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Coefficients Bounds on a Certain Class of Multivalent Analytic Functions

Laxmipriya Parida[1], Ashok Kumar Sahoo[2]*

[1] Department of Basic Science and Humanities, School of Engineering and Technology, GIET University, Gunupur, Rayagada, Odisha, India
[2] Department of Mathematics, VSS University of Technology, SidhiVihar, Burla, Sambalpur, Odisha, India

*ashokuumt@gmail.com

Abstract: In the present work a sub-class $R^{b}_{p,n}(a_1,c_1,S,T)$ of $f \in \mathrm{A}_p(n)$ is defined by using a linear operator $L_p(a_1,c_1)$ and obtained sufficient condition in terms of the coefficients of $f \in \mathrm{A}_p(n)$ to be a member of this class. Furthermore, the Fekete-Szego problem is completely solved and found that the functional$|a_{p+3}a_{p+1} - a^2_{p+2}|$ is bounded. Finally, the sharpness of the associated estimates is also studied.

Keywords: Complex order, Hadamard product, Inclusion relationships, Neighbourhood, Subordination.

1 Introduction

Let $\mathrm{A}_p(n)$ be the class of analytic and p-valent function defined in a unit disk $\mathrm{U} = \{z \in \mathbb{C} : |z| < 1\}$ of the form:

$$f(z) = z^p + \sum_{m=n}^{\infty} a_{p+m} z^{p+m} \qquad (p,n \in N) \tag{1}$$

A_p, A are conveniently used for $n=1$ and $n=1, p=1$, respectively. For two functions f , g are analytic in U , the function f is called to be subordinate to the function g , written $f(z) \prec g(z)$,if there exists a function ψ analytic in U with $|\psi(z)| < 1, z \in U$, and $\psi(0) = 0$, such that $f(z) = g(\psi(z))$ for all $z \in U$. In particular, if g is univalent in U then the following equivalent relationship holds true (cf., e.g.,[25]; see also [26]):

$$f(z) \prec g(z) \Leftrightarrow f(0) = g(0) \quad \text{and} \quad f(\mathrm{U}) \subset g(\mathrm{U})$$

Furthermore, consider the functions f, g are analytic in U, $f(z)$ is given by equation (1) and

$$g(z) = z^p + \sum_{m=n}^{\infty} b_{p+m} z^{p+m} \quad (p,n \in N).$$

The convolution product of above functions is defined by

$$(f * g)(z) = z^p + \sum_{m=n}^{\infty} a_{p+m} b_{p+m} z^{p+m} \qquad (z \in \mathrm{U}).$$

Let $\mathrm{S}^*_{p,n}(d,\eta)$ and $\mathrm{C}_{p,n}(d,\eta)$ re the respective p-valently starlike and convex functions of complex order d and type η involving f of $\mathrm{A}_p(n)$ such that f satisfies,

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$$\operatorname{Re}\left\{p+\frac{1}{d}\left(\frac{zf'(z)}{f(z)}-p\right)\right\}>\eta \quad (d\in C^{*}=C\,,\quad \{0\}, 0\le\eta<p; z\in \mathrm{U}) \tag{2}$$

and

$$\operatorname{Re}\left\{p+\frac{1}{d}\left(1+\frac{zf''(z)}{f'(z)}-p\right)\right\}>\eta \quad (d\in C^{*}=C\,,\quad \{0\}, 0\le\eta<p; z\in \mathrm{U}) \tag{3}$$

respectively. From (1) and (3), we get that

$f\epsilon\, C_{p,n}(d,\eta) \Leftrightarrow \frac{zf'(z)}{p}\epsilon S^{*}_{p,n}(d,\eta)$.

For $p=n=1$, *the classes* $\mathrm{S}^{*}_{p,n}(d,\eta)$ and $C_{p,n}(d,\eta)$ reduces to $\mathrm{S}^{*}(d,\eta)$ and $C(d,\eta)$ the corresponding starlike and convex function of complex order d and type $\eta,(d\in C^{*};0\le\eta<p)$, which were studied by Frasin [2].

In $\mathrm{S}^{*}(d,\eta)$ and $C(d,\eta)$ if we take $\eta=0$then the classes are represented by $\mathrm{S}^{*}(d)$ and $C(d)$,which are starlike and convex functions of order d, respectively, and are studied by Nasr and Aouf [15] and Wiatrowski [20] (also, see [18] and [10]).We denote $\mathrm{S}^{*}_{p,1}(1,\eta)=\mathrm{S}^{*}_{p}(\eta)$ and $C_{p,1}(1,\eta)=C_{p}(\eta)$,as the respective classes of p-valentlystarlike and convex functions of order $\eta\ (0\le\eta<p)$ *in* U□*Also,* $\mathrm{S}^{*}_{1}(\eta)=\mathrm{S}^{*}(\eta)$ *and* $C_{1}(\eta)=C(\eta)$, are starlike and convex functions of order $\eta\ (0\le\eta<p)$ in U . Let $R_{p,n}(d,\eta)$ be the family of functions in $\mathrm{A}_{p}(n)$ satisfying the condition,

$$\operatorname{Re}\left\{p+\frac{1}{d}\left(\frac{f'(z)}{z^{p-1}}-p\right)\right\}>\eta \quad (d\in C^{*}=C\,,\quad \{0\}, 0\le\eta<p; z\in \mathrm{U}).$$

Also, $R_{p,n}(1,\eta)$ is the subclass of p-valently close-to-convex functions of order $\eta\ (0\le\eta<p)$, in the unit disk U.

