Quest for Research Excellence On Computing, Mathematics and Statistics

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Chapters in Book

The 2nd International Conference on Computing, Mathematics and Statistics (iCMS2015)

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Content

CHAPTER 9
The Geometry of the Accessible Sets of Vector Fields

A.Y.Narmanov, and I. Ganiev

Abstract. In this paper it is studied geometry of accessible sets of family of vector fields. It is proved that sets of 0- accessible sets are smooth submanifolds of an orbit. It is also proved that on the any manifold there is a system of vector fields consisting of three vector fields, such that each set of 0- accessible sets coincide with the manifold.

Keywords: manifold; vector field; orbit; 0- accessible sets

A.Y.Narmanov (✉)
National University of Uzbekistan,
e-mail: narmanov@yandex.ru

I. Ganiev
International Islamic University Malaysia
e-mail: inam@iium.edu.my

1 Introduction

The study of the geometry of the accessible set is one of the main objectives of quality control theory and is closely related to the geometry of the orbits of vector fields. Study of the structure of the accessible set and orbit systems of smooth vector fields are objectives of many mathematicians because of its importance in optimal control theory, dynamical systems, in geometry and in foliation theory [1-6].

2 Main Results

Let $M-$smooth manifold of dimension n, $V(M)-$ the set of all smooth vector fields defined on M. We denote by $[X,Y]$ Lie bracket of vector fields $X,Y\in V(M)$. With respect to the Lie bracket of the set $V(M)$ is an algebra Lee. Smoothness in this paper is smooth of class C^{∞}.

Consider the set $D\subset V(M)$, through $A(D)$ denote smallest Lie subalgebra containing the set D. Family D may contain a finite and an infinite number of smooth vector fields.

For a point $x\in M$ denote by $t\to X^t(x)$ the integral curve of the vector field X, passing through the point x at $t=0$. The map $t\to X^t(x)$ defined in a certain region $I(x)\subset R$, which generally depends on the field X, and from starting point x.

In the future, everywhere in the formulas of the form $X^t(x)$, we assume that $t\in I(x)$. If for all points $x\in M$ domain $I(x)$ of the curve $t\to X^t(x)$ coincides with the real axis, then vector field X is called a complete vector field. In this case flow of X generates a dynamical system.

Definition 1. Orbit $L(x)$ family D of vector fields, passing through the point x defined as the set of points y from M, for which there exist real numbers $t_1,t_2,\dots,t_k$ and vector fields $X_1,X_2,\dots,X_k$ from D (where $k-$ is an arbitrary natural number) such that

$$y=X_k^{t_k}(X_{k-1}^{t_{k-1}}(\dots(X_1^{t_1}(x))\dots)).$$

Definition 2. Point

$$y = X_k^{t_k}(X_{k-1}^{t_{k-1}}(...(X_1^{t_1}(x))...)) \in L(x)$$

called $T-$ accessible from the point $x \in M$, if $\sum_i t_i = T.$

We denote by $A_x(T)$ the set of $T-$accessible points from the point $x \in M$.

In [Sussmann. H. ., 1973,] it is proved that every orbit of the family of vector fields with the topology Sussmann has a differential structure, with respect to which it is a smooth manifold smoothly immersed in M .

The following theorem shows that the set $A_x(T)$ has a differential structure of a smooth manifold.

Theorem 1. Set $A_x(T)$ for each $x \in M$ for each T is immersed submanifold of orbit codimension one or zero.

Theorem 2. Let M - smooth connected manifold of dimension $n \geq 2$. There exist a system D , consisting of three vector fields, such that $A_x(0) = M$ for each point $x \in M$.

In the event that the Euler characteristic of M is non-zero, there is a stronger statement.

Theorem 3. Let M - be a smooth compact connected manifold of dimension Euler characteristic which is different from zero. There is a system D on M consisting of two vector fields such that $A_x(0) = M$ for each point $x \in M$.

The following example shows that the on the compact connected manifold M with zero Euler characteristic can also be a system consisting of two vector fields such that $A_x(0) = M$ for each point $x \in M$.

Let the three-dimensional sphere $S^3 \subset R^4$ is given by $x^2 + y^2 + z^2 + w^2 = 1,$ where $x, y, z, w-$ the Cartesian coordinates in R^4 .

Let's consider a system on S^3 , consisting of two on the vector fields:

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - \frac{\partial}{\partial z} + z\frac{\partial}{\partial z}, Y = -z\frac{\partial}{\partial x} + x\frac{\partial}{\partial z}$$

It is easy to verify that these fields are vector fields Killing, ie. local diffeomorphisms $x \to X^t(x), \quad x \to Y^t(x)$ for every t are isometries sphere of S^3.

Lie bracket $[X,Y]$ of vector fields X, Y is as follows:

$$[X,Y] = -w\frac{\partial}{\partial x} - z\frac{\partial}{\partial y} + y\frac{\partial}{\partial z} + x\frac{\partial}{\partial w}$$

At the point $p(1,0,0,0) \in S^3$ the vectors $X(p), Y(p), [X,Y](p)$ are linearly independent, subspace $A_p(D) = \{X(p) : X \in A(D)\}$ is three-dimensional. Therefore, the orbit is three-dimensional. Due to the fact that they are the Killing vector fields, the orbit is a closed set [Narmanov.A.Y.,Saitova S, 2014]. On the other hand, by virtue of the maximality of dimension of the orbit it is an open set. Consequently, the orbit is coincident with S^3.

Now consider the sets $A_q(0)$ for $q \in S^3$. If the sets A_q are submanifolds of codi-mension one, owing to the fact that X, Y -are Killing vector fields, they generate two-dimensional Riemannian foliation on [3]. Recall that the foliation is called the Riemannian if every geodesic orthogonal to foliation at some point, it is orthogonal to foliation at all points [3]. How should follows from the results of [Thurston W, 1976], on the three-dimensional sphere doesn't exist two-dimensional Riemannian foliation. Consequently, the set $A_q(0)$ is coincident with S^3.

In the following example sets $A_q(0)$ are submanifolds of orbits $L(p)$ codimension one. Let $M = R^3$, D - the family consists of following two Killing vector fields

$$X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}, Y = \frac{\partial}{\partial z}.$$

In this case, for each point $p(x,y,z) \in M$, such that $x^2 + y^2 > 0$ the orbit $L(p)$ is a cylinder, and the set $A_q(0)$ for each point $q \in L(p)$ is a spiral line, tangent field vector field of which is $Z = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y} - \frac{\partial}{\partial z}$. For points of the axis OZ of orbit $L(p)$ and sets $A_q(0)$ coincides with the axis OZ.

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