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ON SOME PROPERTIES OF A GENERALIZED CLASS OF CLOSE-TO-STARLIKE FUNCTIONS

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ABSTRACT

In this paper, we consider a new class of close-to-starlike functions denoted by $CS^*_{\alpha,\beta}$ *, defined by the Carlson-Shaffer operator* $\mathcal{L}(\alpha,\beta)$. *Let* S *denote the class of analytic univalent functions* f *defined by* $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$, *then* $f\in CS^*_{\alpha,\beta}$ *if* f *satisfy the condition* $\operatorname{Re}\left\{\frac{\mathcal{L}(\alpha,\beta)(f)(z)}{g(z)}\right\}>0, z\in E$ *, where* $\mathcal{L}(\alpha,\beta)f(z)=z+\sum_{n=2}^{\infty}\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}a_n z^n$ *and* $g(z)$ *is a starlike function. Properties of the class* $CS^*_{\alpha,\beta}$ *such as the coefficient bounds, growth and distortion theorems and radius results are investigated.*

Keywords: *close-to-starlike functions, analytic functions, starlike functions, Carlson-Shaffer operator, coefficient bound, growth and distortion theorems, radius results.*

1. Introduction

Let A denote the class of functions of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1.1}$$

that are analytic and normalized in the open unit disk $E=\{z:|z|<1\}$ and S be the subclass of A consisting of functions that are univalent in E.

Denote by P, the class of functions with positive real part, where functions in this class are of the form

$$p(z)=1+\sum_{n=1}^{\infty}p_n z^n \,. \tag{1.2}$$

Let S^*, K and C denote the subclasses of S which are the known class of starlike, convex and close-to-convex functions respectively (Goodman, 1983; Peter et. al., 2018; Rathi et. al., 2018). Reade (1955) defined the class of close-to-starlike functions where functions in this class satisfy

$$\mathrm{Re}\left[\frac{f(z)}{\phi(z)}\right] > 0, \qquad z \in E$$

with $\phi(z)$ belonging to S^*.

Reade (1955) obtained that $f(z)$ is close-to-starlike if and only if the inequality,

$$\int_{\theta_1}^{\theta_2} \mathrm{Re}\left[re^{i\theta} \frac{f'(re^{i\theta})}{f(re^{i\theta})}\right] d\theta > -\pi$$

holds for all $\theta_1 < \theta_2$ and for all $0 \le r < 1$. The class of close-to-starlike functions is denoted by CS^*.

Srivastava and Attiya (2007) introduced a family of linear operator $J_{\mu,b} : A \to A$ by the Hadamard product of the Hurwitz-Lerch Zeta function with analytic functions as

$$J_{\mu,b}(f)(z) = G_{\mu,b} * f(z)$$

where $b \in \mathbb{C}$ with $b \ne 0, -1, -2, -3, \ldots,$ $\mu \in \mathbb{C},$ $z \in E$ and $G_{\mu,b} \in A$ is given by

$$\begin{aligned} G_{\mu,b} &= (1+b)^{\mu}\left[\phi(\mu, b; z) - b^{-\mu}\right] \\ &= z + \sum_{n=2}^{\infty}\left(\frac{1+b}{n+b}\right)^{\mu} a_n z^n \end{aligned}$$

Srivastava et al. (2013), then introduced a subclass of close-to-starlike functions using Srivastava-Attiya operator denoted by $CS^*_{s,b}$ where they gave the following definition.

Definition 1.1 A function f is said to belong to the class $CS^*_{s,b}$ if and only if, there exists a function $g \in S^*$ such that

$$\mathrm{Re}\left(\frac{J_{s,b}(f)(z)}{g(z)}\right) > 0 \quad , \quad z \in E.$$

In the special case when $s = 0$, the class $CS^*_{s,b}$ can be reduced to the class CS^*, studied earlier by Reade (1955).

