Upper Bound of Second Hankel Determinant for Certain Generalized Class of Starlike Functions

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Introduction

The study of Hankel determinant is one of the most famous coefficient inequalities in geometric theory functions that still being study by many researchers. At the end of 2020, the research on Hankel determinant have reach to the fourth Hankel determinant, $H_4(1)$. In this research, the focus is on the inequalities of $|a_2a_4 - a_3^2|$ or better known as second Hankel determinant. Krishna, Venkateswarlu, and RamReddy (2015) considered that A denote the class of analytic functions $f(z)$ of the form (1.1.1) in the open unit disc $E = \{z : |z| < 1\}$. Ehrenborg (2000) found that the Hankel determinant of order $(n+1)$ is the determinant of corresponding Hankel matrix,

$$\det(a_{i+j})_{0\le i,j\le n} = \det\begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{bmatrix}$$

Then, Noonan & Thomas (1976) defined that q^{th} Hankel determinant of f for $q \ge 1$ by

$$H_q(m) = \begin{vmatrix} a_m & a_{m+1} & \cdots & a_{m+q-1} \\ a_{m+1} & a_{m+2} & \cdots & a_{m+q} \\ \vdots & \vdots & \vdots & \vdots \\ a_{m+q-1} & a_{m+q} & \cdots & a_{m+2q-2} \end{vmatrix}$$

The second Hankel determinant with $q = 2$ and $m = 2$, gives $H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix}$ for $f \in A$, so that $|a_2a_4 - a_3^2|$ and by applying triangle inequality $|H_2(2)| \le |a_2a_4 - a_3^2|$. As a conclusion, second Hankel determinant is one of the coefficient inequalities that can be used to achieve the new subclasses of analytic functions in this research.

Methodology

In a nutshell, the aim is to find the second Hankel determinant, $H_2(2)$ on certain subclasses of analytic functions. The method is inspired by the method used by Janteng et al. (2007), Kaharudin et al. (2011) and Yahya et al. (2013). This method used by many researchers and produced plenty of subclasses of analytic function. To find the second Hankel determinant, $H_2(2)$, this methodology steps were utilized:

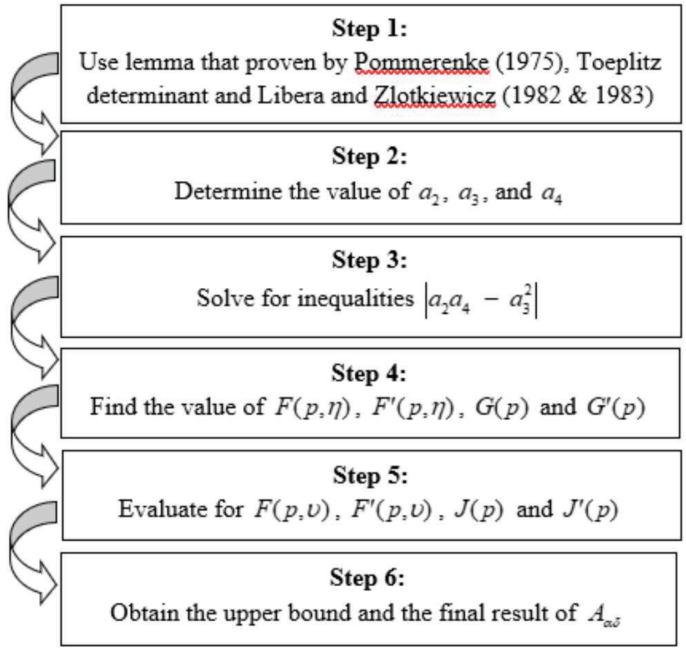

Figure 1: Methodology steps

Results and Discussions

In this research, the new class of function $T(\alpha,\delta,\beta)$ in the unit disk, $E=\{z:|z|<1\}$ is defined as the class of starlike functions. Functions in this class are normalized functions given by

$$\operatorname{Re}\left\{e^{i\alpha}\frac{zf'(z)}{h(z)}\right\}>\delta \tag{4.1}$$

where $|\alpha|\le\pi$, $\cos(\alpha)>\delta$, $h(z)=\dfrac{z}{1-\beta z^2}$ and $-1\le\beta\le1$.

1. Representation Theorem

Theorem 1

Let $f\in S$ be given by Toeplitz determinant. Then, $T(\alpha,\delta,\beta)$ if and only if

$$\frac{e^{i\alpha}\dfrac{zf'(z)}{h(z)}-i\sin\alpha-\delta}{\cos\alpha-\delta}=P$$ where, $z\in E$.

2. Upper Bound of Second Hankel Determinant

Theorem 2

Let $h(z)\in T(\alpha,\delta,\beta)$, $M=\left(\dfrac{64+(9\beta-5)^2}{144}\right)$. If

$$A_{\alpha\delta}=\frac{-\frac{4}{9}\beta\pm\sqrt{\left(\frac{4}{9}\beta\right)^2+4M\left(\frac{1}{9}\beta^2\right)}}{2M}$$

then,

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{9}\beta^2+\frac{4}{9}\beta A_{\alpha\delta}+\left(\frac{64+(9\beta-5)^2}{144}\right)A_{\alpha\delta}^2.$$

Corollary 3

For $T(\alpha,\delta,1)$, let f be functions given in (4.1). If $\beta=1$, then this class will reduced to Yahya et al. (2013). Therefore, the upper bound for second Hankel determinant $T(\alpha,\delta,\beta)$ is

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{9}+\frac{4}{9}A_{\alpha\delta}+\left(\frac{5}{9}\right)A_{\alpha\delta}^2$$

and $A_{\alpha\delta}\le 0.9266$ for $\left|a_2a_4-a_3^2\right|\le 1$ as stated by Janteng et al. (2007) where inequality adheres for the function in which $h(z)=\dfrac{z}{1-z^2}$.

Conclusion

The pinnacle of this research is to determine the coefficient inequality of starlike functions. By going back to the objective of study, this research managed to obtain upper bound of the second Hankel determinant for the new generalized subclasses of analytic functions $T(\alpha,\delta,\beta)$.

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