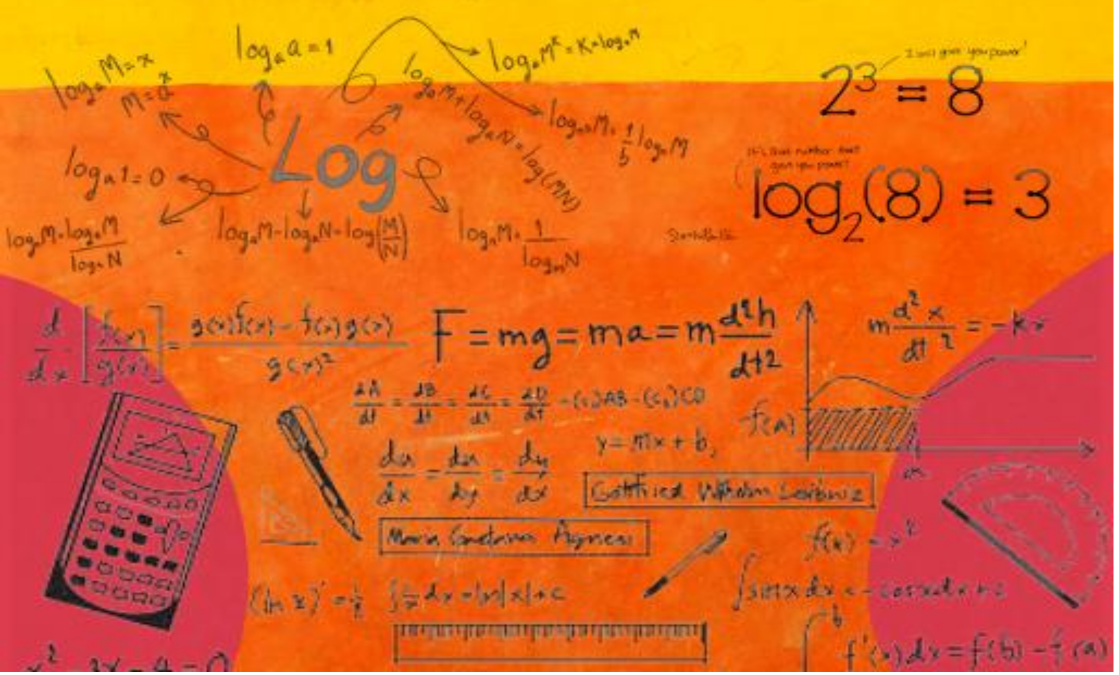
LOGARITHMIC AND EXPONENTIAL FUNCTIONS

AISHAH MAHAT



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PREFACE

This e-book, Logarithmic and Exponential Function aimed to help students in mathematic subject. Targeted users for this module is students who take foundation course. This e-book is divided into two topics which Logarithmic Function and Exponential Function. Mathematical tips and formulas will be placed in accordance to the subtopics whilst each questions will be displayed based on the syllabus carried out during the lesson. At the end of each topic, targeted students should meet up with the lecturer to discuss over the solution of mathematics problem. With the existence of this e-book, hopefully it will be beneficial and give positive impact towards teaching and learning for students and lecturers as a whole.

EXPONENTIAL FUNCTION

Definition of Exponential Function:

The exponential function f with base a is denoted by $f(x) = a^x$ where $a > 0$, $a \neq 1$, and x is any real number.

Evaluate each function below.

Function	Value	Answer
$f(x) = 2^x$	$x = -4.1$	0.058314561
$f(x) = 3^{-x}$	$x = \pi$	0.031701467
$f(x) = 0.5^x$	$x = \frac{5}{2}$	0.176776695
$g(x) = \frac{1}{2} . \frac{1}{4}^x$	$x = 2$	$\frac{1}{32}$
$f(m) = 5 . 2^m$	$\mathrm{m} = -4$	$\frac{5}{16}$
$f(m) = 8 . 3^m$	$\mathrm{m} = 3$	216