This paper will define a new class of close-to-starlike functions using the operator defined by Carlson and Shaffer (1984), where the Carlson-Shaffer linear operator $\mathcal{L}(\alpha, \beta) : A \to A$ is given by

$$\mathcal{L}(\alpha, \beta) f(z) = \varphi(\alpha, \beta; z) * f(z)$$

where

$$\varphi(\alpha,\beta;z)=\sum_{n=0}^{\infty}\frac{(\alpha)_n}{(\beta)_n}z^{n+1}$$
$$=z+\sum_{n=0}^{\infty}\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}z^n,\qquad \beta\neq 0,-1,-2,\ldots$$

The function, $\varphi(\alpha,\beta;z)$ is known as the incomplete beta function. The term $(\lambda)_k$ is the Pochhammer symbol that can be expanded in Gamma functions as

$$(\lambda)_k=\frac{\Gamma(\lambda+k)}{\Gamma(k)}:=\begin{cases}1 & (k=0)\\ \lambda(\lambda+1)\ldots(\lambda+k-1) & (k\in\mathbb{N}=\{1,2,\ldots\}).\end{cases}$$

Thus, Carlson-Shaffer linear operator can be written as,

$$\mathcal{L}(\alpha,\beta)f(z)=z+\sum_{n=2}^{\infty}\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}a_n z_n,\quad z\in E \tag{1.3}$$

We define our new class of functions as follows.

Definition 1.2 A function f is said to belong to the class of $CS^*_{\alpha,\beta}$ if and only if there exists a function $g\in S^*$ such that

$$\operatorname{Re}\left\{\frac{\mathcal{L}(\alpha,\beta)f(z)}{g(z)}\right\}>0,\qquad z\in E \tag{1.4}$$

In the special case, when $\alpha=1$ and $\beta=1$, the class $CS^*_{\alpha,\beta}$ reduces to the class CS^*, studied by Reade (1955).

The main objective of this paper is to find the coefficient inequalities and basic properties of the class $CS^*_{\alpha,\beta}$.

2. Coefficient bounds

Theorem 2.1 *Let f in the form of (1.1) be in the class $CS^*_{\alpha,\beta}$. Then,*

$$|a_n|\le n^2\left|\frac{(\beta)_{n-1}}{(\alpha)_{n-1}}\right|,\qquad n\in\mathbb{N}\setminus\{1\} \tag{2.1}$$

Proof Method of Srivastava et al. (2013) will be used to prove this theorem. Suppose that $f\in CS^*_{\alpha,\beta}$, then from (1.4), there exists a function $g\in S^*$ and a function $p\in P$ such that,

$$\mathcal{L}(\alpha,\beta)f(z)=p(z)g(z),\qquad z\in E \tag{2.2}$$

Let

$$p(z)=1+\sum_{n=1}^{\infty}p_n z_n \text{ and } g(z)=z+\sum_{n=1}^{\infty}b_n z_n \tag{2.3}$$

Then, by substituting (1.3) and (2.3) into (2.2), we have

$$z+\sum_{n=2}^{\infty}\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}a_n z^n = z+\sum_{n=2}^{\infty}\left(b_n+\sum_{k=1}^{n-1}b_k p_{n-k}\right)z^n \ , \quad b_1=1 \tag{2.4}$$

Comparing coefficient of z^n on both sides of (2.4) gives

$$\sum_{n=2}^{\infty}\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}a_n = \sum_{n=2}^{\infty}\left(b_n+\sum_{k=1}^{n-1}b_k p_{n-k}\right). \tag{2.5}$$

Also using the fact that

$$|p_n|\le 2, \qquad n\in\mathbb{N} \qquad \text{and} \qquad |b_n|\le n, \qquad n\in\mathbb{N}\setminus\{1\}$$

equation (2.5) becomes

$$\left|\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}\right||a_n|\le n+2\sum_{k=1}^{n-1}k=n^2 \qquad n\in\mathbb{N}\setminus\{1\}.$$

So, we have that,

$$|a_n|\le n^2\left|\frac{(\beta)_{n-1}}{(\alpha)_{n-1}}\right|, \qquad n\in\mathbb{N}\setminus\{1\}$$

which will complete the proof of Theorem 2.1.
In its specific cases when $\alpha=1$ and $\beta=1$, we obtain

$$|a_n|\le n^2, \quad n\in\mathbb{N}\setminus\{1\}$$

which is the result of Reade (1955).

3. Growth and distortion theorem

Using the coefficient bound we now derive the other properties for the class $CS^*_{\alpha,\beta}$. Firstly we find the growth theorem for functions in the class $CS^*_{\alpha,\beta}$.

