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ON A COMPREHENSIVE CLASS OF ANALYTIC P-VALENT FUNCTIONS ASSOCIATED WITH SHELL-LIKE CURVE AND MODIFIED SIGMOID FUNCTION

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ABSTRACT

In this paper, the authors introduce and study a new class of analytic p-valent functions and its connections with some famous subclasses of analytic and univalent functions associated with shell-like curve and modified sigmoid function in the open unit disk E= {z : z |<1}. In particular, the coefficient condition for function f(z) belonging to the class Bp (λ, β) is investigated using a succinct mathematical approach. In addition, as a special case, convex functions of order 1/4 are shown to be in the aforementioned class Bp(λ, β) in E. With the aid of subordination pri nciple, the authors obtain the first three Taylor-Maclaurin coefficients |ap+1 |, |ap+2 | and |ap+3 | as well as the Fekete-Szegö functional |ap+2 -ηa2p+1| for functions f(z) belonging to the class Bp(λ, β, σ;p̃) involving modified sigmoid function and associated with shell-like curve.

Keywords: *Analytic Function, Bounded Turning Function, Convex Function, Starlike Function, Univalent Function.*

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1. Introduction

Let the function $f(z)$ be analytic in the open unit disk $E=\{z : z|<1\}$. Also let A denote the class of all analytic function $f(z)$ having the form

$$f(z)=z+\sum_{k=2}^{\infty} a_k z^k \tag{1}$$

in *E*. Suppose that S denote the class of all functions in A which are univalent in E. Then the function $f(z)\in A$ is said to be starlike, convex and bounded turning function of order β respectively, if the following geometric conditions are satisfied:

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$$\Re e\left\{\frac{z\,f'(z)}{f(z)}\right\} > \beta, \qquad (z \in E), \qquad (0 \le \beta < 1) \tag{2}$$

$$\Re e\left\{1 + \frac{z\,f''(z)}{f'(z)}\right\} > \beta, \ (z \in E), \qquad (0 \le \beta < 1) \tag{3}$$

and

$$\Re e\{f'(z)\} > \beta, \qquad (z \in E), \qquad (0 \le \beta < 1) \tag{4}$$

Here, we denote the classes of starlike, convex and bounded turning functions of order β by $S^*(\beta)$, $C(\beta)$ and $R(\beta)$ respectively, where

$$C(\beta) \subset S^*(\beta) \subset S \tag{5}$$

For brevity, let A_p denote the class of all analytic *p*-valent function of the form

$$f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad p \in N\,(set\ of\ all\ natural\ numbers\,). \tag{6}$$

A function $f(z)$ is said to be analytic *p*-valent in the open unit disk *E*, if it is analytic and assumes no value more than p-times for $|z| < 1$.

2. Methodology

In this article, an analytic *p-valent* function is used to model two classes of analytic functions namely: $B_p(\lambda, \beta)$ and $B_p(\lambda, \beta, \sigma; \widetilde{p})$ of shell-like curve. Differentiation, binomial expansion and subordination principle through the modified sigmoid function are used in order to obtain the sharp bounds on the first three Taylor-Maclaurin coefficients $|a_{p+1}|$, $|a_{p+2}|$, $|a_{p+3}|$, and the Fekete-Szegö functional $|a_{p+2} - \eta a_{p+1}^2|$. Also, we employ logarithmic differentiation and certain Lemma due to Jack (1971) to established the sufficient condition for functions $f(z)$ of the form (6) to be in the class $B_p(\lambda, \beta)$.

Now for function f and g in *E*, there exists a function μ with the condition that

$$\mu(0) = 0, \qquad |\mu(z)| < 1 \text{ and } \ f(z) = g(\mu(z)),\ z \in E.$$

Then we say that f is subordinate to g and it is denoted mathematically by

$$f \prec g, \ \ z \in E.$$

In particular, when g is univalent in E, then

$$f \prec g \Leftrightarrow f(0) = g(0) \text{ and } f(E) \subset g(E).$$

For recent studies on subordination, refer to (Hamzat & El-Ashwah, 2020). As usual, let P denote the class of all Caratheodory functions having the form

$$q(z) = 1 + \sum_{k=1}^{\infty} c_k z^k, \qquad z \in E \tag{7}$$

which are analytic such that $\Re e\{q(z)\} > 0$, see (Duren, 1983). It is necessary to note that the correspondence between the class of Caratheodory functions P and the class of Schwarz functions (functions with unit bound) w, exists and well known. That is,

