 \sum_{m=0}^{\infty} p_{m,n-1}(t)\,(1 - e^{-\alpha t})$$

$$\lambda_1 \sum_{m=0}^{\infty} p_{m,n}(t) + \lambda_2 \sum_{m=0}^{\infty} p_{m,n}(t) + \delta p_{0,n}(t) + \gamma \sum_{m=0}^{\infty} p_{m,n}(t) = \gamma \sum_{m=1}^{\infty} p_{1,n}(t) + \Big(\delta p_{0,n+1}(t) + \lambda_2 p_{0,n-1}(t)\Big)(1 - e^{-\alpha t}) + \lambda_1 \sum_{m=0}^{\infty} p_{m-1,n}(t)$$

$$\lambda_2 \sum_{m=0}^{\infty} p_{m,n}(t) + \delta p_{0,n}(t) = \Big(\delta p_{0,n+1}(t) + \lambda_2 p_{0,n-1}(t)\Big)(1 - e^{-\alpha t}), \tag{60}$$

By induction on (59), (60) becomes:

$$\lambda_2 \sum_{m=0}^{\infty} p_{m,n}(t) = \delta p_{0,n+1}(t)(1 - e^{-\alpha t}) \tag{61}$$

By adding (61) member to member over n and the fact that $\sum_{m=0}^{\infty} p_{m,n}(t) = 1$, we obtain:

$$\lambda_2 = \delta p_{0,n+1}(t)(1 - e^{-\alpha t}) \tag{62}$$

Firstly,

$$\mathbb{P}[X(t) = 0, Y(t) > 0] = \sum_{n=1}^{\infty} p_{0,n}(t) = \frac{\lambda_2}{\delta}(1 - e^{-\alpha t}) = \rho_2(1 - e^{-\alpha t}), \tag{63}$$

$$\mathbb{P}[X(t) = 0] = \left(1 - \frac{\lambda_1}{\gamma}\right)(1 - e^{-\alpha t}) = (1 - \rho_1)(1 - e^{-\alpha t}) \tag{64}$$

On the other hand,

$$p_{0,0,}(t) = \mathbb{P}[X(t) = 0, Y(t) = 0] = \mathbb{P}[X(t) = 0] - \mathbb{P}[X(t) = 0, Y(t) > 0] = (1 - \rho_1)(1 - e^{-\alpha t}) - \rho_2(1 - e^{-\alpha t}) = (1 - \rho)(1 - e^{-\alpha t}) \tag{65}$$

These results above allow us to calculate the performance parameters of non-priority customers.

https://doi.org/10.24191/jcrinn.v9i1

1. *Server utilization rate at a time* $\boldsymbol{t}$,

$$\mathrm{u}(t) = 1 - p_{0,0}(t) = 1 - (1-\rho)(1-e^{-\alpha t}) = \rho + (1-\rho)e^{-\alpha t}\ , \quad (66)$$

2. *System throughput on a date* $\boldsymbol{t}$:

$$d(t) = \left(1 - p_{0,0}(t)\right)\mu = (\rho + (1-\rho)e^{-\alpha t})\mu = \lambda + \alpha e^{-\alpha t}, \quad (67)$$

3. **Theorem**. *The average number of non-priority customers in the system at the moment* $\boldsymbol{t}$ is given by the expression $\overline{N}_{s_2}(t) = \frac{\rho_2}{1-\rho}\left(1 + \frac{\delta}{\gamma}\frac{\rho_1}{1-\rho_1}\right)(1-e^{-\alpha t})$

Proof

In the equilibrium state, this number is given by:

$$\overline{N}_{s_2} = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} np_{m,n}(t) \quad (68)$$

Let us note by

$$G_m = \sum_{n=0}^{\infty} np_{m,n}(t) = \sum_{n=1}^{\infty} np_{m,n}(t) \quad (69)$$