Properties of Exponents

$$x^a \ . \ x^b = x^{a+b}$$

$$\frac{x^a}{x^b} = x^{a-b}$$

$$(x^a)^b = x^{ab}$$

$$x^{-a} = \frac{1}{x^a}$$

$$x^0 = 1$$

$$a^{\frac{x}{y}} = \sqrt[y]{a^x} = \left(\sqrt[y]{a}\right)^x$$

Properties of Exponents

Rules	Examples
$x^a \,.\, x^b = x^{a+b}$	$a^3 \times a^4 = a^{3+4} = a^7$
$\frac{x^a}{x^b} = x^{a-b}$	$a^8 \div a^2 = a^{8-2} = a^6$
$(x^a)^b = x^{ab}$	$(a^2)^3 = a^{2\times3} = a^6$
$x^{-a} = \frac{1}{x^a}$	$a^{-6} = \frac{1}{a^6}$
$x^0 = 1$	$a^0 = 1$
$a^{\frac{x}{y}} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x$	$a^{\frac{4}{5}} = \sqrt[5]{a^4} = (\sqrt[5]{a})^4$

Graph of Exponential Function

Sketch the graph of each function in the same coordinate plane.

a) $f(x) = 3^x$

b) $g(x) = 5^x$

Solution:

x	-3	-2	-1	0	1	2
3^x	$\frac{1}{27}$	$\frac{1}{9}$	$\frac{1}{3}$	1	3	9
5^x	$\frac{1}{125}$	$\frac{1}{25}$	$\frac{1}{5}$	1	5	25

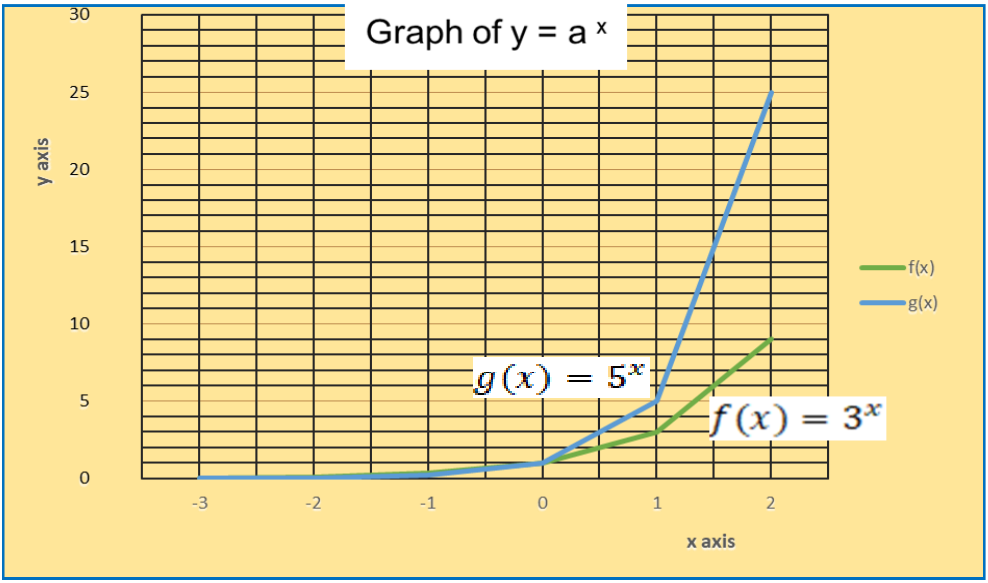


Figure 1.1 Graph of function $y = a^x$

Figure 1.1 shows the graph of function f(x) and g(x). It shows that both graphs are increasing.

Sketch the graph of each function in the same coordinate plane.

a) $f(x) = 3^{-x}$

b) $g(x) = 5^{-x}$

x	-3	-2	-1	0	1	2
3^{-x}	27	9	3	1	$\frac{1}{3}$	$\frac{1}{9}$
5^{-x}	125	25	5	1	$\frac{1}{5}$	$\frac{1}{25}$