$$q(z) = \frac{1 + w(z)}{1 - w(z)}, \qquad q(z) \in \mathrm{P} \tag{8}$$

In the recent time, the classes $SL(\widetilde{p})$ and $KSL(\widetilde{p})$, of starlike shell-like and convex shell-like functions, which are characterized by

$$\frac{z f'(z)}{f(z)} \prec \widetilde{p}(z) = \frac{1 + \sigma^2 z^2}{1 - \sigma z - \sigma^2 z^2}, \qquad \sigma = \frac{1 - \sqrt{5}}{2} \tag{9}$$

and

$$1 + \frac{z f''(z)}{f'(z)} \prec \widetilde{p}(z) = \frac{1 + \sigma^2 z^2}{1 - \sigma z - \sigma^2 z^2}, \qquad \sigma = \frac{1 - \sqrt{5}}{2} \tag{10}$$

have been studied by different authors, see among others (Dziok*et al*., 2011a & 2011b; Orhan *et al*., 2020; Raina & Sokol, 2016; Sokol, 1999). However, the function $\widetilde{p}(z)$ is non-univalent in E but univalent in the disk $|z| = \frac{3 - \sqrt{5}}{2} \approx 0.38$. The image of the circle $|z| = 1$ under $\widetilde{p}(z)$ is a curve represented by the equation:

$$\left(10x - \sqrt{5}\right) y^2 = \left(\sqrt{5} - 2x\right)\left(\sqrt{5x} - 1\right)^2$$

which translated and revolved trisectrix of Maclaurin. Raina &Sokol, (2016) showed that

$$\widetilde{p}(z) = 1 + \sum_{k=2}^{\infty} \left(u_{n-1} + u_{n+1}\right) \sigma^k z^k \tag{11}$$

where $u_k = \frac{(1-\sigma)^k - \sigma}{\sqrt{5}}$ and $\sigma = \frac{1 - \sqrt{5}}{2}$, $k = 1, 2, 3, \ldots$.

Obviously, the result in (11) above has established the relationship between the function $\widetilde{p}(z)$ and the sequence of Fibonacci numbers u_k, such that

$$u_0 = 0,\ u_1 = 1,\ u_{k+2} = u_k + u_{k+1}, \qquad k = 0,\ 1,\ 2,\ \dots \quad .$$

Therefore, we can write that

$$\begin{aligned}\widetilde{p}(z) &= 1 + \sum_{k=1}^{\infty} \widetilde{p}_k\, z^k \\ &= 1 + (u_0 + u_2)\sigma\, z + (u_1 + u_3)\sigma^2 z^2 + \sum_{k=3}^{\infty} (u_{k-3} + u_{k-2} + u_{k-1} + u_k)\sigma^k z^k \\ &= 1 + \sigma\, z + 3\sigma^2 z^2 + 4\sigma^3 z^3 + 7\sigma^4 z^4 + 11\sigma^5 z^5 + \dots\ .\end{aligned} \tag{12}$$

The theory of special function (logistic sigmoid function) has found its application in many physical problems such as in aerodynamics, thermodynamics and electrostatic potential to mention just a few. Logistic activation function is an information system that is inspired by the way nervous systems like the brain processes information. the most widely used sigmoid function is the logistic function which has the following series denotation

$$\varphi(z) = \frac{1}{1 - e^{-z}} = \frac{1}{2} + \frac{1}{4}z - \frac{1}{48}z^3 + \frac{1}{480}z^5 - \dots$$

and with the following properties Fadipe-Joseph *et al.* (2013) and Oladipo & Gbolagade (2014):

(i) it outputs real numbers between 0 and 1

(ii) it maps a very large input domain to small range of outputs

(iii) it never losses information because it is a one-to-one function

(iv) it increases monotonically.

Recently, the modified sigmoid function $\phi(z)$ was defined such that

$$\phi(z) = 2\varphi = \frac{2}{1 - e^{-z}},$$

was shown to belongs to the family of Caratheodory function P. Interestingly, it has the series representation

$$\phi(z) = 1 + \frac{1}{2}z - \frac{1}{24}z^3 + \frac{1}{240}z^5 - \dots \tag{13}$$

see also Hamzat (2017) and Hamzat &Olayiwola (2017).In the present work, the following definitions shall be necessary.