And so (68) becomes:

$$\overline{N}_{s_2} = \sum_{m=0}^{\infty} G_m = G_0 + G_1 + \cdots \quad (70)$$

By multiplying equation (55) by $n$and summing over n, both sides of the equation, we obtain:

$$(\lambda_1 + \lambda_2 + \delta)\sum_{n=1}^{\infty} n\, p_{0,n}(t) = \gamma\sum_{n=1}^{\infty} np_{1,n}(t) + \left(\delta\sum_{n=1}^{\infty} np_{0,n+1}(t) + \lambda_2\sum_{n=1}^{\infty} np_{0,n-1}(t)\right)(1-e^{-\alpha t})$$

$$(\lambda_1 + \lambda_2 + \delta)G_0 = \gamma G_1 + \left(\delta G_0 - \delta\sum_{n=0}^{\infty} np_{0,n+1}(t) + \lambda_2 G_0 + \lambda_2\sum_{n=0}^{\infty} np_{0,n-1}(t)\right)(1-e^{-\alpha t})$$

$$(\lambda_1 + \lambda_2 + \delta)G_0 = \gamma G_1 + \delta G_0 - \delta\left(\frac{\lambda_2}{\delta}\right)(1-e^{-\alpha t}) + \lambda_2 G_0 + \lambda_2(1-\rho_1)(1-e^{-\alpha t}).$$

$$\lambda_1 G_0 = \gamma G_1 - \lambda_2(1-e^{-\alpha t}) + \lambda_2(1-e^{-\alpha t}) - \lambda_2\rho_1(1-e^{-\alpha t})$$

Eventually,

$$G_1 = \rho_1 G_0 + \frac{\lambda_2}{\gamma}\rho_1(1-e^{-\alpha t}) \quad (71)$$

Multiplying (56) by n and summing to $n = 1,2,..$give:

$$(\lambda_1 + \lambda_2 + \gamma)\sum_{n=1}^{\infty} n\, p_{m,n}(t) = \gamma\sum_{n=1}^{\infty} np_{m+1,n}(t) + \lambda_1\sum_{n=1}^{\infty} np_{m-1,n}(t) + \left(\lambda_2\sum_{n=1}^{\infty} np_{m,n-1}(t)\right)(1 - e^{-\alpha t})\ ,$$

Let,

https://doi.org/10.24191/jcrinn.v9i1

$$(\lambda_1+\lambda_2+\gamma)G_m = \gamma G_{m+1}+\lambda_1 G_{m-1}+\lambda_2\sum_{n=1}^{\infty} np_{m,n-1}(t)\,(1-e^{-\alpha t}), \tag{72}$$

From (57) we know that $\sum_{n=0}^{\infty} p_{m,n}(t) = (1-\rho_1)\,\rho_1^m(1+e^{-\alpha t})$, in (72) we obtain:

$$(\lambda_1+\gamma)G_m = \gamma G_{m+1}+\lambda_1 G_{m-1}+\lambda_2(1-\rho_1)\rho_1^m(1-e^{-\alpha t})^2 \tag{73}$$

For $m=1$, we have:

$$(\lambda_1+\gamma)G_1 = \gamma G_2+\lambda_1 G_0+\lambda_2(1-\rho_1)\rho_1(1-e^{-\alpha t})^2 \tag{74}$$

Substituting (70) into (74) we obtain:

$$(\lambda_1+\gamma)\left(\rho_1 G_0+\frac{\lambda_2}{\gamma}\rho_1(1-e^{-\alpha t})\right) = \gamma G_2+\lambda_1 G_0+\lambda_2(1-\rho_1)\rho_1(1-e^{-\alpha t})^2$$

$$\lambda_1\rho_1 G_0+\lambda_2\rho_1^2(1-e^{-\alpha t})+\gamma\rho_1 G_0+\lambda_2\rho_1(1-e^{-\alpha t}) = \gamma G_2+\lambda_1 G_0+\lambda_2\rho_1(1-e^{-\alpha t})-\lambda_2\rho_1^2(1-e^{-\alpha t})$$

$$\gamma G_2 = \lambda_1\rho_1 G_0+2\lambda_2\rho_1^2(1-e^{-\alpha t})+\gamma\rho_1 G_0-\lambda_1 G_0$$

