DISPERSION RELATION EQUATION OF SHALLOW WATER: WAVELENGTH ESTIMATOR

Nor Azni Shahari [1*], Maizatur Najihah Azlan [2], Siti Nuramyra Mohd Abd Razak [3] and Nurain Nadhirah mohamad [3]
[1,2,3,4] Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA (UiTM) Cawangan Negeri Sembilan, Kampus Seremban, Persiaran Seremban Tiga/1, 70300 Seremban, Negeri Sembilan
*Corresponding author:norazni@uitm.edu.my

*Keywords:*Shallow water;dispersion relation equation; wavelength; numerical method

1. Introduction

Waves are a familiar occurrence on the ocean and may be observed on nearly any water surface that is exposed to the atmosphere. Water waves are generated by wind blowing over the water surface, it may also be caused by air pressure disturbances, flow obstructions, landslides, or literally anything else that disrupts the water surface. Waves may be characterized by a wave height, H, and wavelength, L. A wave height, H is defined as the elevation difference between the wave crest and wave trough, and a wavelength, L, which is the horizontal distance between successive crests or troughs as illustrated in Figure 1 (Kennedy, 2019). Waves will also have a period, T, which is defined as the time for successive crests or troughs to pass a fixed point. The study of water waves has been crucial for both theoretically and practically in domains ranging from mathematical sciences to coastal facility design and harbor structures, ecological networks, and tsunami and storm inundation prevention.

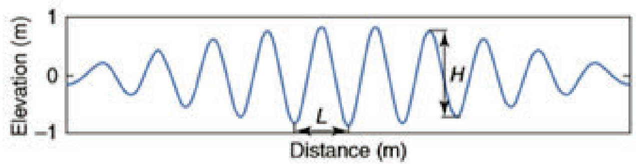

Figure 1: Definition for wave height and length

The water wave problem is a complex problem since it includes huge amplitude motion over a moving surface, and the full nonlinear equations have yet to be solved analytically. In addition, calculating wavelength is an essential parameter in wave attenuation (Hardaway and Gunn, 2010). As in general, the wave is attenuated by using a wave attenuator or also known as a breakwater. Essentially, the breakwater helps to break the coastal waves so that the force is reduced to an acceptable level before it reaches the marina. When the distance between two diffraction endpoints reaches twice the incident wavelength or more, the coastline behind each breakwater acts independently, as if there is no contact between the breakwaters. Consequently, this situation might put marinas in an unsafe position to successfully operate floating docks. Therefore, wavelength calculations should be taken into consideration to reduce the likelihood of an accident in marinas. In addition to that, a similar purpose has been investigated but in more specific, which is an investigation of a nonlinear transformation of tsunami wave attenuation by mangrove forest (Adytia and Husrin, 2019). Although this investigation not only depends on wavelength value, but the value itself acts as the normalization value in nonbreaking solitary waves.

Besides, since most of the large towns are located nearer to the coast, waves constitute an integral part of human civilization. When a wave reaches the shallow water, the wave touches the bottom, friction causes the wave to slow down. As one wave slows down, the one behind it catches up to it, thus decreasing the wavelength so while the wavelength decreases, the wave height increases. When waves are exceptionally large, they might pose a threat to public infrastructure and residents who live near the coastal area. Therefore, some alternative solutions have been implemented to manage this situation such as creating a lighthouse. One of the functions of the lighthouse is it acts like a traffic sign on the sea, and it reacts when wavelength value reaches a risky level to alert certain parties such as boats.

Buoys with a sensor are placed beneath the sea and used to record data of wave measurements, i.e., its height, direction, and period. When given wave period and depth via wave sensors, the data must be processed to find a wavelength. The dispersion relation equation is normally used to calculate the wavelength; however, it may not be convenient for engineers to calculate directly using dispersion relation equation. In this paper, simple and effective method for solving the dispersion equation of shallow water waves is developed using Graphical user interface (GUI) based on numerical methods i.e., Newton and improvised Newton methods to find the wavelength to give a quick calculation.