Definition 2.1: The function $f(z) \in A_p$ is said to belong to the class $B_p(\lambda, \beta, \sigma; \widetilde{p})$, $p \in N,\ 0 \le \lambda \le 1,\ 0 \le \beta < 1,\ \sigma = \frac{1-\sqrt{5}}{2}$, of analytic *p*-valent functions, if the following geometric condition is satisfied:

$$\frac{\left(\frac{zf'(z)}{p\,z^p}\right)\left(\frac{z^p}{f(z)}\right)^{\lambda} - \beta}{1-\beta} \prec \widetilde{p}(z) = \frac{1+\sigma\ z^2}{1-\sigma\, z+\sigma^2 z^2},\ (z \in E) \tag{14}$$

Definition 2.2: The function $f(z) \in A_p$ is said to belong to the new class $B_p(\lambda, \beta)$, $p \in N,\ 0 \le \lambda \le 1,\ 0 \le \beta < 1$, of analytic *p*-valent functions, if it satisfies the condition that

$$\left|\left(\frac{zf'(z)}{p\,z^p}\right)\left(\frac{z^p}{f(z)}\right)^{\lambda} - 1\right| < 1-\beta, \qquad (z \in E) \tag{15}$$

For various choices of the parameters p, λ and β, the family $B_p(\lambda, \beta)$, yield known classes of analytic functions as listed below:

(a) If we set $\lambda = 1$ in (15), the following class of analytic functions is obtained

$$B_p(1, \beta) = S_p^*(\beta).$$

(b) Let $p = 1$ in (15), then the following class of analytic functions is obtained

$$B_1(\lambda,\ \beta) = B(\lambda,\ \beta).$$

This class, $B_1(\lambda,\ \beta) = B(\lambda,\ \beta)$ defined in (b), is due to (Frasin & Jahangiri, 2009).

(c) If we set $\lambda = 0$ in (15), the following class of analytic functions is obtained

$$B_p(0,\ \beta) = R_p(\beta).$$

(d) If we set $\lambda = 0$ and $p = 1$ in (15), then we obtain the following

$$B_1(0, \beta) = R_1(\beta).$$

See Frasin & Darus (2001) and Nunokawa (1995) for more details on the class of analytic function defined in (d).

3. Main Results

In this section, sufficient conditions for functions $f(z)$ of the form (6) to be in the class $B_p(\lambda, \beta)$ are established using a succinct. In addition, as a special case, convex functions of order $\frac{1}{4}$ are shown to be in the aforementioned family $B_p(\lambda, \beta)$ in *E*. Also, coefficient bounds for functions belonging to the class $B_p(\lambda, \beta, \sigma; \widetilde{p})$, associated with shell-like curves are considered. However, before proceeding to the main results, the following Lemmas shall be necessary, the foremost is due to (Jack, 1971). See also Darus *et al.* (2015) and Murugusundaramoorthy & Magesh (2011) among others.

Lemma 2.1 (Jack, 1971): Let $\omega(z)$ be analytic in E such that $\omega(0) = 0$. Suppose that $|\omega(z)|$ attains it maximum value on the circle $|z| = r < 1$ at a point $z_0 \in E$, then $z_0\omega'(z_0) = k\omega(z_0)$ for $k \geq 1$.

Lemma 2.2: Let $q(z)$ be analytic in E with $q(0) = 1$ and suppose that

$$\Re e\left\{1 + \frac{z\,q'(z)}{q(z)}\right\} > \frac{4\beta\rho + 8\beta - \rho - 1}{4\beta(\rho + 1)}, \qquad (z \in E) \tag{16}$$

Then $\Re e\{q(z)\} > \beta$ for $\frac{1}{4} \leq \beta < 1$ and $0 < \rho \leq 1$.

Proof: Assume

$$q(z) = \frac{1 + (1 - 4\beta)\omega(z)}{1 - \rho\omega(z)}, \qquad \left(\omega(z) \neq \frac{1}{\rho}\right). \tag{17}$$

Then $\omega(z)$ is analytic in E and $\omega(0) = 0$. Using Logarithmic differentiation, we obtain

$$\frac{q'(z)}{q(z)} = \frac{(1 - 4\beta)\,\omega'(z)}{1 + (1 - 4\beta)\,\omega(z)} + \frac{\rho\,\omega'(z)}{1 - \rho\omega(z)}. \tag{18}$$

Let there exists a point $z_0 \in E$ such that

$$\max_{|z| < |z_0|} |\omega(z)| = |\omega(z_0)| = 1, \qquad \left(\omega(z_0) \neq \frac{1}{\rho}\right). \tag{19}$$

