Bound for the Second Hankel Determinant of Certain Generalized Class of Close-to-Convex Functions

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Introduction

Geometric function theory is a study about geometric properties of certain classes of analytic functions. There are many classes of functions in geometric function theory that have been introduced in this field of research. Geometric function theory has a wide area to explore, thus this research will focuses on one of the coefficient inequalities in geometric function theory which is second Hankel determinant on upper bounds for the functional $|a_2a_4 - a_3^2|$. Ehrenborg (2000) declared that the Hankel determinant of order $(n+1)$ is determinant of the corresponding Hankel matrix and it can be defined by

$$\det\left(a_{i+j}\right)_{0\le i,j\le n} = \det\begin{bmatrix} a_0 & a_q & \cdots & a_n \\ a_1 & a_2 & \cdots & a_{n+1} \\ \vdots & \vdots & \ddots & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n} \end{bmatrix}.$$

Next, the q^{th} Hankel determinant of f for $q \ge 1$ as

$$H_q(n) = \begin{vmatrix} a & a_n & a_{n+1} & \cdots & a_{n+q+1} \\ & a_{n+1} & a_{n+2} & \cdots & a_{n+q+2} \\ & \vdots & \vdots & \ddots & \vdots \\ & a_{n+q-1} & a_{n+q} & \cdots & a_{n+2q-2} \end{vmatrix},$$

had been stated by (Noonan and Thomas ,1976). In this research, the second Hankel determinant is used in the case of $q = 2$ and $n = 2$ namely,

$$H_2(2) = \begin{vmatrix} a_2 & a_3 \\ a_3 & a_4 \end{vmatrix} = \left|a_2a_4 - a_3^2\right|.$$

In conclusion, second Hankel determinant plays important role in the theory of univalent function. In this research, a new subclass in univalent functions will be discover and the coefficient inequalities of second Hankel determinant will be determined.

Methodology

In this research, analytics proving is choosen instead of computational proving. Precisely, it is focusing on generalizing the new subclasses of analytic functions, then solving the problem of upper bound for second Hankel determinant as stated by Janteng, Halim, and Darus (2007), Kaharudin et al. (2011) and Yahya et al. (2013). The researchers implement this methodology steps to find upper bound for second Hankel determinant:

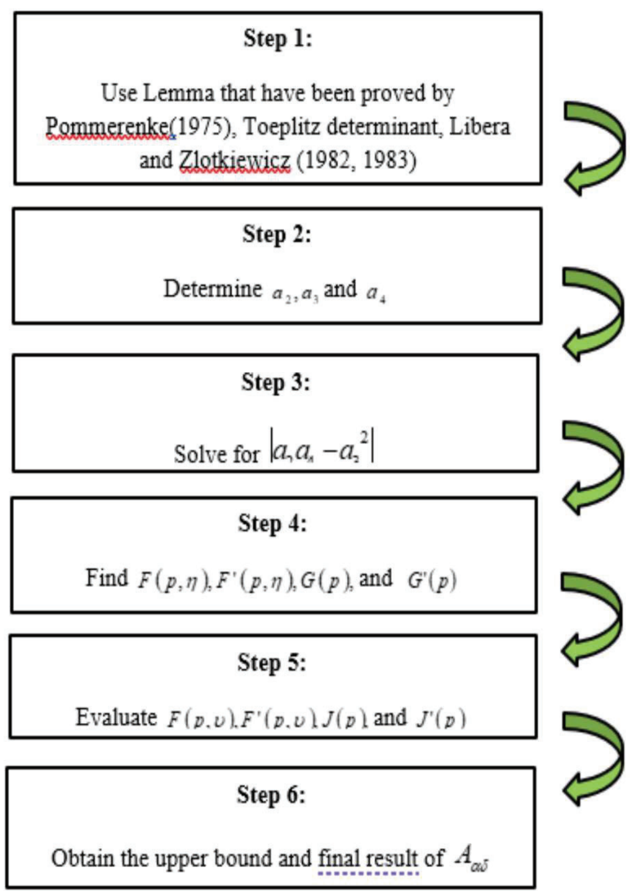

Figure 1: Methodology steps

Results and Discussions

The class of $\alpha-$close-to-convex functions is denoted as $W(\alpha,\delta,\beta)$. This class of functions are normalized functions $f \in A$ satisfy the condition

$$\operatorname{Re}\left\{e^{ia}\frac{zf'(z)}{M(z)}\right\} > \delta, \quad (z \in U)$$

Where $|a| < \pi, \cos(a) > \delta, M(z) = \dfrac{z}{1-\beta z}$ and $-1 \le \beta \le 1$.

1. Representation Theorem

Theorem 1

Let $f \in S$ and $f \in W(\alpha,\delta,\beta)$, then

$$\frac{e^{i\alpha}\dfrac{zf'(z)}{M(z)} - i\sin\alpha - \delta}{\cos\alpha - \delta} \in P, \; z \in U.$$

2. Upper Bound of Second Hankel Determinant

Theorem 2

Let $f \in W(\alpha,\delta,\beta)$ then $B = -\dfrac{14\beta}{27} + \dfrac{2\beta^2}{3} + 2, K = \dfrac{-14}{16-18\beta-18}$ and $L = 8 - 9\beta$. Then,

$$\begin{aligned}\left|a_2a_4 - a_3^2\right| = \frac{1}{72}\Bigg[&\left(9\beta - 8\beta^2\right) + \Bigg[\left(\sqrt{-\frac{14\beta}{27}+\frac{2\beta^2}{3}+2}\right)\left[-7\beta - 8\left(\sqrt{-\frac{14\beta}{27}+\frac{2\beta^2}{3}+2}\right)\beta + 9\beta^2\right]\\ &+\left(\sqrt{-\frac{14\beta}{27}+\frac{2\beta^2}{3}+2}\right)\left[\frac{9}{2}\sqrt{-\frac{14\beta}{27}+\frac{2\beta^2}{3}+2} + \frac{9}{4}\left(-\frac{14\beta}{27}+\frac{2\beta^2}{3}+2\right)\right]\\ &+\left[2+\frac{14\beta}{27}-\frac{2\beta^2}{3}\right]\left[\frac{9}{2} - 8\beta + \frac{27}{4}\left(\sqrt{-\frac{14\beta}{27}+\frac{2\beta^2}{3}+2}\right)\right]\Bigg]A_{\alpha\delta}\\ &+\Bigg[\left(\frac{-14\beta}{16-18\beta-18}\right)^2(8-9\beta) - 9\left(\frac{-14}{16-18\beta-18}\right)^2\\ &+14\left(\frac{-14}{16-18\beta-18}\right)+32\Bigg]A_{\alpha\delta}^{\;2}\Bigg].\end{aligned}$$

Corollary 3

For $W(\alpha,\delta,\beta)$, let f be the functions given in Theorem 2. For $\beta=1$,then

$$\left|a_2a_4-a_3^2\right|\le\frac{1}{72}+A_{\alpha\delta}\left(\frac{29\sqrt{174}}{972}-\frac{7}{36}\right)+A_{\alpha\delta}^2\left(\frac{41}{80}\right)$$

where $A_{\alpha\delta}\le1.206$. The inequality takes holds of the function where $M(z)=\frac{z}{1-\beta z}$. Hence, this inequality obtained is coincides to the result of Kaharudin et al. (2011).

Conclusion

The focus of this research is to find the upper bound of the second Hankel determinant for $W(\alpha,\delta,\beta)$. The objective has been achieved by implementing Lemma of Pommerenke (1975), Lemma of Toeplitz determinants and Lemma of Libera and Zlotkiewicz (1982,1983).

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