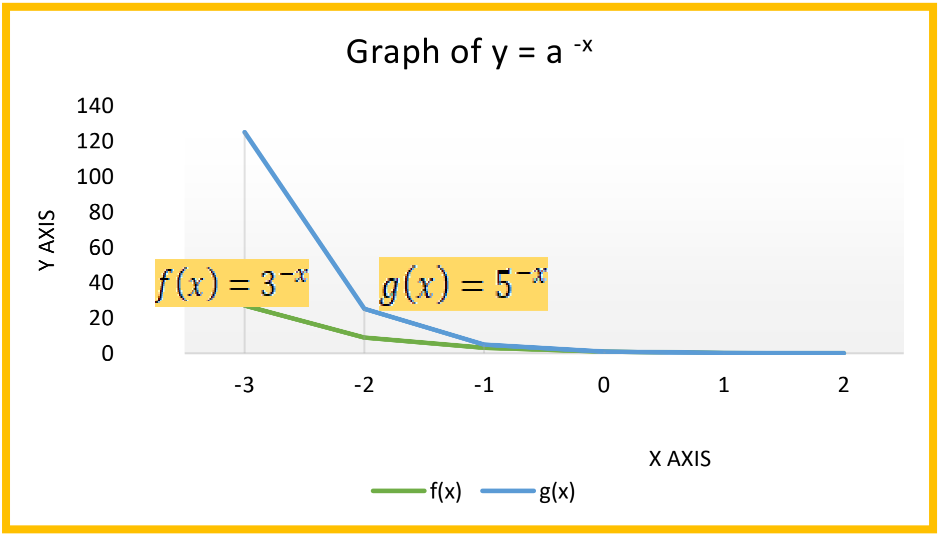


Figure 1.2 Graph of function $y = a^{-x}$

Figure 1.2 shows the graph of function f(x) and g(x). It shows that both graphs are decreasing.

Graph Exponential Function

Graph of $y = a^x$, $a > 1$

Domain: $(-\infty, \infty)$ Range: $(0, +\infty)$

Figure 1.3 shows that graph is continuous and increasing at y – intercept (0,1)

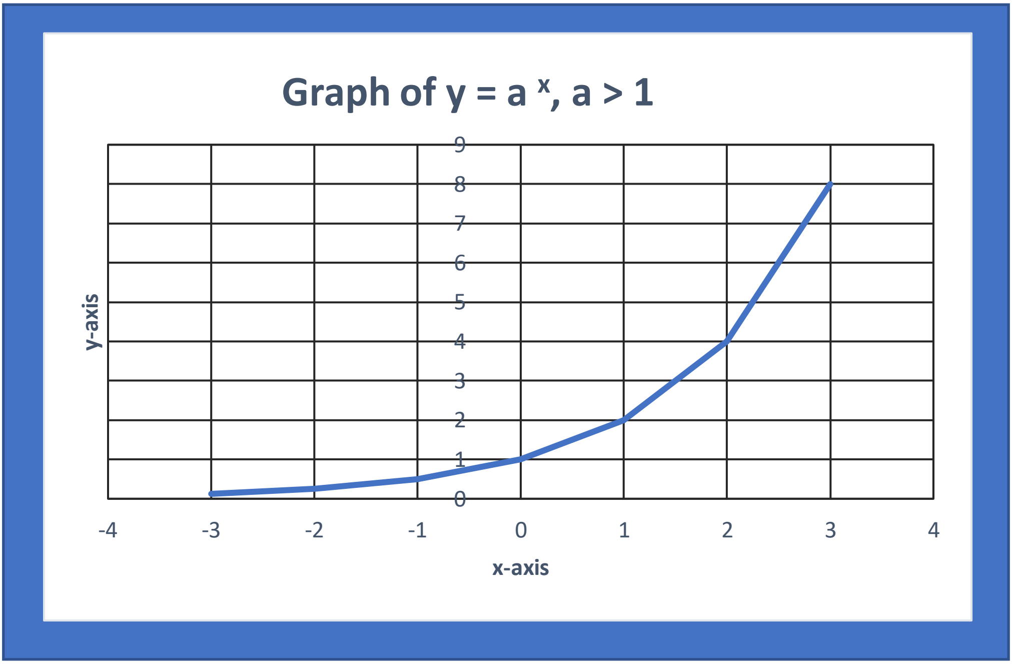


Figure 1.3 Graph of function $y = a^x$, $a > 1$

Graph of $y = a^{-x}$, $a > 1$

Domain: $(-\infty, \infty)$ Range: $(0, +\infty)$

Figure 1.4 shows that graph is continuous and decreasing at y – intercept (0,1)