From where

$$G_2 = \rho_1^2\left(G_0+2\frac{\lambda_2}{\gamma}(1-e^{-\alpha t})\right) \tag{75}$$

For $m=2$;

in (73) and by similar reasoning, we obtain:

$$G_3 = \rho_1^3\left(G_0+3\frac{\lambda_2}{\gamma}(1-e^{-\alpha t})\right) \tag{76}$$

In general

$$G_m = \rho_1^m\left(G_0+m\frac{\lambda_2}{\gamma}(1-e^{-\alpha t})\right) \tag{77}$$

By summing over $m$equation (76), we obtain:

$$\sum_{m=0}^{\infty} G_m = \overline{N}_{s_2} = G_0\sum_{m=0}^{\infty}\rho_1^m+\frac{\lambda_2}{\gamma}(1-e^{-\alpha t})\sum_{m=0}^{\infty} m\rho_1^m \tag{78}$$

Knowing that the geometric series $\sum_{m=0}^{\infty}\rho_1^m = \frac{1}{1-\rho_1}, (\rho_1<1)$and that the series $\sum_{m=0}^{\infty} m\rho_1^m = \sum_{m=1}^{\infty} m\rho_1^m = \frac{\rho_1}{(1-\rho_1)^2}$, therefore equation (78) becomes:

$$\overline{N}_{s_2} = G_0\frac{1}{1-\rho_1}+\frac{\lambda_2}{\gamma}\frac{\rho_1}{(1-\rho_1)^2}(1-e^{-\alpha t}) = \frac{1}{1-\rho_1}\left(G_0+\frac{\lambda_2}{\gamma}\frac{\rho_1}{(1-\rho_1)}(1-e^{-\alpha t})\right), \tag{79}$$

We observe that $\overline{N}_{s_2}$is a function of G_0. To do this, let us multiply equation (61) by n and sum to n, we obtain:

$$\lambda_2\sum_{m=0}^{\infty}\sum_{n=1}^{\infty} np_{m,n}(t) = \delta\sum_{n=1}^{\infty} np_{0,n+1}(t)(1-e^{-i\omega t}) \tag{80}$$

https://doi.org/10.24191/jcrinn.v9i1

$$\lambda_2 \sum_{m=0}^{\infty} G_m = \delta \sum_{n=0}^{\infty} n\, p_{0,n}(t) - \delta \sum_{n=0}^{\infty} np_{0,n+1}\,(t)(1-e^{-i\omega t}) \tag{81}$$

$$\lambda_2 \bar{N}_{s_2} = \delta G_0 - \delta \frac{\lambda_2}{\delta}(1-e^{-\alpha t}) = \delta G_0 - \lambda_2(1-e^{-\alpha t})$$

And so

$$G_0 = \rho_2\left(\bar{N}_{s_2} + (1-e^{-\alpha t})\right) \tag{82}$$

Substituting (82) into (79), we obtain:

$$\bar{N}_{s_2} = \frac{1}{1-\rho_1}\left(\rho_2\left(\bar{N}_{s_2} + (1-e^{-\alpha t})\right) + \frac{\lambda_2}{\gamma}\frac{\rho_1}{(1-\rho_1)}(1-e^{-\alpha t})\right) \tag{83}$$

$$\bar{N}_{s_2} - \frac{\rho_2}{1-\rho_1}\bar{N}_{s_2} = \frac{1}{1-\rho_2}\left(\rho_2 + \frac{\lambda_2}{\gamma}\frac{\rho_1}{1-\rho_1}\right)(1-e^{-\alpha t})$$

$$\bar{N}_{s_2}\left(1 - \frac{\rho_2}{1-\rho_1}\right) = \frac{1}{1-\rho_2}\left(\rho_2 + \frac{\lambda_2}{\gamma}\frac{\rho_1}{1-\rho_1}\right)(1-e^{-\alpha t})$$

$$\bar{N}_{s_2} = \frac{1}{1-\rho}\left(\rho_2 + \frac{\lambda_2}{\gamma}\frac{\rho_1}{1-\rho_1}\right)(1-e^{-\alpha t})$$