Then, it follows from Lemma 2.1 that

$$z_0\omega'(z_0) = k\omega(z_0), \qquad (k \geq 1). \tag{20}$$

Suppose that $\omega(z_0)=e^{i\phi}(\phi\neq 0)$, then

$$\Re e\left\{1+\frac{z_0\,q'(z_0)}{q(z_0)}\right\}=1-\frac{\rho\,k}{\rho+\cos\phi}+\frac{(1-4\beta)\,k\,[1-4\beta+\cos\phi]}{2\,(1-4\beta)\cos\phi+1+(1-4\beta)^2}$$
$$\leq 1-\frac{\rho\,k}{\rho+1}-\frac{(1-4\beta)\,k}{4\beta}\qquad .$$
$$\leq \frac{4\beta\,\rho+8\beta-\rho-1}{4\beta(\rho+1)}.$$

This contradicts the hypothesis of Lemma 2.2. Hence, we have $|\omega(z)|<1$ in E and therefore $\Re e\{q(z)\}>\beta$ for $z\in E$ and this ends the proof.

Theorem 2.3: Suppose that $f(z)\in A_p$. If

$$\Re e\left\{\left(1+\frac{z\,f''(z)}{f'(z)}\right)+\lambda\left(p-\frac{z\,f'(z)}{f(z)}\right)\right\}>\frac{(4\,\beta\,p-1)(\rho+1)+4\beta}{4\beta(\rho+1)},\quad (z\in E) \tag{21}$$

then $f\in B_p(\lambda,\beta)$, where $\lambda\geq 0$, $p\in N$, $\frac{1}{4}\leq\beta<1$ and $0<\rho\leq 1$.

Proof: Let

$$q(z)=\left(\frac{zf'(z)}{p\,z^p}\right)\left(\frac{z^p}{f(z)}\right)^{\lambda}. \tag{22}$$

Then, $q(z)$ is analytic in E with $q(0)=1$. It follows from (22) that

$$\frac{q'(z)}{q(z)}=\lambda\left(p-\frac{z\,f'(z)}{f(z)}\right)+\left(\frac{z\,f''(z)}{f'(z)}-p+1\right). \tag{23}$$

and

$$1+\frac{q'(z)}{q(z)}=1+\lambda\left(p-\frac{z\,f'(z)}{f(z)}\right)+\left(\frac{z\,f''(z)}{f'(z)}-p+1\right) \tag{24}$$

Using Lemma 2.2 in (24), we can conclude that

$$\Re e\left\{\left(\frac{zf'(z)}{p\,z^p}\right)\left(\frac{z^p}{f(z)}\right)^{\lambda}\right\}>\beta,\qquad (z\in E)\ . \tag{25}$$

Therefore, we say that $f\in B_p(\lambda,\beta)$ and this completes the proof of Theorem 2.3. At this juncture, it is noteworthy to state some of the consequences of the above result.

Corollary 2.4: Suppose that $f(z) \in A_p$. If

$$\Re e\left\{\left(1+\frac{z f''(z)}{f'(z)}\right)+\left(p-\frac{z f'(z)}{f(z)}\right)\right\} > \frac{(p-1)(\rho+1)+1}{(\rho+1)},$$

then

$$\Re e\left\{\frac{zf'(z)}{p\, f(z)}\right\} > \frac{1}{4} \qquad (z \in E)\cdot$$

Therefore $f \in B_p\left(1, \frac{1}{4}\right)$.

Corollary 2.5: Suppose that $f(z) \in A_p$. If

$$\Re e\left\{\left(1+\frac{z f''(z)}{f'(z)}\right)+\left(p-\frac{z f'(z)}{f(z)}\right)\right\} > \frac{2p-1}{2},$$

then

$$\Re e\left\{\frac{zf'(z)}{p\, f(z)}\right\} > \frac{1}{4} \qquad (z \in E)\cdot$$

Therefore, $f \in B_p\left(1, \frac{1}{4}\right)$.

Corollary 2.6: Suppose that $f(z) \in A_1 = A$. If

$$\Re e\left\{\left(1+\frac{z f''(z)}{f'(z)}\right)+\left(1-\frac{z f'(z)}{f(z)}\right)\right\} > \frac{1}{2},$$

then

$$\Re e\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{4} \qquad (z \in E).$$

Therefore, $f \in B_1\left(1, \frac{1}{4}\right)$. That is, $f(z)$ is starlike of order $\frac{1}{4}$.

Corollary 2.7: Suppose that $f(z) \in A_p$. If

$$\Re e\left\{\left(1+\frac{z f''(z)}{f'(z)}\right)+\left(p-\frac{z f'(z)}{f(z)}\right)\right\} > \frac{(2p-1)+(\rho+1)+2}{2(\rho+1)},$$

then

$$\Re e\left\{\frac{zf'(z)}{p\, f(z)}\right\} > \frac{1}{2} \qquad (z \in E)\cdot$$

Therefore $f \in B_p\left(1, \frac{1}{2}\right)$.