Let θ_p be the incomplete beta function defined by

$$\theta_p(a_1,c_1;z)=z^p+\sum_{m=n}^{\infty}\frac{(a_1)_m}{(c_1)_m}z^{p+m} \quad (z\in\ \mathrm{U}), \tag{4}$$

where $a_1\in C, c_1\in C$, $\mathrm{Z}_0^-, Z_0^-=\{0,-1,-2,\ldots\}$) and $(x)_m$ denotes the Pochhammer symbol (or the shifted factorial) defined in terms of the Gamma function by

$$(x)_m=\begin{cases}1, & (m=0, x\in C^{*}=C\,,\quad \{0\})\\ x(x+1)\cdots(x+m-1), & (m\in N, x\in C).\end{cases}$$

Using θ_p given by (4) and the convolution product, Saitoh [11] considered a linear operator $L_p(a_1,c_1): A_p(n)\to A_p(n)$

given by

$$L_p(a_1,c_1)f(z)=\theta_p(a_1,c_1;z)*f(z) \qquad (z\in \mathrm{U}). \tag{5}$$

If f is given by (1), then, from (5) it gets that

$$L_p(a_1,c_1)f(z)=z^p+\sum_{m=n}^{\infty}\frac{(a_1)_m}{(c_1)_m}a_{p+m}z^{p+m} \qquad (z\in \mathrm{U}). \tag{6}$$

The investigation of certain sub classes of starlike, convex and prestarlike hypergeometric functions was first designed by Carlsonand Shaffer [3]. We also note that for $f\in A_p$,

(i) $L_p(a_1,a_1)f(z)=f(z)$

(ii) $L_p(p+1,p)f(z)=z^2f''(z)+2zf'(z)/p(p+1);$

(iii) $L_p(p+2,p)f(z)=zf'(z)/p;$

(iv) $L_p(t+p,1)f(z)=D^{t+p-1}f(z)\,(t\in\mathbb{R},t>-p)$, the operator studied by R.M. Goel and N.S. Sohi [23]. In the case p=1, $D^t f$ is the familiar St. Ruscheweyh derivative [24] of $f\in A$.

(v) $L_p(\tau+p,1)f(z)=D^{\tau,p}f(z)\,(\tau>-p)$, the extended linear derivative operator of St. Ruscheweyh type studied by R.K. Raina and H.M. Srivastava [22]. In particular, when $\tau=m$, we get operator $D^{m+p-1}f(z)\,(m\in\mathbb{N},m>-p)$, studied by R.M. Goel and N.S. Sohi [23].

(vi) $L_p(p+1,t+p)f(z)=I_{t,p}f(z)\,(t\in\mathbb{R},t>-p)$, the extended Noor integral operator introduced by L. Liu and K.I. Noor [13].

(vii) $L_p(p+1,p+1-\nu)f(z)=\Omega_z^{(\nu,p)}f(z)\,(-\infty<\nu<p+1)$, the extended fractional differintegral operator first designed by J. Patel and A.K. Mishra,[12].

Note that

$$\Omega_z^{0,p}f(z)=f(z),\ \Omega_z^{1,p}f(z)=\frac{zf'(z)}{p}\ \text{and}\ \Omega_z^{2,p}f(z)=\frac{z^2f''(z)}{p(p-1)}\ (p\geq 2;\ z\in \mathrm{U}).$$

Now, we use the operator $L_p(a_1,c_1)$ and introduce a new subclass of p-valent analytic functions in the unit disk U.

Definition

A class $R_{p,n}^d(a_1,c_1,S,T)$ is the subclass of analytic p-valent functions consisting f of the form equation (1) and satisfies the subordination condition.

$$1+\frac{1}{d}\left\{\frac{\left(L_p(a_1,c_1)f\right)'(z)}{z^{p-1}}-p\right\}\prec\frac{1+Sz}{1+Tz} \tag{7}$$

where $-1\leq T<S\leq 1,\ p\in N,\ d\in C^*.\ and\ z\in \mathrm{U}$. Equivalently, we say $f\in A_p(n)$ is a member of $R_{p,n}^d(a_1,c_1,S,T)$ if

$$\left|\frac{z(L_p(a_1,c_1)f)'(z)-pz^p}{d(S-T)z^p-T\left\{z(L_p(a_1,c_1)f)'(z)-pz^p)\right\}}\right|<1 \quad (z\in U) \tag{8}$$

For $n = 1$, we denote the class by $R_p^d(a_1,c_1,S,T)$ It may be noted that by suitably choosing the parameters involved in definition (1), the class $R_{p,n}^d(a_1,c_1,\nu,\eta)$ extends several subclasses of p-valent holomorphic functions in$U$.

- The class $R_p^d(a_1,c_1,S,T)$ generalizes many other sub-classes, for example, by considering $n=1, d=pe^{-i\phi}\cos\phi, S=1-2\eta/p, T=-1$ in definition (1.1), then, we get
- $R_p^{pe^{-i\phi}\cos\phi}\left(a_1,c_1,1-\dfrac{2\eta}{p},-1\right)=R_p(a_1,c_1,\phi,\eta)$
- $=\left\{f\in A_p:\operatorname{Re}\left[e^{i\phi}\left(\dfrac{(L_p(a_1,c_1)f)'(z)}{z^{p-1}}\right)\right]>\eta\cos\phi\right\},$

Where $0\le\eta<p,\ \phi|<\pi/2\, and\, z\in U$ Taking different restrictions on parameters, we get many subclasses of $R_p(a_1,c_1,\phi,\eta)$ as follows:

(i) For $a_1 = c_1$in the above subclass $R_p(a_1,c_1,\phi,\eta)$, we get

$$R_p(a_1,c_1,\phi,\eta)=R_p(\phi,\eta)=\left\{f\in A_p:\operatorname{Re}\left[e^{i\phi}\left(\frac{f'(z)}{z^{p-1}}\right)\right]>\eta\cos\phi\right\}.$$

The subclass $R_p(0,\eta)$ is recently studied by Krishna and Shalini [30] and found the third Hankel determinant.

(ii) For $a_1=p+1, c_1=p+1-\nu$ in the above subclass $R_p(a_1,c_1,\phi,\eta)$, , we obtained

$$R_p^{pe^{-i\phi}\cos\phi}\left(p+1,p+1-\nu,1-\frac{2\eta}{p},-1\right)=R_{p,\nu}(\phi,\eta)$$
$$=\left\{\left[e^{i\phi}\left(\frac{(\Omega_z^{\nu,p}(a_1,c_1)f)'(z)}{z^{p-1}}\right)\right]>\eta\cos\phi\right\}.$$

Where $0\le\eta<p,-\infty<\nu<p+1,|\phi|<\pi/2\, and\, z\in U$.

