NUMERICAL SOLUTION OF THE DERIVATIVE LINEAR GOURSAT PROBLEM

[1]Mohd Agos Salim bin Nasir and [2]Ahmad Izani bin Md. Ismail
[1]Department of Engineering Science, Universiti Teknologi MARA
Penang Campus Branch, 13500 Pulau Pinang
[2]School of Mathematical Sciences, Universiti Sains Malaysia
11800 Minden, Pulau Pinang

Abstract: Studies have been carried out on the effect on accuracy when different means are used to compute function values in the finite difference solution of the Goursat problem. One study found that the use of harmonic mean averaging resulted in more accurate results compared with arithmetic mean averaging whilst another study found the opposite to be the case. In this paper, we investigate this issue further by focusing on two Goursat problem involving a derivative terms.

Keywords: Finite difference schemes, Partial differential equations, Arithmetic mean, Harmonic mean

INTRODUCTION

Many natural laws of physics and models of physical phenomena can be described using partial differential equations. These equations relating space and time derivatives, needs to be solved in order to gain fuller insight into the underlying physical problem. Many of these equations are such that analytical methods cannot be utilized and numerical methods (such as the finite difference method) needs to be used.

Both arithmetic mean and harmonic mean averaging of function values have been used in finite difference schemes for the Goursat problem. A comparative study carried out by Wazwaz [3] concluded that the use of harmonic mean averaging yielded more accurate results. However a comparative study (including a Goursat equation studied by Wazwaz) carried out by Nasir and Ismail [2] concluded that arithmetic mean averaging resulted in more accurate results. In this paper we conduct a further study of another two Goursat problem using both types of averaging.

DISCUSSIONS

The Goursat Problem and Finite Difference Schemes

The Goursat problem is of the form (3):

$$\begin{aligned}&u_{xy} = f(x,y,u,u_x,u_y)\\&u(x,0)=\phi(x),\ u(0,y)=\psi(y),\ \phi(0)=\psi(0)\\&0\le x\le a,\ 0\le y\le b\end{aligned} \qquad \ldots(1)$$

The sine-Gordon equation $u_{xt} = -\sin u$ is an example of a Goursat problem. This equation arises in Cauchy problem [1].

The established finite difference scheme is based on arithmetic mean (AM) averaging of functional values and is given by Wazwaz [3]:

$$\frac{u_{i+1,j+1}+u_{i,j}-u_{i+1,j}-u_{i,j+1}}{h^2}=\frac{1}{4}\left(f_{i+1,j+1}+f_{i,j}+f_{i+1,j}+f_{i,j+1}\right) \qquad \ldots(2)$$

Henceforth, we shall refer to the finite difference scheme (2) as the AM scheme. Here, the function value at grid location $(i+1/2, j+1/2)$ is approximated by:

$$\frac{1}{4}\left(f_{i+1,j+1}+f_{i,j}+f_{i+1,j}+f_{i,j+1}\right) \qquad \ldots(3)$$

Wazwaz [3] presented a new algorithm for the Goursat problem. This algorithm is based on harmonic mean averaging of functional values and is given by:

$$\frac{u_{i+1,j+1}+u_{i,j}-u_{i+1,j}-u_{i,j+1}}{h^2}=\frac{4f_{i+1,j+1}f_{i,j}f_{i,j+1}f_{i+1,j}}{f_{i+1,j+1}f_{i,j}(f_{i+1,j}+f_{i,j+1})+f_{i+1,j}f_{i,j+1}(f_{i+1,j+1}+f_{i,j})} \qquad \ldots(4)$$

Henceforth, we shall refer to the finite difference scheme (4) as the HM scheme. The harmonic mean (HM) of any two real numbers a and b is $\frac{2ab}{a+b}$. The function value at location $(i+1/2, j+1/2)$, i.e. the r.h.s of equation (4), is obtained from:

HM (HM of $f_{i,j+1}$ and $f_{i+1,j}$; HM of $f_{i+1,j+1}$ and $f_{i,j}$) ...(5)

Wazwaz [3] stated that he investigated the application of the AM and HM scheme over a wide range of examples and concluded that the HM scheme appears to give better results (in terms of accuracy). However, results were only presented for the non-linear Goursat problem (with h = 0.05):

$$\begin{aligned} &u_{xy}=e^{2u} \\ &u(x,0)=\frac{x}{2}-\ln(1+e^x) \\ &u(0,y)=\frac{y}{2}-\ln(1+e^y) \\ &0\le x\le 4, 0\le y\le 4 \end{aligned} \qquad \ldots(6)$$

The results presented were the relative errors at 16 selected grid points. However, from an examination of the displayed results, Nasir and Ismail [2] observed that the AM scheme was more accurate than the HM scheme for all 16 grid points. The question that arises is whether the conclusion that the HM scheme was more accurate was based on performance at grid points not displayed (for h = 0.05, there are 6400 grid points). Nasir and Ismail [2] compared the accuracy of the AM and HM schemes, for a particular grid size, by computing the number of grid points at which one schemes is more accurate than the other and the average relative error over all grid points. This was implemented over a range of grid sizes on problem (6) as well as the linear Goursat problem:

$$\begin{aligned} &u_{xy}=u \\ &u(x,0)=e^x \\ &u(0,y)=e^y \\ &0\le x\le 2, 0\le y\le 2 \end{aligned} \qquad \ldots(7)$$

Based on an analysis of the results, Nasir and Ismail [2] concluded that the AM scheme was more accurate than the HM scheme. In this study we compare the accuracy of the AM and HM scheme for the following linear Goursat problems which involves a derivative term:

$$\begin{aligned}
&u_{xy} = -1 + y + u_x \\
&u(x,0) = -1 + e^x \\
&u(0,y) = -1 + e^y \\
&0 \le x \le 2,\ 0 \le y \le 2
\end{aligned} \qquad \ldots(8)$$

and

$$\begin{aligned}
&u_{xy} = \frac{u_x + u_y + u}{3} \\
&u(x,0) = e^x \\
&u(0,y) = e^y \\
&0 \le x \le 4,\ 0 \le y \le 4
\end{aligned} \qquad \ldots(9)$$

The analytical solution for the Goursat problem (8) and (9) can be found in Wazwaz [4] and Day respectively.

