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The Effect of Mosquito Biting Rate on the Malaria Transmission Model using Bifurcation Analysis

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Abstract: This paper analyses a simple malaria transmission model to investigate the effect of parameter changes in mosquitoes per capita biting rate on human population, β, by using the bifurcation analysis. For further investigation, we first formulated the malaria transmission model into a non-dimensionalized equation. Next, we employed the stability analysis of disease-free equilibrium and endemic equilibrium point of malaria transmission. Using the same non-dimensionalized equation, the one-parameter bifurcation analysis was conducted. A few graphs of the bifurcation diagram, phase plane and time series plot are displayed with the help of mathematical software such as XPPAUT, Maple, and Matlab. Findings reveal that a transcritical bifurcation occurred due to changes in the stability of the system's equilibrium. When the parameter value of β increases, the infected human and infected mosquito population decreases. This finding demonstrates that changes in mosquito biting rates could have an effect on both infected humans and mosquito population.

Keywords: Malaria model, stability analysis, bifurcation analysis, transcritical bifurcation.

1 Introduction

Malaria is a mosquito-borne infectious disease caused by protozoa of the genus plasmodium. Plasmodium falciparum, plasmodium malariae, plasmodium vivax and plasmodium ovale are four dissimilar species of the parasite that infect humans [1]. Following the bite from an infectious mosquito, parasites enter into the bloodstream of the victim and a variety of liver cells are invaded. After replication, the parasites exit the liver cells and begin invading red blood cells after re-entering the bloodstream. This invasion of parasites results into thousands of cells that are parasite-infected in the bloodstream which leads to an illness and later develops into complications of malaria which, if not treated, can last for months. The malaria disease is commonly present in tropical states such as South Africa and Sudan. World Health Organisation [2] undergoes special initiatives in achieving zero malaria cases in these states from 2020 onwards.

Ronald Ross [3] realized that mathematical modelling and the role of mathematics can provide an explicit frame work in order to understand disease transmission dynamics between hosts and parasites. The mathematical modelling of malaria transmission starts from the Ross' model with many improvements done on the model throughout the years [4]. The Ross' model was enhanced by Macdonald and formed a Ross-Macdonald model where he introduced superinfection, reinfection of those who were infected hence carrying multiple types of parasites, and quantitatively synthesized malaria epidemiology [5]. Abboubakar et al. [6] used the bifurcation analysis onto the malaria model and applied the basic reproduction number, R_0 of the malaria disease as the parameter. Result shows that the basic reproduction number, R_0, has no impact when varying the relative biting rate of infectious mosquitoes. Besides, the time delay can also be a bifurcation parameter which results in significant impact on vector-borne epidemic model. The magnitude of the delay can affect the existence and the direction of Hopf bifurcation [7].

The bifurcation analysis on malaria model was employed by several of the researchers in their studies [8-11].

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Feng et al. [12] found that backward bifurcation may occur as a result of disease-free equilibrium coexisting with two endemic equilibria by using the model to stimulate reported malaria cases in China. In backward bifurcation, when reproduction number, $R_0 < 1$, a small positive will result into an unstable equilibrium while large positive equilibrium are locally asymptotically stable [13].

Authors discussed an epidemic model for malaria by showing the occurence of backward bifurcation which implies the existence of a stable disease-free equilibrium and stable endemic equilibrium. If $R_0 > 1$ and with certain conditions, it is observed that a unique endemic equilibrium is globally asymptotically stable [14].

For this research, an earlier stage of the simple malaria model by Ndanusa and Busari [5] based on the Ross-Macdonald model has been employed. Ndanusa and Busari [5] applied the Runge-Kutta integration scheme method to investigate the effect of various parameters on malaria transmission. Motivated by their work, we are mainly interested in the investigation using the bifurcation analysis. In particular, we seek to answer the following question: when does the biting rates of mosquitoes matter in determining the dynamics of infected human and infected mosquito population? This research only focuses on the parameter of per capita biting rate of mosquitoes on human population, β, for our bifurcation analysis. We chose to vary that parameter since we want to investigate the changing effect of per capita biting rate of mosquitoes on human population, β, towards both infected human and mosquito population.