(iii) $R_p^{\frac{2pe^{-i\phi}\cos\phi(1-\frac{\alpha}{p})}{1+\beta}}\left(p+1,p,1,,-\beta\right)=R_{p,\alpha,\beta}^{\phi}\ (0\le\alpha\le p, 0\le\beta\le1,|\phi|\le\pi/2)$

$$=\left\{f\in A_p:\left|\frac{\dfrac{f'(z)}{p}+\dfrac{zf''(z)}{p}-pz^{p-1}}{\dfrac{f'(z)}{p}+\dfrac{zf''(z)}{p}-pz^{p-1}+2(p-\alpha)e^{-i\phi}z^{p-1}cos\phi}\right|<\beta; z\in U\right\}.$$

Further, taking $S = p-\eta,\ T = 0$ in definition (1), we get the following subclass $R^{d}_{p,n}(a_1, c_1, \eta)$ of $A_p(n)$.

- A function $f \in A_p(n)$ is said to be in the class $R^{d}_{p,n}(a_1, c_1, \eta)$,if it satisfies the following inequality:

- $$\left|\frac{1}{d}\left\{\frac{\left(L_p(a_1,c_1)f\right)'(z)}{z^{p-1}} - p\right\}\right| < p-\eta \quad (d \in C^*, 0 \le \eta < p; z \in U) \tag{9}$$

- $R^{d}_{p,n}(p+1, p+1-\nu, \eta) = R^{d}_{p,n}(\nu, \eta)\left(d \in C^*, -\infty < \nu < p\right)$, special cases of the parameters $p, \nu$and ηin the class $R^{d}_{p}(\nu, \eta)$ yield the following subclasses of A_p .

(i) $R^{d}_{p,n}(0,\eta) = R^{d}_{p,n}(\eta) = \left\{ f \in A_p : \left|\frac{1}{d}\left(\frac{f'(z)}{z^{p-1}} - p\right)\right| < p-\eta, 0 \le \eta < p; z \in U \right\}.$

(ii) $R^{d}_{p,n}(1,\eta) = \mathsf{P}^{d}_{p,n}(\eta) = \left\{ f \in A_p : \left|\frac{1}{d}\left((1+(1-p))\frac{f'(z)}{pz^{p-1}} + \frac{f''(z)}{pz^{p-2}} - p\right)\right| < p-\eta, 0 \le \eta < p; z \in U \right\}.$

(iii) $R^{d}_{1,n}(1, 1-\beta) = R^{d}_{n}(\beta) = \left\{ f \in A_p : \left|\frac{1}{d}\left(f'(z) + zf''(z) - 1\right)\right| < \beta, 0 < \beta \le 1; z \in U \right\}.$

Let P denote theclass of analytic functions θnormalized by

$$\theta(z) = 1 + p_1 z + p_2 z^2 + \cdots \quad (z \in U), \tag{10}$$

such that $\mathrm{Re}\{\theta(z)\} > 0$ *in* U.

Noonan and Thomas [14] defined the q −th Hankel determinant of a complex sequence $a_n, a_{n+1}, a_{n+2}, \cdots$ defined by

$$H_q(n) = \begin{vmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ a_{n+1} & a_{n+2} & \cdots & a_{n+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix} \quad (n \in N, q \in N, \ \ \{1\})$$

In a particular case, for $q = 2, n = 1, a\_1 = 1$ *and* $q = 2, n = 2.$ the Hankel determinant simplifies to

$H_2(1) = |a_3 - a_2^2|$ and $H_2(2) = |a_2 a_4 - a_3^2|$,

respectively. We refer to $H_2(2)$as the second Hankel determinant. Also, recall here that, if

$$f(z) = z + \sum_{m=2}^{\infty} a_k z^m \quad (z \in U), \tag{11}$$

is regular in a unit disc U, then the inequality $H_2(1) = |a_3 - a_2^2| \le 1$ holds true (see [18]). For a class F of holomorphic functions of the form equation (7) the classical theorem of Fekete-szego considered to be the Hankel determinant for $H_2(1)$ with well-known result for the estimation of $|a_3 - \mu a_2^2|$, when μ is real or complex.. The problem arising out of the co-efficient $H_2(1)$ for the familiar class of univalent mapping such as starlike functions, convex functions and close-to-convex functions were settled thoroughly by different researchers (see [16],[1],[9],[28],[29]). Tang et al. [31] defined a new subclass of analytic function and then derive the fourth Hankel determinant bound for this class.

In the ongoing presentation, sharp upper bound of Fekete-Szego functional and the second Hankel determinant for functions belonging to the subclass $\mathrm{R}^{d}_{p,n}(a_1,c_1,S,T)$ is determined by following a technique devised by Libera and Zlotkiewicz ([17],[21]). Relevant connections of the results obtained here with some earlier known works are also pointed out. To establish our results, we use the following lemma.

2. Preliminary Lemmas

To establish our main results, we shall need the following lemmas. The first lemma is the well-known Caratheoradory′s lemmaa (see also [5, corollary 2.3.]):

Lemma 2.1.[4] If $P \in \mathrm{P}$ and given by (10), then $|p_k| \leq 2$ for all $k \geq 1$, and the result is best possible for the function $P_*(z) = \dfrac{1+\rho z}{1-\rho z}, |\rho| = 1$.