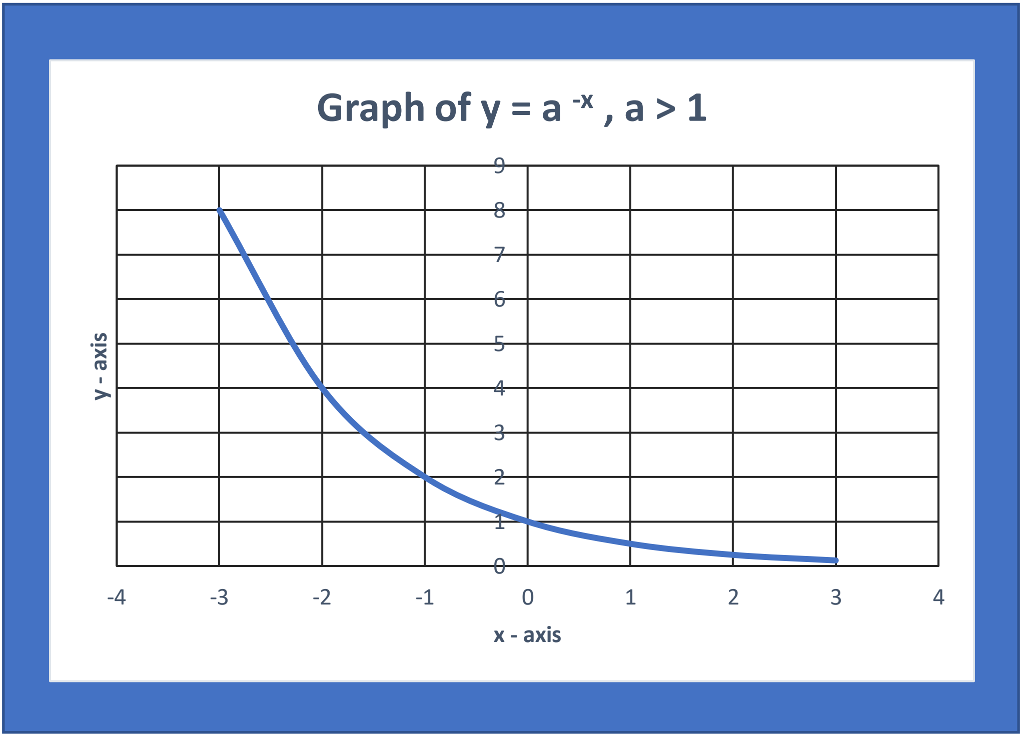


Figure 1.4 Graph of function $y = a^{-x}$, $a > 1$

Tutorial Exponential Function

Solve for x for each of the equation.

1. $4^{x}+7=23$
2. $(5^{x})^{2}=5$
3. $-4^{x}=-256$
4. $7^{X}-10=-9$
5. $4(2^{x})=\frac{4}{16}$
6. $5^{x}=\frac{1}{125}$
7. $6^{7+x}=6$
8. $2^{x}-100=156$
9. $8^{x+1}=8$
10. $6^{3x+2}=6^{x-1}$
11. $4^{x}=2^{x=1}$
12. $8^{2x+1}=(\frac{1}{16})^{3-2x}$
13. $10^{x}=0.1\times 1000^{x-1}$
14. $3^{3}\times 3^{3(2x-3)}=3^{4(3x-5)}$
15. $2^{2}\bullet 2^{x+1}=2^{-3(2x-3)}$
16. $\left(\frac{1}{8}\right)^{-x}=\left(\frac{1}{32}\right)^{1-x}$
17. $\left(\frac{1}{243}\right)^{2x}=81^{1-x}$
18. $\left(\frac{1}{125}\right)^{-3x-1}=25^{-x-1}$

Solution Tutorial Exponential Function

1. $4^x + 7 = 23$ $4^x = 16$ $4^x = 4^2$ $x = 2$	**2.** $(5^x)^2 = 5$ $5^{2x} = 5^1$ $2x = 1$ $x = \frac{1}{2}$	**3.** $-4^x = -256$ $4^x = 4^4$ $x = 4$
4. $7^x - 10 = -9$ $7^x = 1$ $7^x = 7^0$ $x = 0$	**5.** $4(2^x) = \frac{4}{16}$ $2^x = \frac{1}{16}$ $2^x = 2^{-4}$ $x = -4$	**6.** $5^x = \frac{1}{125}$ $5^x = 5^{-3}$ $x = -3$
7. $6^{7+x} = 6$ $7 + x = 1$ $x = -6$	**8.** $2^x - 100 = 156$ $2^x = 256$ $2^x = 2^8$ $x = 8$	**9.** $8^{x+1} = 8$ $8^{x+1} = 8^1$ $x + 1 = 1$ $x = 0$