Corollary 2.8: Suppose that $f(z) \in A_p$. If

$$\Re e\left\{\left(1+\frac{z f''(z)}{f'(z)}\right)+\left(p-\frac{z f'(z)}{f(z)}\right)\right\} > p,$$

then

$$\Re e\left\{\frac{zf'(z)}{p\, f(z)}\right\} > \frac{1}{2} \qquad (z \in E)\cdot$$

Therefore, $f \in B_p\left(1, \frac{1}{2}\right)$.

Corollary 2.9: Suppose that $f(z) \in A_1 = A$. If

$$\Re e\left\{\left(1+\frac{z f''(z)}{f'(z)}\right)+\left(1-\frac{z f'(z)}{f(z)}\right)\right\} > 1,$$

then

$$\Re e\left\{\frac{zf'(z)}{f(z)}\right\} > \frac{1}{2} \qquad (z \in E).$$

Therefore, $f \in B_1\left(1, \frac{1}{2}\right)$. That is, $f(z)$ is starlike of order $\frac{1}{2}$.

Corollary 2.10: Suppose that $f(z) \in A_p$. If

$$\Re e\left\{1+\frac{z f''(z)}{f'(z)}\right\} > \frac{(p-1)(\rho+1)+1}{(\rho+1)},$$

then

$$\Re e\left\{\frac{zf'(z)}{p\, z^p}\right\} > \frac{1}{4} \qquad (z \in E)\cdot$$

Therefore

$$f \in B_p\left(0, \frac{1}{4}\right).$$

Corollary 2.11: Suppose that $f(z) \in A_p$. If

$$\Re e\left\{1+\frac{z\,f''(z)}{f'(z)}\right\}>\frac{2p-1}{2},$$

then

$$\Re e\left\{\frac{zf'(z)}{p\,z^{p}}\right\}>\frac{1}{4} \qquad (z\in E).$$

Therefore, if $f(z)$ is convex of order $\frac{1}{2}$, then $f\in B_p\left(0,\frac{1}{4}\right)$.

Corollary 2.12: Suppose that $f(z)\in A_1=A$. If

$$\Re e\left\{1+\frac{z\,f''(z)}{f'(z)}\right\}>\frac{1}{2},$$

then

$$\Re e\{f'(z)\}>\frac{1}{4}\ (z\in E).$$

Therefore, if $f(z)$ is convex of order $\frac{1}{2}$, then $f\in B_1\left(1,\frac{1}{4}\right)=R_{\frac{1}{4}}$.

Corollary 2.13: Suppose that $f(z)\in A_p$. If

$$\Re e\left\{1+\frac{z\,f''(z)}{f'(z)}\right\}>\frac{(2p-1)(\rho+1)+2}{2(\rho+1)},$$

then

$$\Re e\left\{\frac{zf'(z)}{p\,z^{p}}\right\}>\frac{1}{2} \qquad (z\in E)\cdot$$

Therefore $f\in B_p\left(0,\frac{1}{2}\right)$.

Corollary 2.14: Suppose that $f(z)\in A_p$. If

$$\Re e\left\{1+\frac{z\,f''(z)}{f'(z)}\right\}>p,$$

then

$$\Re e\left\{\frac{zf'(z)}{p\,z^{p}}\right\}>\frac{1}{2} \qquad (z\in E)\cdot$$

Therefore $f \in B_p\left(0, \frac{1}{2}\right)$. That is convex functions of order p.

Corollary 2.15: Suppose that $f(z) \in A_p$. If

$$\Re e\left\{1+\frac{z f''(z)}{f'(z)}\right\} > 1,$$

then

$$\Re e\{f'(z)\} > \frac{1}{2} \ (z \in E).$$

Therefore, if $f(z)$ is convex of order 1, then $f \in B_1\left(0, \frac{1}{2}\right) = R_{\frac{1}{2}}$. The next set of results include the bounds on the Fekete-Szego functional $\left|a_{p+2} - \eta\, a_{p+1}^2\right|$ and the first three Taylor-Maclaurin coefficients for functions belonging to the class $B_p(\lambda, \beta, \sigma; \widetilde{p})$ related to shell-like curves. For recent work on Fekete-Szego problems, refer to Hamzat & Sangoniyi, (2021); Orhan *et al*. (2020) and Sokol (1999) among others.