This paper is organized as follows. Section 2 discusses the non-dimensionalized equation for a simple malaria transmission model, as well as steady states, equilibria, and stability analysis for the malaria transmission model. Section 3 highlights the results and discussions of stability analysis and numerical bifurcation analysis while the result and conclusions are discussed in Section 4.

2 Methodology

A The Malaria Transmission Model

The mathematical modelling of a basic malaria transmission model presented below expresses the key elements of malaria epidemiology studied by Roberts and Heesterbeek [15]. Table 1 shows the definition of each parameter in the model.

$$\begin{aligned} \frac{dX}{d\tau} &= ab\frac{M}{N}Y(1-X)-\gamma X\ , \qquad X(0)=X_0 \\ \frac{dY}{d\tau} &= acX(1-Y)-\mu Y\ , \qquad Y(0)=Y_0\ . \end{aligned} \tag{1}$$

Table 1: Definition of parameters in Eq. (1).

Parameter	Definition
X	Fraction of infected human population
Y	Fraction of infected mosquito population
$\frac{M}{N}$	Number of female mosquitoes per human host in an infection free state
a	Per biting rate of mosquitoes on human populations
b	Probability that a bite by an infectious mosquito transmits the agent
c	Probability that a bite of an infected human by a susceptible mosquito results in transmission of the agent to the mosquito
γ	Rate at which humans recover from infection
μ	Per capita death rate of mosquitos

To reduce the total number of parameters, a non-dimensionalized equation is necessary, resulting in simplified equations that will be utilised to study the stability analysis of the malaria model at malaria disease-free equilibrium and endemic equilibrium. Next, dimensionless variables for fraction of infected humans, fraction of infected mosquitoes and time are introduced by letting

$$x = \frac{ac}{\mu} X , \qquad y = \frac{abM}{\gamma N} Y , \qquad t = ac\tau .$$

Therefore, the non-dimensionalized equations of the simple malaria model are:

$$\begin{aligned} \frac{dx}{dt} &= \alpha y - \beta x(1+y) \\ \frac{dy}{dt} &= \sigma x - \nu y(1+x) \end{aligned} \tag{2}$$

where α is $\frac{\gamma}{\mu}$ which is the human recovery rate, β is $\frac{\gamma}{ac}$ which is the per capita biting rate of mosquitoes on human population, σ is $\frac{bM\mu}{\gamma Nc}$ which is the number of female mosquitoes per human host in an infection-free steady state and ν is $\frac{\mu}{ac}$ which is the per capita death rate of mosquitoes are the parameters. From Eq. (2), x and y are dependent variables where x and y are infected human and infected mosquito population, respectively. However, independent variable t is defined as a time.

B Stability Analysis

There are two possible steady states which are malaria disease free equilibrium, $E_1 = (x^*, 0)$ where the value of y is equal to zero to indicate an entirely healthy population in the absence of mosquitoes and the disease at endemic equilibrium, $E_2 = (x^{**}, y^{**})$ to indicate the human contains susceptible and infected. The malaria model Eq. (2) is solved by using Jacobian matrix in order to find the eigenvalues.

i. Disease Free Equilibrium Point

The Jacobian matrix for Eq. (2) is represented by:

$$J_{x,y} = \begin{pmatrix} -\beta(1+y) & \alpha - \beta x \\ \sigma - \nu y & -\nu(1+x) \end{pmatrix}.$$

$E_1 = (x^*, 0)$ is substituted into Jacobian matrix for linearization of Eq. (2). The resulted matrix is

$$J_{x^*,y} = \begin{pmatrix} -\beta & \alpha - \beta x^* \\ \sigma & -\nu(1+x^*) \end{pmatrix}.$$

Here, the parameter was simplified by letting:

$$\begin{aligned} \delta &= \alpha - \beta x^* \\ \tau &= \nu(1+x^*) . \end{aligned}$$