Eventually:

$$\bar{N}_{s_2}(t) = \frac{\rho_2}{1-\rho}\left(1 + \frac{\delta}{\gamma}\frac{\rho_1}{1-\rho_1}\right)(1-e^{-\alpha t}) \quad avec\ (\rho = \rho_1 + \rho_2 < 1) \tag{84}$$

LIttle's theorem we deduce the following parameters:

4. *The average time of a non-priority customer in the system during a period* $\boldsymbol{t}$

$$\bar{T}_{s_2}(t) = \frac{\bar{N}_{s_2}}{d(t)} = \frac{\rho_2}{1-\rho}\left(1 + \frac{\delta}{\gamma}\frac{\rho_1}{1-\rho_1}\right)(1-e^{-\alpha t})\frac{1}{\lambda+\alpha e^{-\alpha t}} \tag{85}$$

5. *The average time of a non-priority customer in queue for a period* $\boldsymbol{t}$ **:**

$$\bar{T}_{f_2}(t) = \bar{T}_{s_2}(t) - \frac{1}{\mu} = \frac{\rho_2}{1-\rho}\left(1 + \frac{\delta}{\gamma}\frac{\rho_1}{1-\rho_1}\right)(1-e^{-\alpha t})\frac{1}{\lambda+\alpha e^{-\alpha t}} - \frac{\rho}{\lambda}.$$

$$= \frac{\rho_2\left(1+\frac{\delta}{\gamma}\frac{\rho_1}{1-\rho_1}\right)(1-e^{-\alpha t}) - (1-\rho)\rho}{(1-\rho)\lambda} \tag{86}$$

6. *The average number of non-priority customers in the queue at moment* t.

$$\bar{N}_{f_2} = \lambda \bar{T}_{s_2}(t) = \frac{\rho_2\left(1+\frac{\delta}{\gamma}\frac{\rho_1}{1-\rho_1}\right)(1-e^{-\alpha t}) - (1-\rho)\rho}{(1-\rho)} \tag{87}$$

https://doi.org/10.24191/jcrinn.v9i1

Numerical example

With a commercial bank in Kinshasa (DRC), two categories of customers are served at the single counter of its branch located in Gombe. The priority category called large customers, that is to say having a figure of at least 10 million FC in their account, arrives for operations at the bank following a Poisson rate process $\lambda_1 = 0{,}65$ and the non-priority category called ordinary customers, having less than a million in their respective accounts, arrives to be served according to the same process with a rate $\lambda_2 = 0{,}23$; services are exponential in rate $\mu = 0{,}94$. Within a 60-minute interval, determine the performance parameters for each category of customers:

$$\bar{N}_{s_1}(t) = \frac{\rho_1(1-e^{-\alpha t})}{1-\rho_1} = \frac{0{,}69\left(1-e^{-(0{,}94-0{,}65)60}\right)}{1-0{,}69} \simeq 2$$

$$\bar{T}_{s_1}(t) = \frac{\rho_1(1-e^{-\alpha t})}{(1-\rho_1)(\lambda+\alpha e^{-\alpha t})} = \frac{0{,}69\left(1-e^{-(0{,}94-0{,}65)60}\right)}{(1-0{,}69)\left(0{,}65+(0{,}94-0{,}65)e^{-(0{,}94-0{,}065)60}\right)} = 3{,}45$$

$$\bar{N}_{s_2}(t) = \frac{\rho_2}{1-\rho}\left(1+\frac{\delta}{\gamma}\frac{\rho_1}{1-\rho_1}\right)(1-e^{-\alpha t}) = \frac{0{,}24}{1-0{,}93}\left(1+\frac{0{,}65}{0{,}23}\frac{0{,}69}{1-0{,}69}\right)\left(1-\frac{1}{e^{(0{,}94-0{,}88)60}}\right) = 11$$