Theorem 2.16: Let the function $f(z)$ be of the form (6). If $f(z)$ belongs to the class $B_p(\lambda, \sigma; \widetilde{p})$, $p \in N$, $0 \le \lambda \le 1$, $0 \le \beta < 1$, $\sigma = \frac{1-\sqrt{5}}{2}$, then

$$\left|a_{p+1}\right| \le \frac{|\sigma|(1-\beta)}{4\Phi_1},$$

$$\left|a_{p+2}\right| \le \frac{|\sigma|(1-\beta)\left\{|\sigma|(1-\beta)\left[3\Phi_1^2 + \lambda(1-\beta)\left(\frac{p+1}{p} - \frac{\lambda+1}{2}\right)\right] + \Phi_1^2\right\}}{16\Phi_1^2\Phi_2},$$

$$\left|a_{p+3}\right| \le \frac{|\sigma|(1-\beta)\left\{3|\sigma|^2\left[A-B\right] + C + D\right\}}{192\Phi_1^3\Phi_2\Phi_3},$$

$$\left|a_{p+2} - \eta a_{p+1}^2\right| \le \frac{|\sigma|(1-\beta)\left\{|\sigma|(1-\beta)\left[3\Phi_1^2 + \lambda(1-\beta)\left(\frac{p+1}{p} - \frac{\lambda+1}{2}\right) - \eta\Phi_2\right] + \Phi_1^2\right\}}{16\Phi_1^2\Phi_2}$$

for complex number η , where

$$A=\Phi_1\left[4\Phi_1^2\Phi_2+\lambda(1-\beta)^2\left(\frac{p+2}{p}+\frac{p+1}{p}-(\lambda+1)\right)\left(3\Phi_1^2+\lambda(1-\beta)\left(\frac{p+1}{p}-\frac{\lambda+1}{2}\right)\right)\right].$$

$$B=\lambda(1-\beta)^2\Phi_1\left(\frac{\lambda+1}{2}\right)\left(\frac{p+1}{p}-\frac{\lambda+2}{3}\right),$$

$$C=3|\sigma|\Phi_1^3\left[6\Phi_2+\lambda(1-\beta)\left(\frac{p+2}{p}+\frac{p+1}{p}-(\lambda+1)\right)\right],$$

$$D=4\Phi_1^3\Phi_2 \text{ and } \Phi_i=\left(\frac{p+i}{p}-\lambda\right),\ i=1,2,3,\dots\ .$$

Proof: Suppose that $f(z)\in B_p(\lambda,\ \sigma;\ \widetilde{p})$, then from Definition 1.1, we have that

$$\frac{\left(\frac{zf'(z)}{pz^p}\right)\left(\frac{z^p}{f(z)}\right)^{\lambda}-\beta}{1-\beta}=\widetilde{p}(\phi(z)), \qquad (z\in E). \tag{26}$$

Since $\phi(z)$ is of the form (13) and belongs to class P of Caratheodory functions such that $\phi(z)\prec\widetilde{p}(z)$, then there exists an analytic function $\phi(z)$ with $\phi(0)=1$ and $|\phi(z)|<1$ such that $\phi(z)=\widetilde{p}(\phi(z))$. In view of this we define the function $h(z)$ such that

$$h(z)=\frac{1+\phi(z)}{1-\phi(z)}=1+\frac{1}{2}z-\frac{1}{24}z^3+\frac{1}{240}z^5-\frac{1}{64}z^6+\dots\ . \tag{27}$$

It follows from (27) that

$$\phi(z)=\frac{h(z)-1}{h(z)+1}=\frac{1}{4}z-\frac{1}{16}z^2-\frac{1}{48}z^3-\frac{1}{192}z^4+\dots$$

And

$$\widetilde{p}(\phi(z))=1+\frac{1}{4}\sigma z+\sigma\left(\frac{3}{16}\sigma-\frac{1}{16}\right)z^2+\sigma\left(\frac{1}{16}\sigma^2-\frac{3}{32}\sigma-\frac{1}{48}\right)z^3+\dots\ . \tag{28}$$

Also,

$$\left(\frac{zf'(z)}{pz^p}\right)\left(\frac{z^p}{f(z)}\right)^{\lambda}=1+\left(\frac{p+1}{p}-\lambda\right)a_{p+1}z+\left[\left(\frac{p+2}{p}-\lambda\right)a_{p+2}-\left(\frac{p+1}{p}-\frac{\lambda+1}{2}\right)\lambda a_{p+1}^2\right]z^2$$
$$+\left[\begin{array}{l}\left(\frac{p+3}{p}-\lambda\right)a_{p+3}-\left(\frac{p+1}{p}+\frac{p+2}{p}-(\lambda+1)\right)\lambda a_{p+1}a_{p+2}\\ +\left(\frac{\lambda(\lambda+1)}{2}\frac{p+1}{p}-\frac{\lambda(\lambda+1)(\lambda+2)}{6}\right)a_{p+1}^3\end{array}\right]z^3 \tag{29}$$
$$+\ \ldots$$