Then, a new Jacobian matrix obtained is to be used in finding the eigenvalues. To find the eigenvalues, the determinant was determined and set to equal zero,

$$J_{x^*,0} = \begin{pmatrix} -\beta & \delta \\ \sigma & -\tau \end{pmatrix}$$
$$\det(J - \lambda I) = \det\begin{pmatrix} -\beta - \lambda & \delta \\ \sigma & -\tau - \lambda \end{pmatrix}$$
$$(-\beta - \lambda)(-\tau - \lambda) - \sigma\delta = 0$$
$$\lambda^2 + (\beta + \tau)\lambda + (\beta\tau - \sigma\delta) = 0\,.$$

To illustrate and obtain the exact values of λ, $x^* = 2$ was assumed to indicate the existence of human population. Hence, new point $E_1 = (2,0)$ together with the parameter values as in Table 2 will be used to find the eigenvalues at disease-free equilibrium.

Table 2: Parameter values for malaria Eq. (2)

Parameter	Definition	Value	Source
α	Human recovery rate	0.003704	[8]
σ	The number of female mosquitoes per human host in an infection free steady state	0.5	[5]
β	Per capita biting rate of mosquitoes on human	0.0025	Assumed
ν	Per capita death rate of mosquitoes	0.03	[12]

By substituting all the parameter values into quadratic equation, the eigenvalues, λ, at disease-free equilibrium are $\lambda_1 = -0.0107$ and $\lambda_2 = -0.0818$.

ii. Endemic Equilibrium Point

The endemic equilibrium point, $E_2 = (x^{**}, y^{**})$ is substituted into Jacobian matrix. The resulted matrix is:

$$J_{x^{**},y^{**}} = \begin{pmatrix} -\beta(1 + y^{**}) & \alpha - \beta x^{**} \\ \sigma - \nu y^{**} & -\nu(1 + x^{**}) \end{pmatrix}.$$

Here, the parameter was simplified by letting:

$$\beta_1 = \beta(1 + y^{**})$$
$$\delta_1 = \alpha - \beta x^{**}$$
$$\sigma_1 = \sigma - \nu y^{**}$$
$$\tau_1 = \nu(1 + x^{**})\,.$$

Then, a new Jacobian matrix is obtained to be used in finding the eigenvalues.

$$J_{x^{**},y^{**}} = \begin{pmatrix} -\beta_1 & \delta_1 \\ \sigma_1 & -\tau_1 \end{pmatrix}$$

$$\det(J-\lambda I)=\det\begin{pmatrix}-\beta_1-\lambda & \delta_1\\ \sigma_1 & -\tau_1-\lambda\end{pmatrix}$$
$$(-\beta_1-\lambda)(-\tau_1-\lambda)-\sigma_1\delta_1=0$$
$$\lambda^2+(\beta_1+\tau_1)\lambda+(\beta_1\tau_1-\sigma_1\delta_1)=0$$

To obtain λ, the values assumed for (x^{**},y^{**}) is $(2,2)$ to indicate the existence of human and mosquito population. Hence, a new point $E_2=(2,2)$ together with the parameter values as in Table 2 will be used to find the eigenvalues at endemic equilibrium. By substituting all the parameter values into quadratic equations, the eigenvalues, λ, at endemic equilibrium are $\lambda_1=-0.01511$ and $\lambda_2=-0.08239$.

3 Result and Discussions

A Stability Analysis of Disease-Free Equilibrium Point and Endemic Equilibrium Point