2. Model Formulation

Dispersion relation equation is used to compute wavelength for various wave periods and the water depth. The dispersion relation is the most important aspect of a linear wave theory. The dispersion relation where it relates the relationship between the wave number, k and the frequency, f, is as follows (Goda, 2000) :

$$\omega^2 = gk \tanh(kh) \tag{1}$$

Where the angular frequency denoted by $\omega = 2\pi f$, wave number, $k = \frac{2\pi}{L}$ and h, is the water depth. The surface boundary condition at the water surface generates the equation 1. Hence, the standard relationship between the wavelength, L, the wave period, T, and water depth, h, can be stated by using the dispersion relation equation, which is

$$L = \frac{g}{2\pi} T^2 \tanh\left(\frac{2\pi h}{L}\right) \tag{2}$$

The dispersion relation equation can be used to find wavelength with the known variable of water depth, h and wave period, T. However, an equation 2 does not have an explicit solution. Therefore, it is generally computed numerically to a given precision using Newton's technique, which is an iterative calculation approach. On the other hand, Equation 2 which is also known as transcendental equation is used in calculating wavelength. Equation 2 need to be solved by Newton's Method and Improvised Newton methods to obtain the value of the wavelength, L, where it can be transformed to

$$x \tanh(x) = D \tag{3}$$

where $D = \frac{\omega^2 h}{g} = \frac{2\pi h}{L_0}$, $L = \frac{2\pi g}{\omega^2} = \frac{g}{2\pi} T^2$ and $x = kh = \frac{2\pi h}{L}$. To remove the inflection point, equation 3 needs to be rewrite as

$$y(x) = x - D \coth(x) \tag{4}$$

where $\coth(x) = \frac{\cosh(x)}{\sinh(x)}$. Hence, an iterative solution of the wavelength can be solved through this finalized equation. The two methods employed are the Newton's Method (NM), and the Improvised Newton's Methods.

2.1. *Newton's Method (NM)*

The Newton's method is:

$$x_{n+1} = x_n - \frac{f(x)}{f'(x)} \quad \text{for} \quad n = 0, 1, 2, ... \tag{5}$$

2.2. *Midpoint Newton's Method (MNM)*

The Midpoint Newton's method is an improvised method based on midpoint rule for solving nonlinear equations. The formula is:

$$x_{n+1} = x_n - \frac{f(x)}{f'\left(x_n - \frac{f(x_n)}{2f'(x_n)}\right)} \tag{6}$$

2.3. *Improvise Midpoint Newton's Method (IMNM)*

Then, an improvised Midpoint Newton's method by using Midpoint Newton's method (Alleme and Azad, 2012) is shown below, for any one initial approximation as x_0 of the root. The improvised Midpoint Newton's method formula is:

$$x_{n+1} = x_n - \left(\frac{f(x)(y_n - x_n)}{f(y_n) - f(x_n)}\right) \tag{7}$$

2.4. *Chebychev Method (CM)*

The procedure was based on Newton's method and the expansion of the inverse of the function as a power series. The Chebyshev method is defined by:

$$x_{n+1} = x_n - \frac{f(x)}{f'(x_n)} - \frac{f^2(x_n)f''(x_n)}{2(f'(x_n))^3} \tag{8}$$

3. Graphical User Interface (GUI)

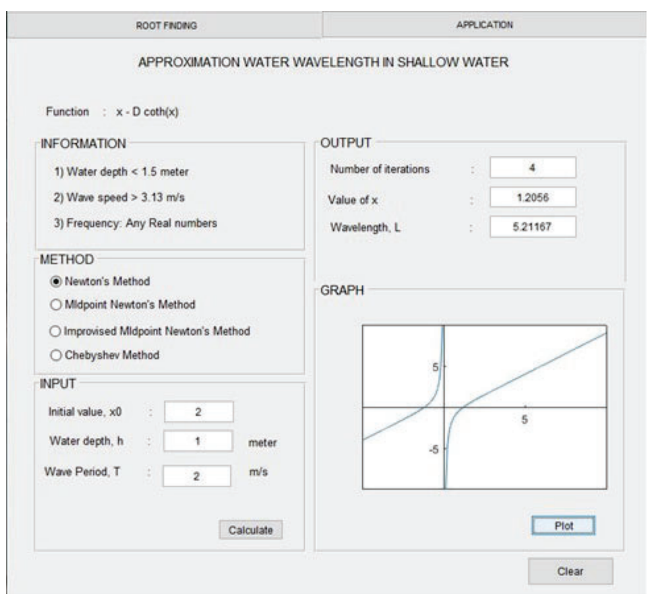

Figure 2: Graphical User Interface (GUI) to generate the results by MATLAB software