10

$$6^{3x+2} = 6^{x-1}$$
$$3x + 2 = x - 1$$
$$3x - x = -1 - 2$$
$$2x = -3$$
$$x = -\frac{3}{2}$$

11.

$$4^{x} = 2^{x=1}$$
$$2^{2x} = 2^{x-1}$$
$$2x = x - 1$$
$$2x - x = -1$$
$$x = -1$$

12.

$$8^{2x+1} = (\frac{1}{16})^{3-2x}$$
$$2^{3(2x+1)} = 2^{-4(3-2x)}$$
$$3(2x+1) = -4(3-2x)$$
$$6x + 3 = -12 + 8x$$
$$6x - 8x = -12 - 3$$
$$-2x = -15$$
$$x = \frac{15}{2}$$

13.

$$10^{x} = 0.1 \times 1000^{x-1}$$
$$10^{x} = 10^{-1} \times 10^{3(x-1)}$$
$$x = -1 + 3x - 3$$
$$x - 3x = -4$$
$$-2x = -4$$
$$x = 2$$

14.

$$3^{3} \times 3^{3(2x-3)} = 3^{4(3x-5)}$$
$$3 + 6x - 9 = 12x - 20$$
$$6x - 12x - 6 = -20$$
$$-6x = -20 + 6$$
$$-6x = -14$$
$$x = \frac{7}{3}$$

15.

$$2^{2} \bullet 2^{x+1} = 2^{-3(2x-3)}$$
$$2 + (x+1) = -6x + 9$$
$$3 + x = -6x + 9$$
$$x + 6x = 9 - 3$$
$$7x = 6$$
$$x = \frac{6}{7}$$

16.

$$\left(\frac{1}{8}\right)^{-x} = \left(\frac{1}{32}\right)^{1-x}$$
$$2^{-3(x)} = 2^{-4(1-x)}$$
$$-3x = -4 + 4x$$
$$-3x - 4x = -4$$
$$-7x = -4$$
$$x = \frac{4}{7}$$

17.

$$\left(\frac{1}{243}\right)^{2x} = 81^{1-x}$$
$$3^{-5(2x)} = 3^{4(1-x)}$$
$$-10x = 4 - 4x$$
$$-10x + 4x = 4$$
$$-6x = 4$$
$$x = -\frac{2}{3}$$

18.

$$\left(\frac{1}{125}\right)^{-3x-1} = 25^{-x-1}$$
$$5^{-3(-3x-1)} = 5^{2(-x-1)}$$
$$9x + 3 = -2x - 2$$
$$9x + 2x = -5$$
$$11x = -5$$
$$x = -\frac{5}{11}$$

LOGARITHMIC FUNCTION

Definition of Logarithmic Functions

Definition of Logarithmic Function with base a

For $x > 0$, $a > 0$, and $a \neq 1$, $y = \log_a x$ if and only if $x = a^y$

Evaluate logarithms

Based on $y = \log_a x$, $a^y = x$, find y for each of the logarithm function.

$f(8) = \log_2 8$ Solution: $y = \log_2 2^3$ $y = 3 \log_2 2$ $y = 3$	$f(3) = \log_9 3$ Solution: $y = \log_9 9^{\frac{1}{2}}$ $y = \frac{1}{2} \log_9 9$ $y = \frac{1}{2}$
$f(1) = \log_4 1$ Solution: $y = \log_4 4^0$ $y = 0 \log_4 4$ $y = 0$	$f(\frac{1}{1000}) = \log_{10} \frac{1}{1000}$ Solution: $y = \log_{10} 10^{-3}$ $y = -3 \log_{10} 10$ $y = -3$

Evaluate the function $f(x) = \log x$ by using calculator.

function $f(x) = \log x$	Value	Answer
$f(84)$	$f(84) = \log 84$	1.924279286
$f(\frac{1}{5})$	$f(\frac{1}{5}) = \log \frac{1}{5}$	-0.698970004
$f(3.5)$	$f(3.5) = \log 3.5$	0.544068044
$f(-4)$	$f(-4) = \log -4$	error
$f(1000)$	$f(1000) = \log 1000$	3