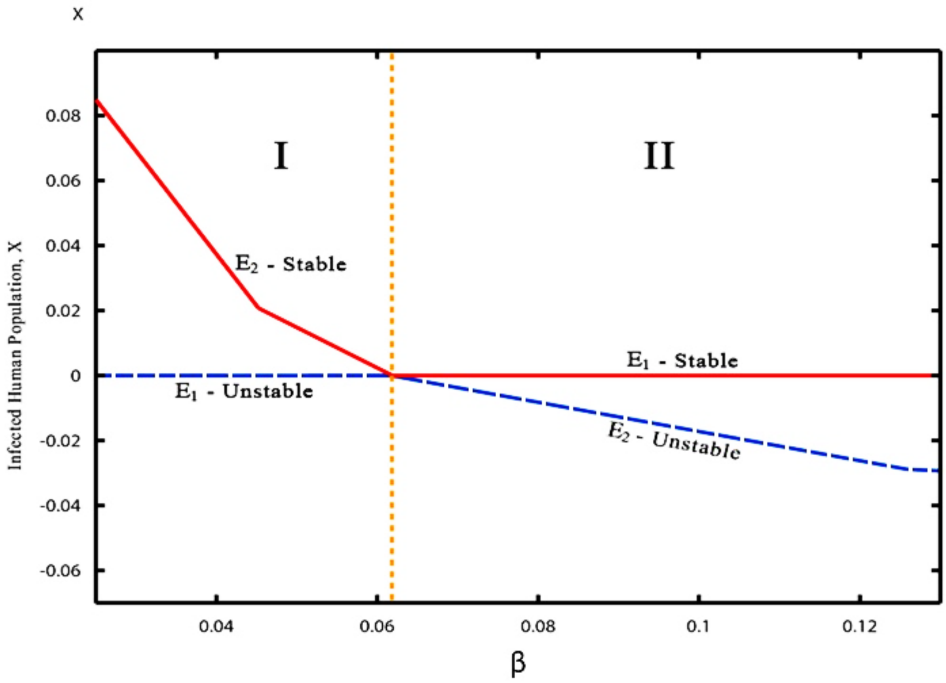
The stability of an equilibria obtained through a malaria model has been investigated in detail. According to Ndanusa and Busari [5], the equilibrium attained by an entirely healthy population where there is an absence of mosquitoes is a disease-free equilibrium. For this model, the disease-free equilibrium point is $E_1=(x^*,0)$. The stability obtained for disease-free equilibrium is stable when the value of x^* is assumed as 2 so that x^* is more than 1. It is concluded that, to be stable, the eigenvalues $\lambda_1=-0.0107$ and $\lambda_2=-0.0818$ are distinct real values with both negative signs. Also, according to the stability properties of linear systems in [5], when λ_1 is less than λ_2, and both are less than 0, $\lambda_1<\lambda_2<0$, the equilibrium point is an asymptotically stable improper node.

Next, another equilibrium point that is the endemic equilibrium point is also attained from [5]. The human population is both susceptible and infected during the state of endemic equilibrium. The endemic equilibrium point is $E_2=(x^{**},y^{**})$. The acquired stability for endemic equilibrium from computing the determinant of the Jacobian matrix is stable as the eigenvalue, $\lambda_1=-0.01511$ and $\lambda_2=-0.08239$ are distinct real eigenvalues with the same negative signs. In addition to that, the endemic equilibrium point is an asymptotically stable improper node as λ_1 is less than λ_2, and both of these eigenvalues are less than 0, $\lambda_1<\lambda_2<0$.