The Graphical User Interface (GUI) in MATLAB programming software is shown in Figure 2 for the purpose of computing the root of the nonlinear water wavelength equation and the number of iterations required for the approaches to converge to the exact roots. The application's function is $f(x) = x - D\coth(x)$. The solution begins by selecting a method among four available options at the radio button function: Newton's Method (NM), Midpoint Newton's Method (MNM), Improvised Midpoint Newton's Method (IMNM), or Chebyshev's Method (CM). Following that, the user must populate the input box with the initial value, water depth, and wave periods, T. After inputting all the required information, the user must click the calculate button, which will compute the water wavelength based on the provided data. Additionally, the output portion includes the number of iterations required for the algorithm to converge to the exact roots, x and the water wavelength, L. Finally, if the user clicks the plot button in the interface, the function's graph will emerge.

4. Results and Discussions

Since the formula for the iterative process in an approximation water wavelength requires water depth values , this study was carried out with multiple water depths values (Goda, 2000), including 2m, 1.5m, 0.5m and 0.1m. The depth range chosen for the water is the depth range for water in shallow areas. Table 1 shows the comparison with Goda (2000) and Lee (2019) with wave period, T=2 for different depth. Consequently, these findings support Lee (2019) hypothesis that the wavelength approximation value gradually decreases when the water depth is decreased as shown in Table 1. This has also confirmed that the numerical method could be used to compute the wavelength of shallow water.

Table 1: Comparison of water wavelength with other paper

	Wavelength(L)		
Water depth h(m)	Goda (2000)	This study	Lee (2019) $L = T\sqrt{gh}$
2	6.05	6.04048	8.85
1.5	5.78	5.77848	7.668
0.5	4.05	4.05398	4.42
0.1	1.97	1.94662	1.97

Thus, moving to the numerical process, this study is furthered by comparing all methods in terms of the number of iterations produced for each method to converge to determine the optimum method. Table 2 below shows the comparison result of iterations for water wavelength between all methods. The findings reveal that the Improvised Newton's Method which are MNM, IMNM as well as CM outperforms Newton's Method to converge on the actual root. Al-

Table 2: Comparison number of iterations to find the water wavelength

Water depth(h)	Initial value	Iteration			
		NM	MNM	IMNM	CM
2	1	4	3	3	3
	0.5	5	3	4	3
	2	2	2	2	2
1.5	1	4	2	3	2
	0.5	5	3	4	3
	2	3	2	3	2
0.1	1	4	3	4	3
	0.5	6	2	3	4
	2	5	3	5	4

though the findings reveal little difference in the iteration value, this minor difference shows that the Improvised Newton's Method can provide better performance than the Newton's Method. If these three improvised methods were compared, it was discovered that MNM required less iterations than IMNM and CM.

According to the findings, numerical methods can be used to find wavelengths of shallow water. As a result, there are certain suggestions that should be taken into consideration for future research such as might look at water wavelengths in transitional water and deep water rather than shallow water to have a better understanding of the significance of these findings. This may help someone get a better understanding of the wavelength behavior in waves.

References

Adytia, D. and Husrin, S. (2019). Numerical simulations of nonbreaking solitary waveattenuation by a parameterized mangrove forest model. *International Journal of Engineering and Technology*, 8:10–16.

Alleme, M. and Azad, N. (2012). On modified newton's method for solving a nonlinear algebraic equations by mid-point. *Interntional Journal of Scientific & Engineering Research*, 17(12):1546–1548.

Goda, Y. (2000). *Random Seas and Design of Maritime StructuresBio-inspired Routing Protocols for Vehicular Ad-Hoc Networks*, chapter In Advanced Series on Ocean Engineering. World Scientific.

Hardaway, C. S. J. and Gunn, J. R. (2010). Design and performance of headland bays in chesapeake bay, usa. *Coastal Engineering*, 57(2):203–212.

Kennedy, A. (2019). *Fundamentals of Water Waves*, chapter Encyclopedia of Water: Science, Technology, and Society. John Wiley& Son.

Lee, C. (2019). Calculating water wavelength using dispersion relation and approximation. *Undergraduate Journal of Mathematical Modelling: One+Two*, 9(2):1–11.