The next lemma gives us a majorant for the coefficients of the functions of class P, and more details may be found in [27, Lemma 1]:

Lemma 2.2. [21] Let the function P is given by (10) be a member of the class P. Then,

$$\left|p_2 - \nu p_1^2\right| \leq 2\max\{1;|2\nu - 1|\}, \text{ where } \nu \in C$$

The result is sharp for the function given by

$$P^*(z) = \frac{1+\rho^2 z^2}{1-\rho^2 z^2} \quad \text{and} \quad P_*(z) = \frac{1+\rho z}{1-\rho z}, |\rho| = 1$$

Lemma 2.3.[21] Let the function P is given by (10) be a memberof the class P. Then,

$$p_2 = \frac{1}{2}\left[p_1^2 + \left(4 - p_1^2\right)x\right],$$

and

$$p_3 = \frac{1}{4}\left[p_1^3 + 2\left(4 - p_1^2\right)p_1 x - \left(4 - p_1^2\right)p_1 x^2 + 2\left(4 - p_1^2\right)(1 - |x|^2)z\right],$$

for some complex numbers x, $z$satisfying $|x| \leq 1$and $|z| \leq 1$.

Other details regarding the above lemma may be found in the relations (3.9) and (3.10) mentioned in [21].

3 Main Results

Unless otherwise mentioned, we assume throughout the sequel that,

$$d \in C^*, p \in N, a_1 > 0, c_1 > 0, -1 \leq T < S \leq 1, z \in \mathrm{U} .$$

And the powers appearing in different expression are known as principle values. We derive a sufficient condition for a function $f \in \mathrm{A}_p$ to be in the class $R^d_{p,n}(a_1,c_1,S,T)$.

Theorem 3.1. If a functional f given by (1) satisfies

$$\sum_{m=n}^{\infty} \frac{(a_1)_m}{(c_1)_m} |a_{p+m}|(p+m) \le \frac{|d|(S-T)}{(1+|T|)} \tag{12}$$

then $f \in R_{p,n}^{d}(a_1, c_1, S, T)$

Proof. To prove that f given by (1) is a member of $R_{p,n}^{d}(a_1, c_1, S, T)$ it needs to satisfy (8).
For $|z| = 1$, we have

$$\left| \frac{z(L_p(a_1,c_1)f)'(z) - pz^p}{d(S-T)z^p - T\left\{z(L_p(a_1,c_1)f)'(z) - pz^p\right\}} \right| = \frac{\left| \sum_{m=n}^{\infty} \frac{(a_1)_m}{(c_1)_m} a_{p+m}(p+m)z^m \right|}{\left| d(S-T) - T\sum_{m=n}^{\infty} \frac{(a_1)_m}{(c_1)_m} |a_{p+m}|(p+m)z^m \right|}$$

$$\le \frac{\sum_{m=n}^{\infty} \frac{(a_1)_m}{(c_1)_m} |a_{p+m}|(p+m)z^m}{|d|(S-T) - |T| \sum_{m=n}^{\infty} \frac{(a_1)_m}{(c_1)_m} |a_{p+m}|(p+m)z^m} \quad (z \in \mathrm{U}),$$

The last expression is needed to be bounded above by 1, which requires

$$\sum_{m=n}^{\infty} \frac{(a_1)_m}{(c_1)_m} |a_{p+m}|(p+m) \le \frac{|d|(S-T)}{(1+|T|)}.$$

By the claim of maximum modulus theorem, (8) is justified for $z \in \mathrm{U}$ and the proof of Theorem 3.1 is completed.

Corollary 3.1. For $f \in \mathrm{A}_p, |\phi| < \frac{\pi}{2}, 0 \le \eta < p,$

$$\sum_{m=1}^{\infty} \frac{(a_1)_m}{(c_1)_m} |a_{p+m}|(p+m) \le (p-\eta)\cos\phi,$$

is the sufficient condition to be member of $R_p(\phi, a_1, c_1, \eta).$

Theorem 3.2. If the function f , given by (1) belongs to the family $R_{p,n}^{d}(a_1, c_1, S, T)$, then

$$|a_{p+m}| \le \frac{|d|(S-T)(c_1)_m}{(p+m)(a_1)_m} \quad (m \ge n \in \mathbb{N}). \tag{13}$$

The estimate (13)is sharp.

Proof. Since $f \in R_{p,n}^{d}(a_1, c_1, S, T),$ we have

$$\frac{z(L_p(a_1,c_1)f)'(z) - pz^p}{z^p} = \frac{d(S-T)\omega(z)}{1+T\omega(z)} \quad (z \in \mathrm{U}),. \tag{14}$$

Where $\omega(z) = w_1 z + w_2 z^2 + \cdots$ is analytic in U satisfying the condition $|\omega(z)| \le |z|$ for $z \in \mathrm{U}$.

In (14) we replace the series from of f and ω after doing some simplification we reach at

$$\sum_{m=n}^{\infty}\frac{(a_1)_m}{(c_1)_m}a_{p+m}(m+p)z^m=\left\{d(S-T)-T\sum_{m=n}^{\infty}\frac{(a_1)_m}{(c_1)_m}a_{p+m}(m+p)z^m\right\}\sum_{m=1}^{\infty}w_m z^m \quad (z\in \mathrm{U}). \tag{15}$$

By simplifying the coefficient of both sides of (15) we get a_{p+m} depend on

$a_{p+n}, a_{p+(n+1)}, \cdots, a_{p+m-1}, m\geq n\in N.$

Hence, for $m\geq n$, it follows from (15) that,

$$\sum_{m=n}^{t}\frac{(a_1)_m}{(c_1)_m}(m+p)a_{p+m}z^m+\sum_{m=t+1}^{\infty}d_m z^m=\left\{d(S-T)-T\sum_{m=n}^{t-1}\frac{(a_1)_m}{(c_1)_m}(m+p)a_{p+m}z^m\right\}\omega(z),$$

Where the series $\sum_{m=t+1}^{\infty}d_m z^m$ converges in U. .Since $|\omega(z)|<1|$ for $z\in \mathrm{U}$, we get

$$\left|\sum_{m=n}^{t}\frac{(a_1)_m}{(c_1)_m}(m+p)a_{p+m}z^m+\sum_{m=t+1}^{\infty}d_m z^m\right|\leq\left|\left\{d(S-T)-T\sum_{m=n}^{t-1}\frac{(a_1)_m}{(c_1)_m}(m+p)a_{p+m}z^m\right\}\right|. \tag{16}$$