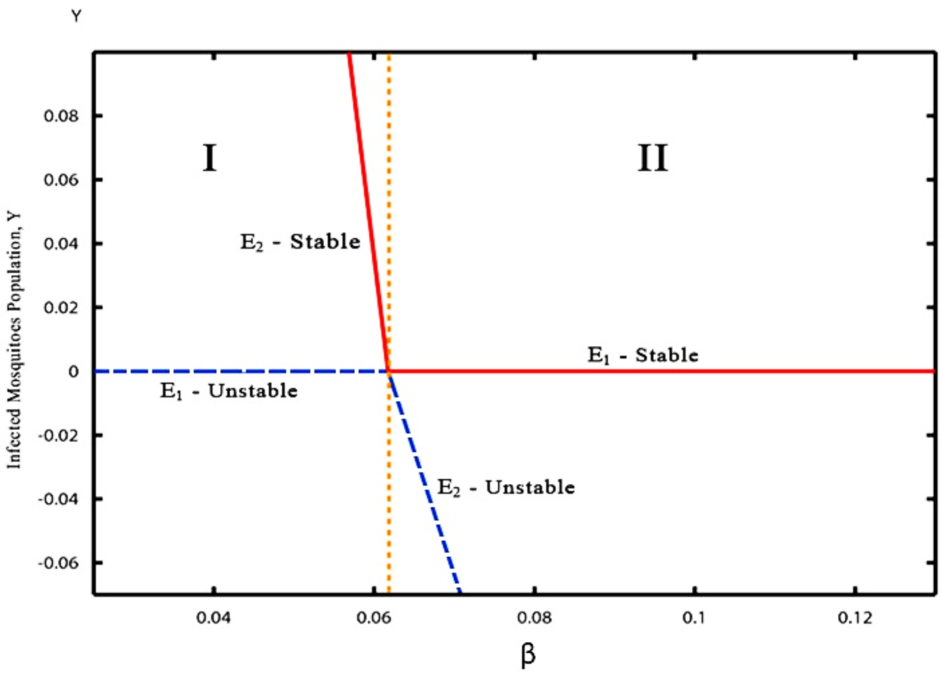
B Numerical bifurcation analysis

The behaviors of the malaria transmission model were investigated by performing bifurcation analysis using parameter variation technique. The malaria system Eq. (2) was analyzed and the bifurcation diagrams were obtained using a numerical software, XPPAUT. The process in this software began by coding the bifurcation analysis using Auto (in XPPAUT) where this will include the non-dimensionalized Eq. (2) and the parameters were set according to Table 2 for simplicity. Further steps were conducted to obtain bifurcation diagrams and bifurcation point. Since the parameter of biting rate of mosquitoes on humans needs to be varied, two other values were determined based on the bifurcation point. Consequently, the stability of the steady states, E_1 and E_2 for both values lesser than β and more than β can be analysed with the help of Maple software. We investigated the effects of per capita biting rate of mosquitoes on human population parameter, β, found in the system. Figure 1 illustrates the steady states, E_1 and E_2, in the bifurcation diagrams with respect to the per capita biting rate of mosquitoes on human parameter, β. In both diagrams, the red solid line represents the stable steady states, the blue dashed line represents the unstable steady states and the vertical orange dashed line represents the transcritical bifurcation point.

Based on Figure 1, it can be observed that there are occurrences of transcritical bifurcation where the steady state branches, E_1 and E_2, interchange with each other after passing through the bifurcation point, $\beta = 0.06173$. This is because an intermediate change in parameter β affected the stability and the equilibrium of the system. This change in stability and equilibrium are also illustrated in Table 3 where it can be seen clearly that the steady state, E_1, is an unstable saddle node and after the occurrence of transcritical bifurcation, E_2 changed into asymptotically stable node while steady state, and E_2 interchanged from asymptotically stable node to unstable saddle node after the transcritical bifurcation point.



(a)



(b)

Figure 1: Bifurcation diagrams of Eq. (2) with respect to the per capita biting rate of mosquitoes on human, β, with $\alpha = 0.003704$, $\sigma = 0.5$ and $\nu = 0.03$ for (a) infected human population X and (b) infected mosquito population Y, respectively.

Table 3: Stability and bifurcation analysis results.

Bifurcation parameter	Steady states	Eigenvalues	Characteristics	Figure
$\beta = 0.05$	$E_1 = (0,0)$	$\lambda_1 = -0.08418$ $\lambda_1 = 0.00418$	Unstable saddle node	2
	$E_2 = (0.01328, 0.21848)$	$\lambda_1 = -0.08729$ $\lambda_1 = -0.00403$	Asymptotically stable node	
$\beta = 0.0617$	Transcritical bifurcation point			
$\beta = 0.08$	$E_1 = (0,0)$	$\lambda_1 = -0.10477$ $\lambda_1 = -0.00523$	Asymptotically stable node	3
	$E_2 = (-0.01293, -0.21825)$	$\lambda_1 = -0.09776$ $\lambda_1 = 0.00561$	Unstable saddle node	

It also shows that both population densities of infected humans and infected mosquitoes decrease as mosquitoes' per capita biting rate on human population increases. This situation happens because, as the biting rate of mosquitoes increases, the mortality of mosquitoes increases, hence lesser mosquitoes are likely to survive to become infectious. Therefore, humans are also less likely to be infected when there are lesser infected mosquitoes as lesser vectors carry the parasite. This scenario can be described by the red lines in region (I) of Figure 1, wherein, in this case, both infected human and infected mosquito population depletes in number. However, when mosquitoes' per capita biting rate on human population continuously increases until it exceeds a certain threshold $\beta = 0.06173$, the steady state of E_1 and E_2 interchange with each other where eradicating both infected human and infected mosquito population occurs as illustrated in the region (II).