Theorem 3.1 *Let* $f\in CS^*_{\alpha,\beta}$. *Then,*

$$r-r^2\sum_{n=2}^{\infty}n^2\left|\frac{(\beta)_{n-1}}{(\alpha)_{n-1}}\right|\le|f(z)|\le r+r^2\sum_{n=2}^{\infty}n^2\left|\frac{(\beta)_{n-1}}{(\alpha)_{n-1}}\right| \qquad z\in E,\ |z|=r.$$

Proof Let $f\in CS^*_{\alpha,\beta}$. By taking absolute values on both sides of (1.1), we have

$$|f(z)|=\left|z+\sum_{n=2}^{\infty}a_n z^n\right|$$

Hence, by using triangular inequalities, we have,

$$|z|-\left|\sum_{n=2}^{\infty}a_n z^n\right|\le|f(z)|\le|z|+\left|\sum_{n=2}^{\infty}a_n z^n\right| \qquad :|z|=r$$

$$r-\sum_{n=2}^{\infty}|a_n|r^n \le |f(z)| \le r+\sum_{n=2}^{\infty}|a_n|r^n . \qquad (3.1)$$

By substituting (2.1) into (3.1), we have

$$r-r^2\sum_{n=2}^{\infty}n^2\left|\frac{(\beta)_{n-1}}{(\alpha)_{n-1}}\right| \le |f(z)| \le r+r^2\sum_{n=2}^{\infty}n^2\left|\frac{(\beta)_{n-1}}{(\alpha)_{n-1}}\right|.$$

Thus this completes the proof of Theorem 3.1.

We next derive the distortion theorem for functions in the class $CS^*_{\alpha,\beta}$.

Theorem 3.2 *Let* $f \in CS^*_{\alpha,\beta}$. *Then,*

$$1-r\sum_{n=2}^{\infty}n^3\left|\frac{(\beta)_{n-1}}{(\alpha)_{n-1}}\right| \le |f'(z)| \le 1+r\sum_{n=2}^{\infty}n^3\left|\frac{(\beta)_{n-1}}{(\alpha)_{n-1}}\right| \qquad z\in E,\ |z|=r.$$

Proof Let $f \in CS^*_{\alpha,\beta}$. For distortion theorem, we differentiate (1.1) to obtain

$$f'(z)=1+\sum_{n=2}^{\infty}na_nz^{n-1}$$

By taking absolute values on both sides, we have

$$|f'(z)|=\left|1+\sum_{n=2}^{\infty}na_nz^{n-1}\right|$$

Hence, by using triangular inequalities, we have,

$$|1|-\left|\sum_{n=2}^{\infty}na_nz^{n-1}\right| \le |f'(z)| \le |1|+\left|\sum_{n=2}^{\infty}na_nz^{n-1}\right| \qquad :|z|=r$$

$$1-\sum_{n=2}^{\infty}n|a_n|r^{n-1} \le |f'(z)| \le 1+\sum_{n=2}^{\infty}n|a_n|r^{n-1} \qquad (3.2)$$

By substituting (2.1) into (3.2), we have

$$1-r\sum_{n=2}^{\infty}n^3\left|\frac{(\beta)_{n-1}}{(\alpha)_{n-1}}\right| \le |f'(z)| \le 1-r\sum_{n=2}^{\infty}n^3\left|\frac{(\beta)_{n-1}}{(\alpha)_{n-1}}\right|.$$

Thus this completes the proof for Theorem 3.2.

4. Radius of convexity, starlikeness and close-to-convexity

Results on radius of convexity, starlikeness and close-to-convexity for the class $CS^*_{\alpha,\beta}$ will be obtained in this section.

A function $f \in S$ is said to be convex of order $\delta(0 \le \delta \le 1)$ if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > \delta \qquad (4.1)$$

Theorem 4.1 *Let the functions in the form of (1.1) be in the class of* $CS^*_{\alpha,\beta}$ *. Then,* f *is convex of order* δ *if,*

$$|z| \le \left[\frac{1-\delta}{n^3(n-\delta)} \left| \frac{(\alpha)_{n-1}}{(\beta)_{n-1}} \right| \right]^{\frac{1}{n-1}}, \qquad n \in \mathbb{N} \setminus \{1\}$$

Proof

It is adequate to show that the values for $\left(1+\dfrac{zf''(z)}{f'(z)}\right)$ lie in a circle centered at $w=1$ with radius $1-\delta$.