Writing $z=re^{i\phi}\ (r<1),$ squaring both sides of (16) and then integrating, we obtain

$$\sum_{m=n}^{t}\frac{(a_1)_m^2}{(c_1)_m^2}(m+p)^2|a_{p+m}|^2r^{2m}+\sum_{m=t+1}^{\infty}|d_m|^2r^{2m}\leq|d|^2(S-T)^2+|T|^2\sum_{m=n}^{t-1}\frac{(a_1)_m^2}{(c_1)_m^2}(m+p)^2|a_{p+m}|^2r^{2m}.$$

Having $r\to 1^-$ in the above discrimination, we obtain

$$\frac{(a_1)_t^2}{(c_1)_t^2}(t+p)^2|a_{p+t}|^2\leq|d|^2(S-T)^2-(1-|T|^2)\sum_{m=1}^{t-1}\frac{(a_1)_m^2}{(c_1)_m^2}(m+p)^2|a_{p+m}|^2\leq|d|^2(S-T)^2,$$

Where we have used the fact that $|T|\leq 1.$ Then the result will be

$$|a_{p+t}|\leq\frac{|d|(S-T)(c_1)_t}{(t+p)(a_1)_t} \quad (t\geq n\in N). \tag{17}$$

It is easily seen that the estimate (17) is sharp for the functions

$$f_m(z)=\theta_p(c_1,a_1;z)\,\mathring{a}\,z^p\left[\frac{(m+p)+\{T(m+p)+d(S-T)\}z^m}{(m+p)(1+Tz^m)}\right] \quad (m\in N; z\in \mathrm{U}).$$

From the above theorem 3.2, we can further draw different conclusions in the form of following corollaries.

Corollary 3.2.

$R_{p,n}^d(a_1+1,c_1,S,T)\subset R_{p,n}^d(a_1,c_1,S,T)$

and

$R_{p,n}^d(a_1,c_1,S,T)\subset R_{p,n}^d(a_1,c_1+1,S,T).$

Letting $d = pe^{-i\phi}, S = 1-2\eta/p, T = -1$ in Theorem 3.2, we get

Corollary 3.3 If the function $f \in \mathrm{A}_p$ is in the class $R_p(a_1, c_1, \phi, \eta)$, then

$$|a_{p+m}| \leq \frac{2p(1-\frac{\eta}{p})(c_1)_m}{(p+m)(a_1)_m} \quad (m \geq n \in N).$$

Remark 3.1.

(i) Upon taking $d=1, p=1$ and $a_1 = c_1$ in the Corollary 3.3, the inequality coincides with the Theorem -3.5 of [8], in addition with the above restrictions if we take $\eta = 0$, then, the result will agree with theorem -5 of [19] with $\alpha = 0$.

(ii) Choosing $d=1, p=1$ and $a_1 = c_1$ in Theorem 3.2, we get the same result as in theorem -2.1 of [6] with $\tau = 1$ and $\gamma = 0$.

4 Hankel Determinant

In this section, we solved the Fekete-Szegö problem and determine the sharp upper bound to the second Hankel determinant for the family $R_p^d(a_1, c_1, S, T)$ We first prove the following theorem.

Theorem 4.1. If the function $f \in \mathrm{A}_p$, from the family, $R_p^d(a_1, c_1, S, T)$ then for any $\nu \in \mathrm{C}$

$$|a_{p+2} - \nu a_{p+1}^2| \leq \frac{|d|(S-T)}{(p+2)} \frac{(c_1)_2}{(a_1)_2} \max\left\{1, \left| T + \frac{\nu d(S-T)(p+2)}{(p+1)^2} \frac{(c_1)(a_1+1)}{a_1(c_1+1)} \right| \right\}. \tag{18}$$

The estimate (18) is sharp.

Proof. Since $f \in R_p^d(a_1, c_1, S, T)$, we can find $\theta \in \mathrm{P}$ of the from (4) such that

$$\frac{(\mathrm{L}_p(a_1, c_1)f)'(z)}{z^{p-1}} - p = \frac{d(S-T)(\theta(z)-1)}{(1-T)+(1+T)\theta(z)} \quad (z \in \mathrm{U}). \tag{19}$$

Writing the series expansion of both sides, we obtain

$$\left(\sum_{m=1}^{\infty} \frac{(a_1)_{p+m}}{(c)_{p+m}} (p+m) a_{p+m} z^m \right) \left(2 + (1+T) \sum_{m=1}^{\infty} q_m z^m \right) = d(S-T) \sum_{m=1}^{\infty} q_m z^m. \tag{20}$$

Equating coefficient of z, z^2 and z^3, we get

$$a_{p+1} = \frac{c_1}{a_1} \frac{d(S-T)q_1}{2(p+1)}, \tag{21}$$

$$a_{p+2} = \frac{(c_1)_2}{(a_1)_2} \frac{d(S-T)}{2(p+2)} \left\{ q_2 - \left(\frac{1+T}{2} \right) q_1^2 \right\}, \tag{22}$$

and

$$a_{p+3} = \frac{(c_1)_3}{(a_1)_3} \frac{d(S-T)}{2(p+3)} \left\{ q_3 - \left(\frac{1+T}{2} \right) q_1 q_2 + \left(\frac{1+T}{2} \right)^2 q_1^3 \right\}. \tag{23}$$

We have $a_{p+2}-\nu a_{p+1}^2=\frac{d(S-T)}{2(p+2)}\frac{(c_1)_2}{(a_1)_2}\left\{q_2-\left[\frac{1+T}{2}+\frac{\nu d(S-T)(p+2)}{2(p+1)^2}\frac{c_1(a_1+1)}{a_1(c_1+1)}\right]q_1^2\right\}.$

This expansion gives $\gamma=\left[\frac{1+T}{2}+\frac{\nu d(S-T)(p+2)}{2(p+1)^2}\frac{c_1(a_1+1)}{a_1(c_1+1)}\right]$ and consequently using Lemma 2.2