The results for this study are compared with Abboubakar et al. [6], where it was proven that the results in Figure 1 are logical. As the relative biting rate of mosquitoes increases, the proportion of both infectious and mosquitoes decreases. The increased biting rate of mosquitoes is cancelled by the shorter lifespan of infectious mosquitoes, which causes a decrease in the infected mosquito population. Hence, when there are lesser infected mosquitoes, humans are less likely to be infected, which then causes a decrease in the infected human population.

Furthermore, region (II) depicts an excessive per capita mosquito biting rate on human population. As stated in [6], increasing the biting rate of mosquitoes causes a decrease in the proportion of infected mosquitoes. This is because more mosquitoes are more likely to die due to their higher death rate, thus causing both the population of infected mosquitoes and infected humans to be eradicated. The phase portraits of each parameter value of β in Table 2 are plotted in Figures 2 and 3.

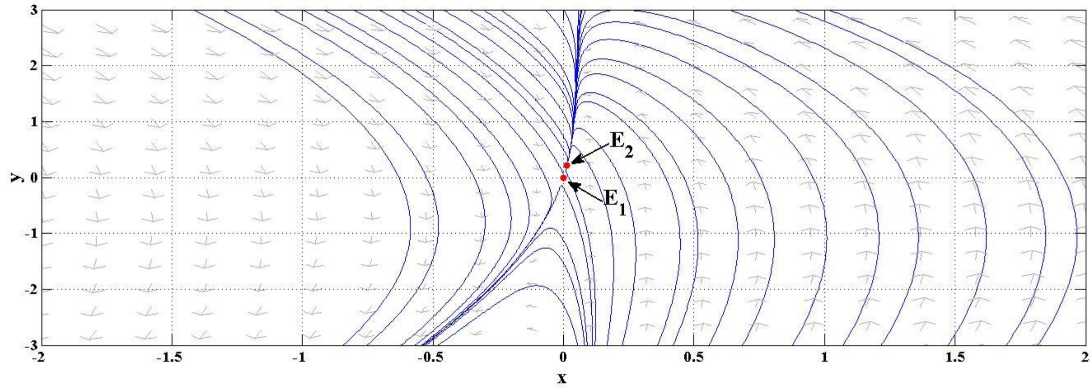


Figure 2: Phase portrait for system Eq. (2) with $\beta = 0.05$ and initial conditions $x(0) = 1$ and $y(0) = 1$. The equilibrium point E_1 is unstable and E_2 is stable.

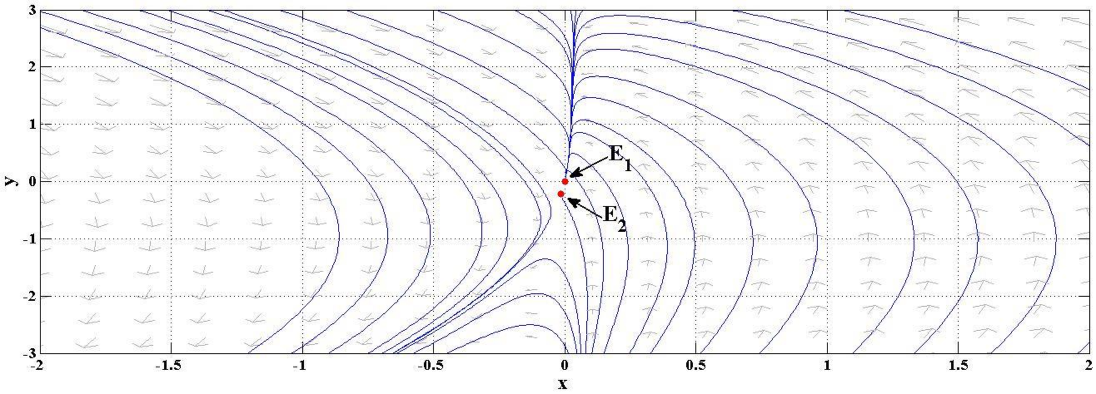


Figure 3: Phase portrait for system Eq. (2) with $\beta = 0.08$ and initial conditions $x(0) = 1$ and $y(0) = 1$. The equilibrium point E_1 is unstable and E_2 is unstable.