Numerical Experiments

We developed computer programs for the application of both the AM and HM schemes to problems (8) and (9).

Results for problem (8):
For h = 0.030

Relative Errors For AM Scheme

	X=0.6	X=1.2	X=1.8	X=2.4
Y=0.6	3.0126364e-003	6.3260590e-003	9.6058501e-003	1.2893705e-002
Y=1.2	5.2266200e-003	1.0955582e-002	1.6462257e-002	2.1804715e-002
Y=1.8	6.6047630e-003	1.3719158e-002	2.0581933e-002	2.7215324e-002
Y=2.4	7.4481525e-003	1.5303516e-002	2.2944321e-002	3.0367177e-002

Relative Errors For HM Scheme

	X=0.6	X=1.2	X=1.8	X=2.4
Y=0.6	6.7342808e-002	4.4556374e-002	6.2442560e-002	1.9458721e-001
Y=1.2	6.3473698e-002	4.9524936e-002	7.2298361e-002	2.0199172e-001
Y=1.8	4.8903378e-002	4.3057746e-002	6.4391000e-002	1.5176452e-001
Y=2.4	3.6032073e-002	3.5539789e-002	5.3744795e-002	9.7252826e-002

Number of grid points where AM scheme superior = 6190
Number of grid points where HM scheme superior = 210
Average error of AM scheme = 9.9341687e-003
Average error of HM scheme = 5.6393552e-002

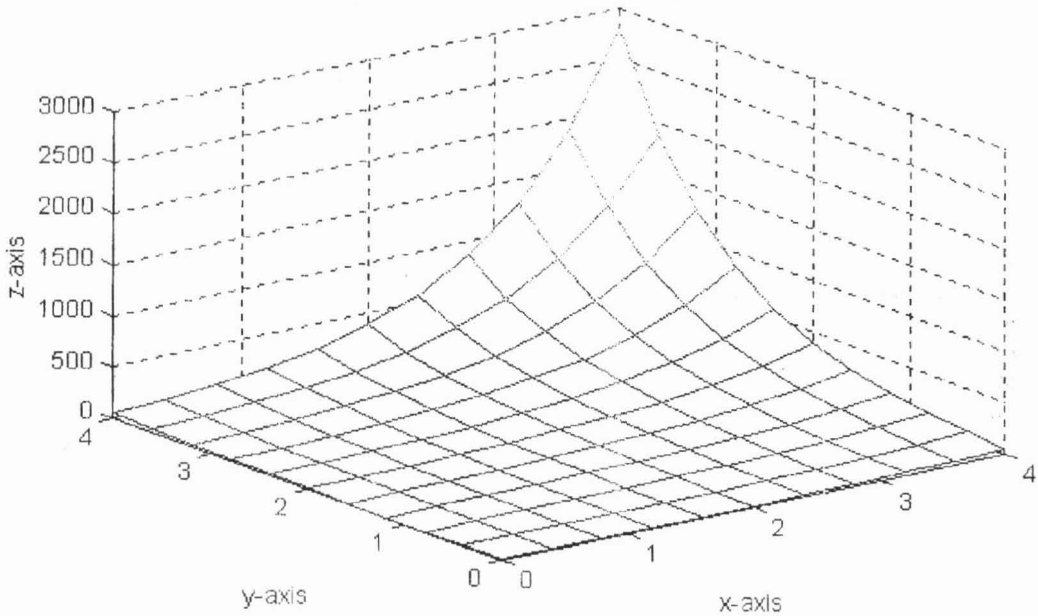

Figure 2: Solution of problem (9) in graphic form using AM scheme; h = 0.05

It is clear that, for the Goursat problems (8) and (9) and for the grid sizes investigated, the AM scheme is more accurate than the HM scheme.

CONCLUSIONS

In this paper we have studied the AM and HM finite difference schemes for the solution of two linear Goursat problems involving derivatives. Our findings reinforce the conclusions of our previous study [2] that the AM scheme seems to be more accurate than the HM scheme for the finite difference solution of Goursat problems.

ACKNOWLEDGEMENTS

This project was funded by URDC grant Universiti Teknologi Mara.

REFERENCES

1. Kaup, D.J. & Newell, A.C. 1978. The Goursat and Cauchy problems for the Sine-Gordon Equation, SIAM Journal of Applied Mathematics.

2. Nasir, M.A.S. & . Ismail, A.I.B. 2003. Finite difference schemes for the Goursat problem, To be published in Proceeding of International Conference On Research and Education in Mathematics, Universiti Putra Malaysia.

3. Wazwaz, A.M. 1993. On the Numerical solution for the Goursat Problems, Applied Mathematics and Computation.

4. Wazwaz, A.M. 1995. The decomposition method for approximate solution of the Goursat Problem, Applied Mathematics and Computation.