Therefore, in view of (26), (28) and (29), we obtain

$$\left(\frac{p+1}{p}-\lambda\right)a_{p+1}=\frac{\sigma(1-\beta)}{4}, \tag{30}$$

$$\left(\frac{p+2}{p}-\lambda\right)a_{p+2}-\lambda\left(\frac{p+1}{p}-\frac{\lambda+1}{2}\right)a_{p+1}^2=\frac{\sigma(3\sigma-1)(1-\beta)}{16}, \tag{31}$$

$$\left(\frac{p+3}{p}-\lambda\right)a_{p+3}-\left(\frac{p+1}{p}+\frac{p+2}{p}-(\lambda+1)\right)\lambda a_{p+1}a_{p+2}+\left(\frac{\lambda(\lambda+1)}{2}\frac{p+1}{p}-\frac{\lambda(\lambda+1)(\lambda+2)}{6}\right)a_{p+1}^3$$
$$=\frac{(6\sigma^3-9\sigma^2-2\sigma)(1-\beta)}{96} \tag{32}$$

and

$$a_{p+2}-\eta a_{p+1}^2=\frac{\sigma(1-\beta)\left\{\sigma(1-\beta)\left[3\left(\frac{p+1}{p}-\lambda\right)^2+\lambda(1-\beta)\left(\frac{p+1}{p}-\frac{\lambda+1}{2}\right)-\eta\left(\frac{p+1}{p}-\lambda\right)\right]+\left(\frac{p+1}{p}-\lambda\right)^2\right\}}{16\left(\frac{p+1}{p}-\lambda\right)^2\left(\frac{p+2}{p}-\lambda\right)}. \tag{33}$$

From (30), (31), (32) and (33), we obtain the desired results as contained in Theorem 2.16 and this ends the proof.

Corollary 2.17: Let the function $f(z)$ be of the form (6). If $f(z)$ belongs to the class $B_p(0, 0, \sigma;\, \tilde{p})$, then

$$\left|a_{p+1}\right|\le\frac{p|\sigma|}{4(p+1)},\qquad \left|a_{p+2}\right|\le\frac{p|\sigma|\{3|\sigma|+1\}}{16(p+2)},\qquad \left|a_{p+3}\right|\le\frac{p|\sigma|\{6|\sigma|^2+9|\sigma|+2\}}{96(p+3)}$$

and

$$\left|a_{p+2}-\eta a_{p+1}^2\right|\le\frac{p|\sigma|\{|\sigma|[3(p+1)^2-p\eta(p+2)]+(p+1)^2\}}{16(p+1)^2(p+2)}.$$

Corollary 2.18: Let the function $f(z)$ be of the form (6). If $f(z)$ belongs to the class $B_1(0, 0, \sigma;\, \tilde{p})$, then

$$|a_2| \le \frac{|\sigma|}{8}, \qquad |a_3| \le \frac{|\sigma|\{3|\sigma|+1\}}{48}, \qquad |a_{p+3}| \le \frac{|\sigma|\{6|\sigma|^2+9|\sigma|+2\}}{96(4)}$$

and

$$\left|a_{p+2} - \eta\, a_{p+1}^2\right| \le \frac{|\sigma|\{3|\sigma|[4-\eta]+4\}}{48(4)}.$$

Corollary 2.19: Let the function $f(z)$ be of the form (6). If $f(z)$ belongs to the class $B_1\left(0,\, 0,\, \frac{1-\sqrt{5}}{2};\, \widetilde{p}\right)$, then

$|a_2| \le 0.0773$, $|a_3| \le 0.0367$, $|a_4| \le 0.0159$ and $\left|a_3 - \eta\, a_2^2\right| \le 0.0367 + 0.0060\eta$.