We get $\quad 2\gamma-1=T+\frac{\nu d(S-T)(p+2)}{(p+1)^2}\frac{c_1(a_1+1)}{a_1(c_1+1)},$

This resulted the needed estimation (4.1). Sharpness of this estimation can easily be verified by taking the function f , defined in U by

$$f(z)=\begin{cases}\theta_p(c_1,a_1;z)\,å\,z^p\left\{\dfrac{1+\left(T+d\dfrac{(S-T)}{p+2}\right)z^2}{1+Tz^2}\right\}, & \text{if } \left|T+\nu\dfrac{d(S-T)(p+2)(a_1+1)c_1}{(p+1)^2a_1(c_1+1)}\right|\le 1\\[2ex] \theta_p(c_1,a_1;z)\,å\,z^p\left\{\dfrac{(p+2)+\left(T(p+1)+d(S-T)\right)z}{(p+2)+T(p+1)z}\right\}, & \text{if } \left|T+\nu\dfrac{d(S-t)(p+2)(a_1+1)c_1}{(p+1)^2a_1(c_1+1)}\right|>1.\end{cases}$$

This completes the proof of Theorem 4.1.

For ν to be real, we obtain the following result.

Corollary 4.1. If the function $f\in \mathrm{A}_p$, from the family $R_p^d(a_1,c_1,S,T)$, then for an $\nu\in R$.

$$|a_{p+2}-\nu a_{p+1}^2|\le\begin{cases}\dfrac{|d|(S-T)}{(p+2)}\dfrac{(c_1)_2}{(a_1)_2}, \text{ for } \dfrac{-(1+T)(p+1)^2a_1(c_1+1)}{d(S-T)(p+2)c_1(a_1+1)}\le\nu\le\dfrac{(1-T)(p+1)^2a_1(c_1+1)}{d(S-T)(p+2)}\\[2ex] \dfrac{|d|(S-T)}{(p+2)}\dfrac{(c_1)_2}{(a_1)_2}\left\{T+\dfrac{\nu d(S-T)(p+2)}{(p+1)^2}\dfrac{c_1(a_1+1)}{a_1(c_1+1)}\right\}, \text{ Otherwise.}\end{cases}$$

Putting $d=pe^{-i\phi}\cos\phi, S=1-2\eta/p, T=-1$ in theorem (4.1), we get the following result.

Corollary 4.2. If $f\in R_{p,}^{pe^{-i\phi}\cos\phi}(a_1,c_1,1-\frac{2\eta}{p},-1),$ then

$$|a_{p+2}-\nu a_{p+1}^2|\le\frac{2(p-\eta)\cos\phi}{(p+2)}\frac{(c_1)_2}{(a_1)_2}\max\left\{1,\left|\frac{2\nu e^{-i\phi}\cos\phi(p-\eta)(p+2)}{(p+1)^2}\frac{c_1(a_1+1)}{a_1(c_1+1)}-1\right|\right\}..$$

The estimate is sharp.

Remark 4.1. (i) Upon taking $d=p=\nu=1, a_1=c_1$ and $\phi=0$ in the Corollary 4.2, the inequality coincides with the theorem-3.3 of [8],in addition with above restrictions if we take $\eta=0$, then the result will agree with theorem-4 of [19] with $\alpha=0$.

(ii) Choosing $d=1, p=1$ and $a_1=c_1$ in Theorem 4.1, we get the result which is an agreement to the theorem -2.3 of [6] with $\tau=1$ and $\gamma=1$.

Theorem 4.2 If the function $f\in R_p^d(a_1,c_1,S,T)$, and $a_1\geq c_1>0$, then

$$|a_{p+3}a_{p+1}-a_{p+2}^2|\leq\left\{\frac{|d|(S-T)(c_1)_2}{(p+2)(a_1)_2}\right\}^2. \tag{24}$$

Proof. Using equation (21), (22) and (23), we get

$$a_{p+3}a_{p+1}-a_{p+2}^2=\frac{d^2(S-T)^2}{4}\frac{c_1(c_1)_2}{a_1(a_1)_2}\left\{\frac{1}{(p+3)(p+1)}\frac{c_1+2}{a_1+2}q_1q_3-\frac{(c_1+1)}{(a_1+1)(p+2)^2}q_2^2\right.$$

$$\left.+\left[\frac{(c_1+1)}{(a_1+1)(p+2)^2}-\frac{1}{(p+3)(p+1)}\frac{c_1+2}{a_1+2}\right](1+T)q_1^2q_2+\left[\frac{1}{(p+3)(p+1)}\frac{c_1+2}{a_1+2}-\frac{(c_1+1)}{(a_1+1)(p+2)^2}\right]\left(\frac{1+T}{2}\right)^2q_1^4\right\}.$$

Also, from Lemma (2.3), we get

$$a_{p+3}a_{p+1}-a_{p+2}^2=$$

$$\frac{d^2(S-T)^2}{4}\frac{(c_1)(c_1)_2}{a_1(a_1)_2}\left\{\frac{1}{4(p+3)(p+1)}\frac{c_1+2}{a_1+2}\left[q_1^4+2(4-q_1^2)q_1^2x-(4-q_1^2)q_1^2x^2+2q_1(4-q_1^2)(1-|x|^2z)\right]\right.$$

$$-\frac{(c_1+1)}{(a_1+1)(p+2)^2}\left[q_1^4+2(4-q_1^2)q_1^2x+(4-q_1^2)x^2\right]$$

$$+\left[\frac{(c_1+1)}{(a_1+1)(p+2)^2}-\frac{1}{(p+3)(p+1)}\frac{c_1+2}{a_1+2}\right]\frac{(1+T)}{2}\left[q_1^4+(4-q_1^2)q_1^2x\right]$$

$$\left.+\left[\frac{1}{(p+3)(p+1)}\frac{c_1+2}{a_1+2}-\frac{(c_1+1)}{(a_1+1)(p+2)^2}\right]\left(\frac{1+T}{2}\right)^2q_1^4\right\}.$$