Properties of Logarithms

$$log(ab) = log(a) + log(b)$$

$$\log_m m = 1$$

$$\log \frac{a}{b} = \log a - \log b$$

$$\log a^b = b \log a$$

$$\log_x 1 = 0$$

Rules	Examples
$a^x = y \rightarrow \log_a y = x$ **Base** $\log_a (a^x)$	$2^5 = 32$ $\log_2 32 = 5$ $\log_2 2^5$ $5\log_2 2$ 5
$\log_b x = \frac{\log_a x}{\log_a b}$	$\log_{25} 125$ $\frac{\log_5 125}{\log_5 25}$ $\frac{\log_5 5^3}{\log_5 5^2}$ $\frac{3}{2}$

Examples Solutions of Equations

$$\begin{aligned}\log_5 12.5 + \log_5 2 &= \log_5 (12.5 \times 2)\\ &= \log_5 25\\ &= \log_5 5^2\\ &= 2\log_5 5\\ &= 2\end{aligned}$$

$$\begin{aligned}\log_{10} 37 - \log_{10} 18.5 &= \log_{10} \frac{37}{18.5}\\ &= \log_{10} 2\\ &= 0.3010\end{aligned}$$

$$\begin{aligned}\log_4 81 &= \log_4 3^4\\ &= 4\log_4 3\end{aligned}$$

$$\log_{10}\frac{1}{8} = \log_{10} 8^{-1}$$
$$= -1\log_{10} 8$$
$$= -1\log_{10}(2 \times 4)$$
$$= -(\log_{10} 2 + \log_{10} 4)$$
$$= -(0.3010 + 0.6021)$$
$$= -0.9031$$

$$\log_2 7 + \log_2 12 - \log_2 21$$
$$\log_2\left(\frac{7 \times 12}{21}\right)$$
$$\log_2 4$$
$$\log_2 2^2$$
$$2\log_2 2$$
$$2$$

$$\log_4 5x - \log_4 (2x-1) = 1$$

$$\log_4 \left(\frac{5x}{2x-1}\right) = 1$$

$$\left(\frac{5x}{2x-1}\right) = 4^1$$

$$5x = 4(2x-1)$$

$$5x = 8x - 4$$

$$5x - 8x = -4$$

$$-3x = -4$$

$$x = \frac{4}{3}$$

Natural Logarithmic Function

The function is defined by

$$f(x) = \log_e x = \ln x, \qquad where \;\; x > 0$$

Properties of Natural Logarithms

$\ln 1 = 0$
$\ln e = 1$
$\ln e^x = x$ and $e^{\ln x} = x$
$\ln x = \ln y$, then $x = y$

Use the properties of natural logarithms to simplify each expression.

$\ln\frac{1}{e} = \ln e^{-1} = -1$	$\frac{\ln 1}{6} = \frac{0}{6} = 0$
$e^{ln\,3} = 3$	$4\ln e = 4(1) = 4$
$\ln e^{-5} = -5$	$3\ln 1 = 3\,(0) = 3$
$8\,ln\,1 = 8\,(0) = 0$	

Graph of Logarithmic Function

Sketch the graph of $f(x) = \log x$

x	$\frac{1}{100}$	$\frac{1}{10}$	1	3	6	7	10
$f(x) = \log x$	-2	-1	0	0.477121	0.778151	0.845098	1