Corollary 2.20: Let the function $f(z)$ be of the form (6). If $f(z)$ belongs to the class $B_1(1,\, 0,\, \sigma;\, \widetilde{p})$, then

$$|a_2| \le \frac{|\sigma|}{4}, \qquad |a_3| \le \frac{|\sigma|\{4\sigma+1\}}{32}, \qquad |a_4| \le \frac{|\sigma|\{57|\sigma|^2+45|\sigma|+8\}}{192(6)}$$

and

$$\left|a_3 - \eta\, a_2^2\right| \le \frac{|\sigma|\{2|\sigma|(2-\eta)+1\}}{32}.$$

Corollary 2.21: Let the function $f(z)$ be of the form (6). If $f(z)$ belongs to the class $B_1\left(1,\, 0, \frac{1-\sqrt{5}}{2};\, \widetilde{p}\right)$, then

$|a_2| \le 0.1545$, $|a_3| \le 0.0671$, $|a_4| \le 0.0309$ and $\left|a_3 - \eta\, a_2^2\right| \le 0.0671 + 0.02387\eta$.

4. Conclusion

Ultimately, it is noteworthy to state that one of the prime significance of the sharp bounds obtained for the initial coefficients $|a_{p+1}|$, $|a_{p+2}|$ and $|a_{p+3}|$ for function $f(z)$ in the class $B_p(\lambda,\, \beta,\, \sigma;\, \widetilde{p})$ is the information about their geometric properties. For instance, the bounds can be used in the discussion of Hankel determinants. In a general sense, these bounds help in putting information into a special code in order to prevent the third party from looking into the information without permission (i.e., data encryption) among others.

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References

Darus M., Aldawish I., & Ibrahims R. M. (2015). Some concavity properties of general integral operators. *Bull. Iranian Math. Soc.,* 41(5), 1085-1092.

Duren P. L. (1983). Univalent functions. *Grundlehren der Mathematischen Wissenschaften Series, 259.* Springer Verlag, New York.

Dziok J., Raina R. K. and Sokol J. (2011a). Certain results for a class of convex functions related to a shell-like curve connected with Fibonacci numbers. *Comput. Math. Appl.,* 61, 2605-2613.

Dziok J., Raina R. K., & Sokol J. (2011b). On $\alpha-$convex functions related to a shell-like curve connected with Fibonacci numbers. *Appl. Math. Comput.,* 218, 996-1002.

Fadipe-Joseph O. A., Oladipo A. T., & Ezeafulukwe U. A. (2013). Modified sigmoid function in univalent theory. *Int. J. Math. Sci. Engr. Appl.,* 7, 313-317.

Frasin B. A., & Jahangiri J. M. (2009). A new and comprehensive class of analytic functions. *Analele Universitatii Oradea, Fasc. Mathematica, Tom XV,* 59-62.

Frasin B. A., & Darus M. (2001). On certain analytic univalent functions. *Int. J. Math. and Math. Sci.,* 25(5), 305-310.

Hamzat J. O., & El-Ashwah R. M. (2020). Application of a subordination theorem associated with certain new generalized subclasses of analytic and univalent functions. *J. Egyptian Math. Soc.* 28:33, 1-10.

Hamzat J. O. (2017). Coefficient bounds for Bazilevic functions associated with modified sigmoid function. *Asian Research J. Math.,* no.33818, 5(3), 1-10.

Hamzat J. O., & Olayiwola M.A. (2017). Initial coefficient estimates for certain generalized class of analytic functions involving sigmoid function. *Asian Research J. Math., no.33819*, 5(3), 1-11.

Hamzat J. O., & Sangoniyi S. O. (2021). Fekete-Szego inequalities associated with certain generalized class of non-Bazilevic function. *Int. J. Math. Sci. Opt.: Theory and Applications.* 6 (2), 874-885.

Jack I. S. (1971). Function starlike and convex of order α. *J. London Math. Soc*. 3, 469-474.

Murugusundaramoorthy G., & Magesh N. (2011). On certain sufficient conditions for analytic univalent functions. *EUROPEAN J. of Pure and Appl. Math.,* 4(1), 76-82.

Nunokawa M. (1995). On some angular estimates of analytic functions. *Math. Japon,* 41(2), 447-452. MR 96c:30013.zbl.30014.

Oladipo A. T., & Gbolagade A. M. (2014). Subordination results for logistic sigmoid activation function in the space of univalent functions in the unit disk, *Adv. Comp. Sci. Engr.* 12(2), 61-79.

Orhan H., Magesh N., & Abirami C. (2020). Fekete-Szego problem for bi-Bazilevic functions related to shell-like curves. *AIMS Math*, 5(5): 4412-4423. DOI:10.3934/math.2020281

Raina R. K., & Sokol J. (2016). Fekete-Szego problems for some starlike functions related to shell-like curves. *Math. Slovaca,* 66, 135-140.

Sokol J. (1999). On starlike functions connected with Fibonacci numbers. *Folia Scient. Univ. Tech. Resoviensis, 175, 111-116.*