For simplicity in the expression, we put

$$\alpha=\frac{d^2(S-T)^2}{4}\frac{c_1(c_1)_2}{a_1(a_1)_2},\ \beta=\frac{c_1+2}{4(p+3)(p+1)(a_1+2)},\text{ and }\Gamma=\frac{(c_1+1)}{4(a_1+1)(p+2)^2}.$$

Then by simple calculation, it can be observed that $0<\Gamma<\beta<2\Gamma.$. Using above notation and triangle inequality, we can write

$$|a_{p+3}a_{p+1}-a_{p+2}^2|\leq|\alpha|\left\{\frac{1}{8}\left[(\beta-\Gamma)(8+T(1+T))\right]q_1^4+\frac{1}{8}\left[(\beta-\Gamma)(15-T)\right](4-q_1^2)q_1^2x\right.$$
$$\left.+\left(\beta q_1^2+\Gamma(4-q_1^2)\right)(4-q_1^2)x^2+\left(2\beta q_1(4-q_1^2)(1-x^2)\right)\right\}. \tag{25}$$

Since the functions $\theta(z)$ and $\theta(e^{i\phi}z)$ $(\phi\in R)$ belong to the family P, we can take $q_1>0$ by which generality of the problem is not lost. Taking $x=v, q_1=u$ in (4.8), we get the function $Q(u,v)$ (say).

$$Q(u,v) = |\alpha| \left\{ \frac{1}{8}\big[(\beta-\Gamma)(8+T(1+T))\big]u^4 + \frac{1}{8}\big[(\beta-\Gamma)(15-T)\big](4-u^2)u^2 v \right.$$
$$\left. + \left(\beta u^2 + \Gamma(4-u^2)\right)(4-u^2)v^2 + \left(2\beta u(4-u^2)(1-v^2)\right) \right\}.$$

We need to find maximum value of $Q(u,v)$ in the interval $0 \le u \le 2,$ (by Lemma 2.1) $0 \le v \le 1.$ We can see by using the fact $0 < \Gamma < \beta < 2\Gamma.$

$$\frac{\partial Q}{\partial v} = |\alpha|(4-u^2)\left\{\frac{1}{8}\big[(\beta-\Gamma)(15-T)\big] + 2(\beta-\Gamma)u^2 v + 4(2\Gamma - \beta u)v\right\} > 0 \ \ (0 \le u \le 2, 0 \le v \le 1).$$

So $Q(u,v)$ cannot attain its maximum value within $0 \le u \le 2,\ 0 \le v \le 1.$ Moreover, for fixed $u \in [0,2],$

$$M(u) = \max_{0 \le v \le 1} Q(u,v) = Q(u,1) = |\alpha| \left\{ \frac{1}{8}\big[(\beta-\Gamma)(8+T(1+T))\big]u^4 \right.$$
$$\left. + \frac{1}{8}\big[(\beta-\Gamma)(15-T)\big](4-u^2)u^2 + \left(\beta u^2 + \Gamma(4-u^2)\right)(4-u^2) \right\},$$

and

$$M'(u) = |\alpha| \left\{ \frac{1}{2}(\beta-\Gamma)\left[T^2 + 2T - 15\right]u^3 + \left((\beta-\Gamma)(23-T) - 8\Gamma\right)u \right\}.$$

Since $M'(u) > 0,$ the maximum value occurs at $u = 0, v = 1.$ Therefore

$$|a_{p+3}a_{p+1} - a_{p+2}^2| \le \left\{ \frac{|d|(S-T)(c_1)_2}{(p+2)(a_1)_2} \right\}^2.$$

Taking $d = pe^{-i\phi}\cos\phi, S = 1 - 2\eta/p, T = -1$ in Theorem (4.2) we get the following result.

Corollary 4.3

If $f \in R_p^{pe^{-i\phi}\cos\phi}(a_1, c_1, 1-2\eta/p, -1),$ then

$$|a_{p+3}a_{p+1} - a_{p+2}^2| \le \left\{ \frac{2\cos\phi(p-\eta)(c_1)_2}{(p+2)(a_1)_2} \right\}^2. \tag{26}$$

The estimate (4.9) is sharp.

Putting $a_1 = p+1, c_1 = p+1+\nu$ in Corollary (4.3), we get following result.

Corollary 4.4. If $f \in R_{p,\nu}(\phi,\eta),$ then

$$|a_{p+3}a_{p+1} - a_{p+2}^2| \le \left\{ \frac{2\cos\phi(p-\eta)(p+1-\nu)_2}{(p+2)(p+1)_2} \right\}^2. \tag{27}$$

The estimate (27) is sharp.

Remark 4.2.

(i) Choosing $d = 1, p = 1$ and $a_1 = c_1$ in Theorem 4.2, we get the result which is an agreement to the theorem-2.4 [6] with $\tau = 1$ and $\gamma = 1,$ in addition with above restrictions if we take $S = 1, T = -1$ we get the result obtained by [19] in theorem-1 with $\alpha = 0$.

(ii) Assign $d = 1, p = 1$ and $a_1 = c_1$ in Theorem 4.2, we get the result which is an agreement to the theorem-2.1 of [7] with $\tau = 1$ and $\gamma = 0$.

5. Conclusion

The new generalized subclass $R_{p,n}^{d}(a_1, c_1, S, T)$ of the class $A_p(n)$ that we have introduced using the $L_p(a_1, c_1)$ convolution operator of Saitoh [11] and the concept of subordination, generalize many well-known subclasses of analytic functions defined and studied by several authors. The sufficient condition for a function $f \in A_p$, is in the class $R_p^d(a_1, c_1, S, T)$ is derived in Theorem-3.1 which simplifies the results for some other subclasses. Sharp upper bound for the absolute value of the coefficient a_{p+m} and some inclusion results based on the designed subclass are derived in Theorem-3.2. Fekete-Szego problem and sharp upper bound of second Hankel determinate is derived in section-4. Moreover, the results obtained which are generalizing other previous results of various authors are mentioned.