$$\bar{T}_{s_2}(t) = \frac{\rho_2}{1-\rho}\left(1+\frac{\delta}{\gamma}\frac{\rho_1}{1-\rho_1}\right)(1-e^{-\alpha t})\frac{1}{\lambda+\alpha e^{-\alpha t}} = \frac{0{,}244}{1-0{,}93}\left(1+\frac{0{,}69}{1-0{,}69}\right)\left(1-e^{-(0{,}94-0{,}88)60}\right)\frac{1}{0{,}88+(0{,}94-0{,}88)e^{-(0{,}94-0{,}88)60}} = 13$$

$$\bar{N}_{f1}(t) = \frac{{\rho_1}^2-({\rho_1}^2-\rho_1+1)e^{-\alpha t}}{1-\rho_1} = \frac{(0{,}69)^2-\left((0{,}69)^2-0{,}69+1\right)e^{-(0{,}94-0{,}65)60}}{1-0{,}69} = 1$$

$$\bar{T}_{f1}(t) = \frac{{\rho_1}^2-({\rho_1}^2-\rho_1+1)e^{-\alpha t}}{(1-\rho_1)(\lambda+\alpha e^{-\alpha t})} = \frac{(0{,}69)^2-\left((0{,}69)^2-0{,}69+1\right)e^{-(0{,}94-0{,}65)60}}{(1-0{,}69)\left(0{,}65+(0{,}94-0{,}65)e^{-(0{,}94-0{,}65)60}\right)} = 0{,}42$$

$$\bar{N}_{f_2} = \frac{\rho_2\left(1+\frac{\delta}{\gamma}\frac{\rho_1}{(1-\rho_1)}\right)(1-e^{-\alpha t})-(1-\rho)\rho}{(1-\rho)\lambda} = \frac{0{,}24\left(1+\frac{0{,}69}{1-0{,}69}\right)\left(1-e^{-(0{,}94-0{,}88)60}\right)-(1-0{,}93)0{,}93}{(1-0{,}93)0{,}88} = \frac{0{,}687838}{0{,}07} = 11$$

$$\bar{T}_{f_2}(t) = \frac{\rho_2\left(1+\frac{\delta}{\gamma}\frac{\rho_1}{(1-\rho_1)}\right)(1-e^{-\alpha t})-(1-\rho)\rho}{(1-\rho)\lambda} = \frac{0{,}24\left(1+\frac{0{,}69}{1-0{,}69}\right)\left(1-e^{-(0{,}94-0{,}88)60}\right)-(1-0{,}93)0{,}93}{(1-0{,}93)0{,}88} = 8{,}64$$

https://doi.org/10.24191/jcrinn.v9i1

Graphical representation of performance parameters a function of time

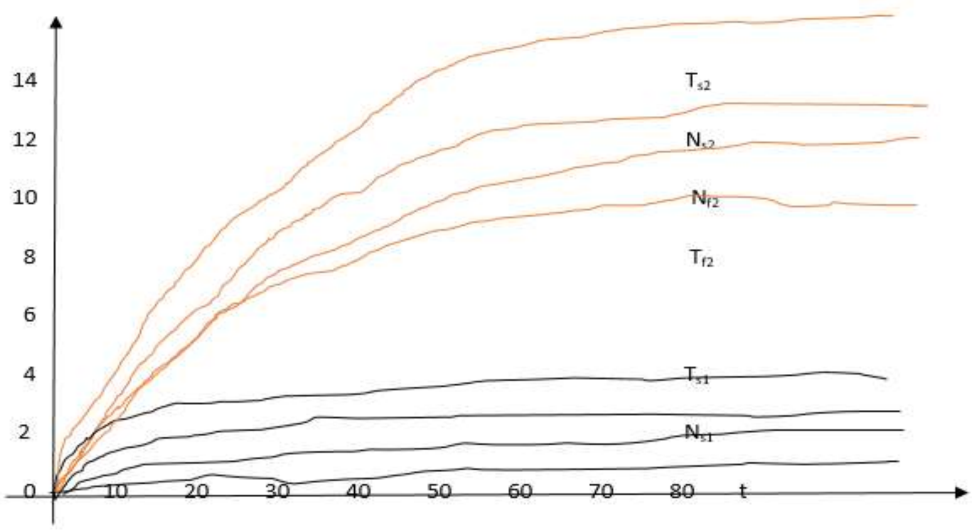

Fig.2. Performance parameters a function of time

3. DISCUSSION

The results obtained through this numerical example reflect the differences in the time spent by priority and non-priority customers in the queue. The same is true of their numbers in the system in a given time interval t. In contrast to the results expressed by (Mabela and et al., 2021) in the transient regime without priority and Lama steady-state with top priority, this model has the advantage of reducing waiting time for priority customers.