Based on the phase potraits in Figures 2 and 3, the occurrence of transcritical bifurcation is further proven as an interchange in stability for both E_1 and E_2 can be observed. The steady state E_1, that is initially stable in Figure 2, interchanged its stability when passing the bifurcation point $\beta = 0.06173$ and became unstable in Figure 3. Similar interchange in stability can also be observed with E_2.

Time series graphs are also plotted using Matlab software to investigate the dynamical behaviors of the system with respect to time as shown in Figures 4 and 5.

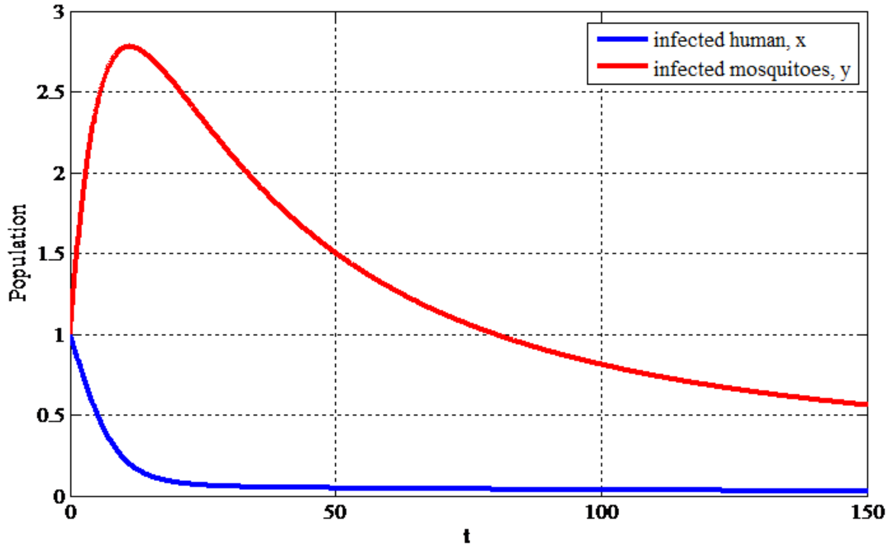


Figure 4: Time series graph of malaria model with parameter values $\alpha = 0.003704$, $\sigma = 0.5$ and $\nu = 0.03$ and initial conditions of $(x_0, y_0) = (1,1)$ with $\beta = 0.05$ (region I).

Figure 4 demonstrates that when the per capita biting rate of mosquitoes on human population parameter β is less than 0.06173, the infected mosquito population increases rapidly at first and reaches a maximum value of $t = 11.4$, before gradually decreasing in value. According to Anderson et al. [16], the decrease in the infected mosquito population is caused by an increase in the mosquito mortality rate whenever mosquitoes bite humans, as infected mosquitoes tend to bite humans more than once to obtain a full blood meal, which increases the risk of the mosquitoes being killed during feeding. However, the infected human population instantly decreases, and after a short period of time, it becomes constant, but not fully eradicated as it does not reach zero. This is the result of the increased mortality rate of mosquitoes. As the infected mosquito population decreases, the infected human population also decreases in number over time as the malaria disease is transmitted through infected mosquitoes.