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References

[1] A. Janteng, S.A. Halim and M. Darus, Coefficient inequality for a function whose derivative has a positive real part, *J. Inequal. Pure and Application Mathematics* 7, Art. 50, 2006.
[2] B. A. Frasin, Family of analytic functions of complex order, *Acta Mathematica. Academiae Paedagogicae Nyiregyhaziensis*. (N.S.) 22, 179-191, 2006.
[3] B.C. Carlson and D.B Shaffer, Starlike and prestarlike hypergeometric functions, *SIAM Journal on Mathematcal Analysis*. 15, 737-745, 1984.
[4] C. Carathëodory, Uber den Variabilitätsbereich der Fourier'schen Konstanten von positiven harmonischen Funktionen, *Rend. Circ. Mat. Palermo*, 32, 193-217, 1911.
[5] Ch. Pommerenke, Univalent Functions, *Vanderhoeck& Ruprecht, Göttingen*, 1975.
[6] D. Bansal, Fekete-Szego Problem and Upper Bound of Second Hankel Determinant for a New Class of Analytic Functions, *KYUNGPOOK Math. J*. 54, 443-452, 2014.
[7] D. Bansal, Upper bound of second Hankel determinant for a new class of analytic functions, *App. Math. Lett*. 26, 103–107, 2013.
[8] D. V. Krishna, B. Venkateswarlu and T. Ram Reddy, Third Hankel determinant for bounded turning functions of order alpha, *J. Nige. Math. Soc*. 34, 121–127, 2015.
[9] F. R. Keogh and E.P. Merkes, A coefficient inequality for certain classes of analytic functions, *Proc. Amer. Math. Soc*. 20, 8-12, 1969.
[10] H. M. Srivastava and S. Owa(Eds.), *Current Topics in Analytic Function Theory*, *World Scientific Publishing Company*, Singapore, New Jersey, London and Hong Kong, 1992.
[11] H. Saitoh , A linear operator and its application of first order differential subordinations, *Math. Japonica* 44, 31-38, 1996.
[12] J. Patel and A.K. Mishra, On certain multivalent functions associated with an extended fractional differintegral operator, *J. Math. Anal. Appl*. 332, 109-122, 2007.
[13] J. L. Liu and K. I. Noor, Some properties of Noor integral operator, *J. Natur. Geom*. 21, 81-90, 2002.
[14] J. W. Noonan and D.K. Thomas, On the second Hankel determinant of areally mean p-valent functions, *Trans. Amer. Math. Soc*. 223, 337-346, 1976.
[15] M. A. Nasr and M.K. Aouf, Starlike functions of complex order, *J. Natur. Sci. Math*. 25, 1-12, 1985.
[16] M. Fekete and G. Szego, Eine bemerkung\"{u}berungeredeschlichtefunktionen, *J. London Math. Soc.*8, 85-89, 1933.

[17] R. J. Libera and E.J. Zlotkiewicz, Early coefficient of the inverse of a regular convex function, *Proc. Amer. Math. Soc*. 85 (2), 225-230, 1982.
[18] P. L. Duren, Univalent Functions, A Series of Comprehensive Studies in Mathematics, Vol. 259, *Springer-Verlag*, New York, Berlin, Heidelberg, and Tokyo, 1983.
[19] P. Sahoo, Third Hankel Determinant for a Class of Analytic Univalent Functions, *Elect. J. of Math. Anal. and Appl*. 6(1), pp. 322-329, 2018.
[20] P. Wiatrowski, Onthe coefficients of some family of holomorphic functions, *ZeszytyNauk. Uniw. LodzNauk. Mat.-Przyrod*. 39 (2), 75-85, 1970.
[21] R. J. Libera and E. J. Zlotkiewicz, Coefficient bounds for the inverse of a function with derivative in P, *Proc. Amer. Math. Soc.,* 87(2), 251-257, 1983.
[22] R. K. Raina and H.M. Srivastava, Inclusion and neighborhood properties of some analytic and multivalent functions, *J. Inequal. Pure Appl. Math*. 7 (1), Art. 5, 2006.
[23] R. M. Goel and N.S. Sohi, A new criterion for p-valent functions, *Proc. Amer. Math. Soc*. 78, 353-357, 1980.
[24] St. Ruscheweyh, New criteria for univalent functions, *Proc. Amer. Math. Soc*. 49, 109-115, 1975.
[25] S. S. Miller and P.T. Mocanu, Differential subordinations and univalent functions, *Michijan. math. J*. 28, 157-171, 1981.
[26] S. S. Miller and P.T. Mocanu, Differential Subordinations, Theory and Applications, *Series on Monographs and Textbooks in Pure and Applied Mathematics,* Vol. 225, Marcel Dekker, New York and Basel, 2000.
[27] W. C. Ma and D. Minda, A unified treatment of some special classes of univalent functions, In: *Proceedings of the Conference on Complex Analysis (Tianjin, 1992*), Z. Li, F. Ren, L. Yang and S. Zhang (Eds.), Int. Press, Cambridge, MA, 157-169, 1994.
[28] W. Koepf, On the Fekete-Szego problem for close-to-convex functions-II, *Arch. Math*. 49, 420-433, 1987.
[29] W. Koepf, On the Fekete-Szeg\"{o} problem for close-to-convex functions, *Proc. Amer. Math. Soc*. 101, 89-95, 1987.
[30] D.V. Krishna and S. Shalini, Third Hankel Diterminant for Multivalant Bounded Turning Functions of Order Alpha*, TWMS J. App. and Eng. Math. V*. 11, M.3, pp 789-794, 2021.
[31] Huo Tang, Muhammad Arif, MirajulHaq, Nazar Khan, Mustaqeem Khan, Khurshid Ahmad and Bilal Khan; Fourth Hankel Determinant Problem Based on Certain Analytic Functions. *Symmetry*, 14, 663. https://doi.org/10.3390/sym1404066, 2022.