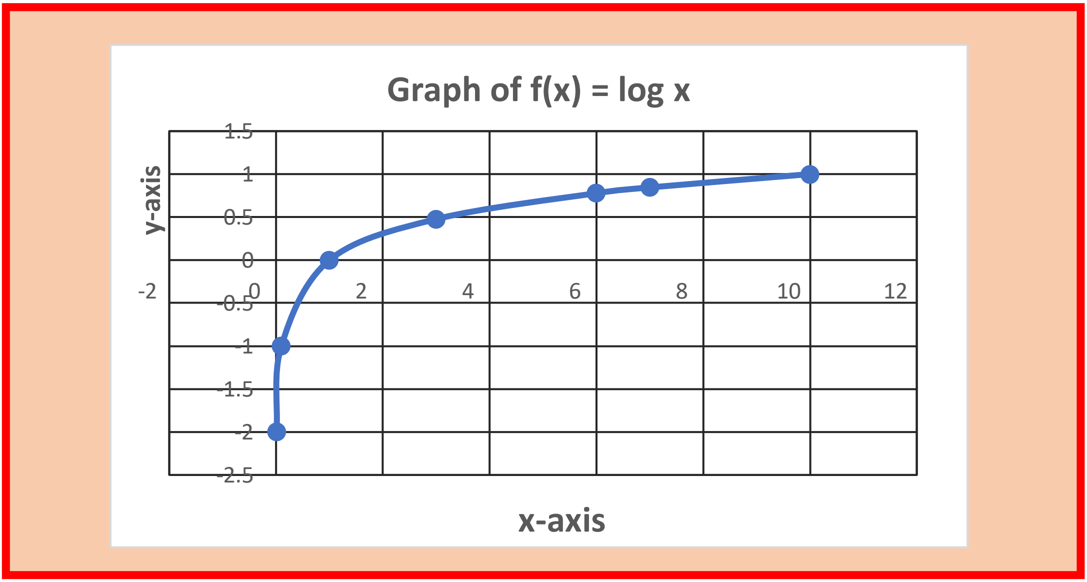


Figure 1.5 Graph of function $f(x) = \log x$

Domain: $(0,\infty)$

Range: $(-\infty, \infty)$

Graph of Exponential and Logarithmic Functions

Sketch the graph of each function.

a) $f(x) = 2^x$

b) $g(x) = \log_2 x$

Answer:

x	-2	-1	0	1	2	3
$f(x) = 2^x$	$\frac{1}{4}$	$\frac{1}{2}$	1	2	4	8

x	0	1	2	3
$f(x) = \log_2 x$	0	1	1.58	2

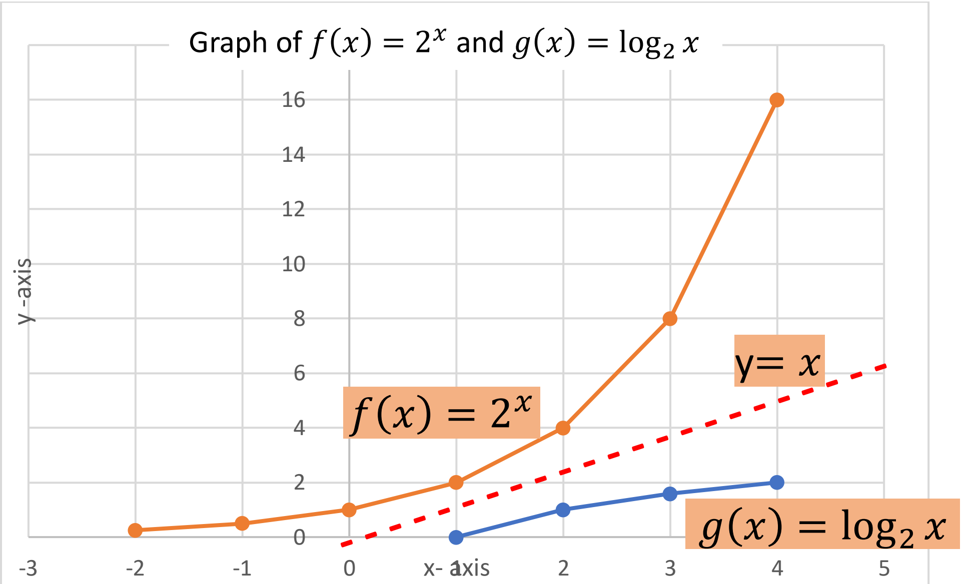


Figure 1.6 Graph of function $f(x) = 2^x$ and $g(x) = \log_2 x$

The graph of $g(x) = \log_2 x$ is the inverse function of $f(x) = 2^x$. Therefore, the graph of $g(x)$ is a reflection of the graph of $f(x)$ in the line $y = x$.

Tutorial Logarithmic Function

Find the value of the logarithm.

1. $\log 1$
2. $\log_{\frac{1}{3}} \frac{1}{9}$
3. $\log_3 27$
4. $\ln e^2 + \ln e^3$
5. $\log 500 - \log 5$
6. $\log 8^8$
7. $\log 200 + \log 5 - \log 100$
8. $\log_8 64$
9. $\log e^3 + \log e^{-3}$

Solve the logarithmic equations.