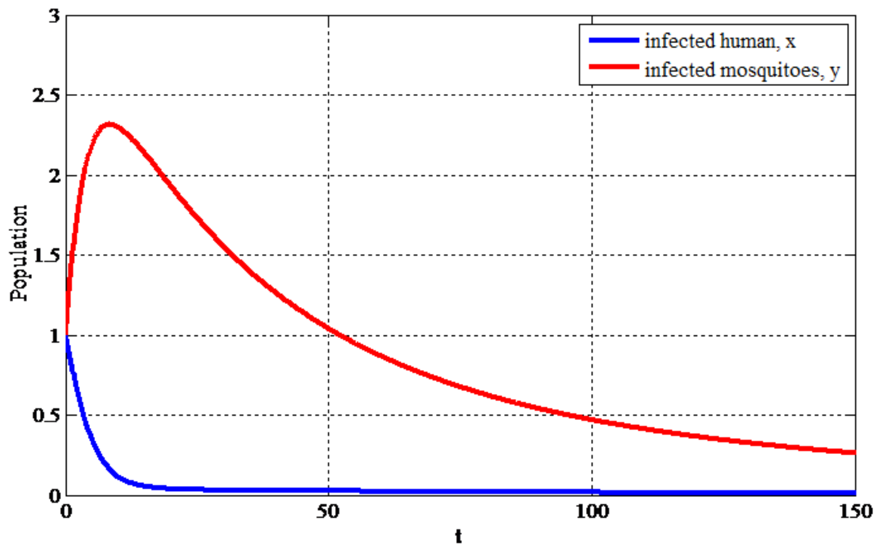


Figure 5: Time series graph of the malaria model with parameter values $\alpha = 0.003704$, $\sigma = 0.5$ and $\nu = 0.03$ and initial conditions of $(x_0, y_0) = (1,1)$ with $\beta = 0.08$ (region II).

As shown in Figure 5, the population of infected mosquitoes increases at a slower rate when the per capita biting rate of mosquitoes on human population parameter, β, exceeds 0.06173. It is also observed that the infected mosquito population increases at a slower rate as compared to Figure 4. The population of infected mosquitoes then maximizes at value $t = 8.45$, and gradually decreases in value. The decreased infected mosquito population then becomes constant after a period of time. This results from the increased mortality rate of mosquitoes, where more infected mosquitoes are more likely to die over time. A higher biting rate indicates an increase in infectious mosquito population. However, the short life span cancels the biting rate and causes a depletion in the number of infected mosquito population. The infected human population, however, instantly reduces in a short period of time and, after a while, it goes near zero and becomes extinct. The continuous reduction in the number of infected mosquito population highly affected the population of infected humans as fewer humans could be infected.

4 Conclusions

In this paper, a simple malaria transmission model that represents the important aspects of malaria epidemiology has been formulated in non-dimensionalized equations. By employing this equation, the behaviour of the system was easy to be further analysed using bifurcation analysis.

Furthermore, the effect of per capita biting rate of mosquitoes on human population, β, was analysed as it is the most essential parameter that has an influence in malaria transmission. By using the same parameter values in Table 2, the type of co-dimension one bifurcation occurring in the malaria model is the transcritical bifurcation point at $\beta = 0.06173$, as there is an exchange of stability analysis between two fixed points, E_1 and E_2. When the value of β is higher than the transcritical bifurcation point, its stability at the steady state E_1 starts out being unstable but later becomes stable. For the other steady state, E_2, the sequence of changes is the opposite. Thus, when parameter value of β increases, infected human and infected mosquito population decreases. This shows that the biting rate of mosquitoes leads to a change in both the populations of infected human and infected mosquitoes.

In general, people should take precautions wherever they go, especially travellers who travel to zones with malaria transmission occurring, in order to protect themselves from being bitten by infected mosquitoes. To illustrate, awareness of the rate of malaria cases in regions to be visited by travellers, the use of mosquito bed nets, wearing long sleeves and long pants as well as the use of insect repellent when required are several precautionary steps that could be easily implemented in people's lives to avoid the risk of being infected with malaria disease. If symptoms of this disease, such as fever during or after arriving home from travel were detected, they should seek treatment from medical professionals. Besides that, a few other steps are through a required increase in water flow and disposal of empty containers that can lead to stagnant water. These actions could eliminate the breeding sites of infected mosquitoes as well as control the transmission of malaria disease. Thus, future research should study the effect of human recovery rate that have been infected by malaria disease using bifurcation analysis. This could ensure that the strategies to eliminate the malaria disease is successful, hence, the target of malaria-free status can be achieved in the future.

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