We have

$$\left| \left(1+\frac{zf''(z)}{f'(z)}\right) - 1 \right| \le \frac{\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty} n|a_n||z|^{n-1}} \le 1-\delta \tag{4.2}$$

$$\sum_{n=2}^{\infty} n(n-1)|a_n||z|^{n-1} \le (1-\delta)\left(1-\sum_{n=2}^{\infty} n|a_n||z|^{n-1}\right)$$

Finally, we get

$$\frac{\sum_{n=2}^{\infty} n(n-\delta)|a_n||z|^{n-1}}{1-\delta} \le 1$$

In view of Theorem 2.1, we have

$$\frac{n(n-\delta)|z|^{n-1}}{1-\delta} \le \frac{1}{n^2} \left| \frac{(\alpha)_{n-1}}{(\beta)_{n-1}} \right|, \qquad n \in \mathbb{N} \setminus \{1\} \tag{4.3}$$

Solving (4.3) for $|z|$ we obtain

$$|z| \le \left[\frac{1-\delta}{n^3(n-\delta)} \left| \frac{(\alpha)_{n-1}}{(\beta)_{n-1}} \right| \right]^{\frac{1}{n-1}}, \qquad n \in \mathbb{N} \setminus \{1\}.$$

This completes the proof of Theorem 4.1.

A function $f \in S$ is said to be starlike of order $\delta (0 \le \delta \le 1)$ if it satisfies

$$\mathrm{Re}\left\{ \frac{zf'(z)}{f(z)} \right\} > \delta. \tag{4.4}$$

Theorem 4.2 *Let the functions in the form of (1.1) be in the class of* $CS^*_{\alpha,\beta}$ *. Then,* f *is starlike of order* δ *if,*

$$|z| \le \left[\frac{1-\delta}{n^2(n-\delta)} \left| \frac{(\alpha)_{n-1}}{(\beta)_{n-1}} \right| \right]^{\frac{1}{n-1}}, \qquad n \in \mathbb{N} \setminus \{1\}.$$

Proof We must show that,

$$\left| \frac{zf'(z)}{f(z)} - 1 \right| \le \frac{\sum_{n=2}^{\infty} (n-1)|a_n||z|^{n-1}}{1-\sum_{n=2}^{\infty} |a_n||z|^{n-1}} \le 1-\delta \tag{4.5}$$

But in view of Theorem 2.1, the inequality (4.5) holds if

$$\frac{(n-\delta)|z|^{n-1}}{1-\delta} \le \frac{1}{n^2}\left|\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}\right|, \qquad n \in \mathbb{N}\setminus\{1\} \qquad (4.6)$$

Solving (4.6) for $|z|$ we obtain

$$|z| \le \left[\frac{1-\delta}{n^2(n-\delta)}\left|\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}\right|\right]^{\frac{1}{n-1}}, \qquad n \in \mathbb{N}\setminus\{1\}$$

This completes the proof of Theorem 4.2.

A function $f \in S$ is said to be close-to-convex of order $\delta (0 \le \delta \le 1)$ if

$$\text{Re}\{f'(z)\} > \delta. \qquad (4.7)$$

Theorem 4.3 *Let the functions in the form of (1.1) be in the class of* $CS^*_{\alpha,\beta}$ *. Then,* f *is close-to-convex of order* δ *if,*

$$|z| \le \left[\frac{1-\delta}{n^3}\left|\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}\right|\right]^{\frac{1}{n-1}}, \qquad n \in \mathbb{N}\setminus\{1\}.$$

Proof We must show that

$$|f'(z)-1| \le \sum_{n=2}^{\infty} n|a_n||z|^{n-1} \le 1-\delta \qquad (4.8)$$

But in view of Theorem 2.1, the inequality (4.8) holds if

$$\frac{n|z|^{n-1}}{1-\delta} \le \frac{1}{n^2}\left|\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}\right|, \qquad n \in \mathbb{N}\setminus\{1\} \qquad (4.9)$$

Solving (4.9) for $|z|$ we obtain

$$|z| \le \left[\frac{1-\delta}{n^3}\left|\frac{(\alpha)_{n-1}}{(\beta)_{n-1}}\right|\right]^{\frac{1}{n-1}}, \qquad n \in \mathbb{N}\setminus\{1\}.$$

This completes the proof of Theorem 4.3.

5. Conclusion

In this paper, a certain class of analytic functions in the complex plane is discussed. The class of analytic univalent functions is denoted by S .This paper is specifically focused on a class of close-to-starlike functions defined using Carlson-Shaffer operator. This class is denoted by $CS^*_{\alpha,\beta}$ .We find the coefficient bounds, growth and distortion theorems and radius properties for the class defined.

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