1. $\log x = 2$
2. $\log x^2 = 2$
3. $\log_x 25 = 2$
4. $\log_6 (4x + 8) = 2$
5. $\log_5 x = 3$
6. $\log x - \log 6 = 2$
7. $\frac{\log x}{\log(5x - 3)} = 1$
8. $\log_3 (x^2 - 8x) = 2$
9. $\log_{81} x = -1$

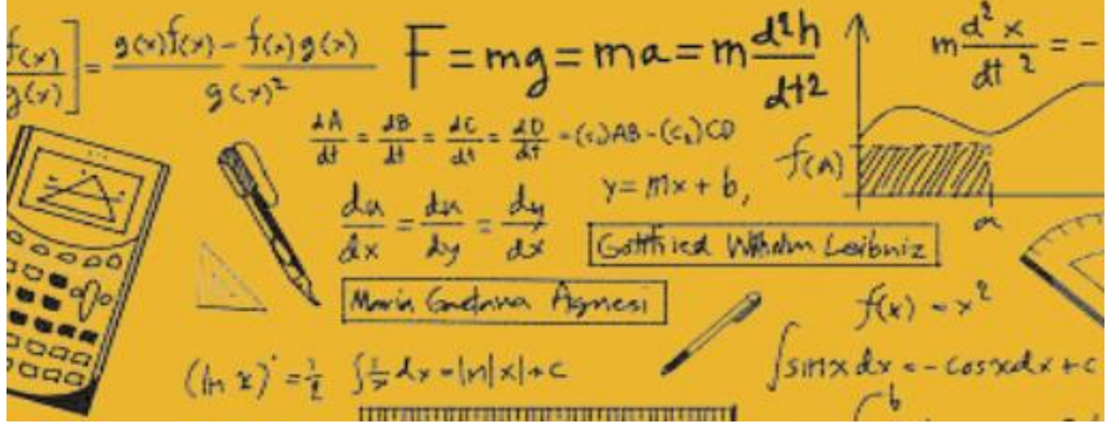
SOLUTION TUTORIAL LOGARITHMIC FUNCTION

1. $\log 1 = 0$	**2.** $\log_{\frac{1}{3}} \frac{1}{9}$ $\log_{\frac{1}{3}} \left(\frac{1}{3}\right)^2$ $2 \log_{\frac{1}{3}} \frac{1}{3}$ 2	**3.** $\log_3 27$ $\log_3 3^3$ $3 \log 3^3$ 3
4. $\ln e^2 + \ln e^3$ $2\ln e + 3\ln e$ $2 + 3$ 5	**5.** $\log 500 - \log 5$ $\log \frac{500}{5}$ $\log 100$ $\log_{10} 100$ $\log_{10} 10^2$ $2 \log_{10} 10$ 2	**6.** $\log 8^8$ 1
7. $\log 200 + \log 5 - \log 100$ $\log \frac{(200 \times 5)}{100}$ $\log \frac{1000}{100}$ $\log 10$ 1	**8.** $\log_8 64$ $\log_8 8^2$ $2\log_8 8$ 2	**9.** $\log e^3 + \log e^{-3}$ $3\log e + (-3) \log e$ $3(1) + (-3)$ 0

1. $\log x = 2$ $x = 10^2$ $x = 100$	**2.** $\log x^2 = 2$ $x^2 = 10^2$ $x = 10$	**3.** $\log_x 25 = 2$ $x^2 = 25$ $x^2 = 5^2$ $x = 5$ $\log_x 25 = 2$
4. $\log_6 (4x + 8) = 2$ $4x + 8 = 6^2$ $4x + 8 = 36$ $4x = 28$ $x = 7$	**5.** $\log_5 x = 3$ $x = 5^3$ $x = 125$	**6.** $\log x - \log 6 = 2$ $\log \frac{x}{6} = 2$ $\frac{x}{6} = 10^2$ $\frac{x}{6} = 100$ $x = 600$
7. $\frac{\log x}{\log(5x - 3)} = 1$ $\log x = \log(5x - 3)$ $x = 5x - 3$ $x - 5x = -3$ $-4x = -3$ $x = \frac{3}{4}$	**8.** $\log_3 (x^2 - 8x) = 2$ $(x^2 - 8x) = 3^2$ $x^2 - 8x = 9$ $x^2 - 8x - 9 = 0$ $(x - 9)(x + 1) = 0$ $x = 9, x = -1$	**9.** $\log_{81} x = -1$ $x = 81^{-1}$ $x = \frac{1}{81}$

Reference

Larson, Ron (2014). *Algebra and Trigonometry* (9th edition). Cengage Technology Edition.


F=mg=ma=m d²h/dt2
y= mx + b,
Gottfried Wilhelm Leibniz
Maria Gaetana Agnesi
f(x)