4. CONCLUSION

At the end of this thought on the performance measurements of a queuing system with absolute priority in a *transient state*, model M/M/1, the main question was to determine the performance parameters of non-priority customers in a transient state. To achieve this, we have relied on Kolmogorov equations, state probabilities, Laplace transforms, transition matrices and Little's theorem. In the existing literature, nothing similar has been established in a transitional regime with absolute priority. This is what marks the originality and substantial contribution of this reflection in the field of queues. The same reflection will be carried out in a fuzzy environment in the long run.

5. REFERENCES

Agnès Lagnoux et Claudie Chabriac. (2012). *Processus stochastiques et modélisation*. Université De Toulouse De Mirial, Master 2.

Alonge w'Omatete. (2021). Evaluation of the performance parameters of the Markovian queueing system M/M/1 in transient state by the Laplace transform method. *Annals of the Faculty of Science and Technology*, UNIKIN, vol 1, pp 115-125.

https://doi.org/10.24191/jcrinn.v9i1

Babu, D., Joshua, V. C., & Krishnanoorthy. (2020). A queueing system with probabilistic joining strategy for priority customers. *International Conference on Information Technologies and Mathematical Modelling* (pp. 37-42). Springer.

Baudouin Adia, L. M., Rostine Mabela, M. M., Jean Pierre, M. K., & Bopatriciat, B. M. (2022). Analysis of the performance measures of a non-Markovian Fuzzy Queue via Fuzzy Laplace Transforms method. *Journal of Computing Research and Innovation (JCRINN)*, *7*(2), 304-315. https://doi.org/10.24191/jcrinn.v7i2.323

Dassa Meriyam (2019). *Equation de Kolmogorov et EDP, Mémoire de Master en Mathématiques*. Université Mohamed Khider Biskra.

Deepak, G., Aarti, S., & Tripathi. (2022). Mathematical analysis of priority Bi-serial Queue Network Model. *Mathematics and Statistics, 10*(5): 981-987.

Lama O., D. (2021). *Analyse des paramètres de performance de systèmes d'attente avec priorité absolue et applications*. Mémoire de DEA Unikin, RDC.

Mabela, M et Alonge W. (2019). *Évaluation des paramètres de performance du système d'attente Markovien M/M/1 en régime transitoire par la méthode de transformée de Laplace*. In Annales De La Faculté des Sciences (pp. 115-125).

Norbert Verdier, Aurélien Gautreau et Pascal Raini (2022). La transformation de Laplace élémentaire dans les formations mathématiques pour l'ingénieur et le technicien. Du calcul intégral à « l'algébrisation des équations différentielles et la production des tables ». *Cahier Francois Viète*, III.13, 187-221.

Ritha, W. & Rajeswari. (2021). An analytical study of the M/M/1/N imprecise queuing system encouraged arrivals. *Int. J. of Aquatic Science*, *12*(2), 1349-1359.

Rostin, M. M. M., Daniel, L. O., Claude, M. M., & Bopatriciat, B. M. (2023). Computing the performance parameters of Fuzzy Markovian Queueing System FM/FM/1 in transient regime by Flexible AlphaCuts Method. *Journal of Computing Research and Innovation (JCRINN)*, *8*(1), 17-34.

Y cart B. (2014). Files d'attente. In *Cahier De Mathématiques Appliquées* (pp. 69-131).

Yin, M., Yan, M., Guo, Y., & Liu, M. (2023). Analysis of a pre-emptive two-priority queueing system with impatient customers and heterogeneous servers. *Mathematics*, *11*(18), 3878. https://doi.org/10.3390/math11183878