JURNAL AKADEMIK

Vol. 1, No. 1, Disember 1999

UNIVERSITI TEKNOLOGI MARA KAMPUS SAMARAHAN

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FINITE ELEMENT SOLUTION TO ENGINEERING PROBLEMS

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Abstract

This paper deals with the solution methodology to the engineering field problems with the application of Finite Element Method (FEM). Various types of boundary value problems that can be solved by FEM are indicated. Formulation of a mathematical model for a field problem and the steps of solution by FEM are explained. A case study of a heat conduction problem has been presented to demonstrate the implementation of an algorithm oriented computer program already developed. General conclusions on the application of FEM have been discussed.

1.0 INTRODUCTION

Finite Element Method was applied initially to the analysis and design of aircraft structures. However, the general nature of its theory makes it applicable to a wide variety of boundary value problems in engineering. A boundary value problem is one in which a solution is sought in the domain or continuum or region of a body, subject to the satisfaction of certain prescribed boundary conditions on the dependant variables or their derivatives. These boundary value problems can be classified into three major categories, viz.,

(i) Equilibrium or Steady state or Time independent problems

(ii) Eigen-Value problems

(iii) Propagation or Transient or Time dependent problems

In equilibrium problems, we need to find the solution for the steady state conditions. Examples of such problems are : Steady state displacements or stress distribution in solid mechanics, steady state temperature or heat flux distribution in heat transfer, and steady state pressure or velocity distribution in fluid mechanics.

Eigen-Value problems may be considered as extensions of equilibrium problems in which critical values of certain parameters are to be determined in addition to the corresponding steady state configurations. The examples are: Natural frequencies or buckling loads in solid mechanics, stability of laminar flows in fluid mechanics, and resonance characteristics in electric circuit.

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The propagation or transient problems are essentially time dependent problems. In these, the solution is sought satisfying the initial conditions in addition to the prescribed boundary conditions. The solution will march forward in time domain starting from the initial state of the field variables. The examples are: Response of a body under time varying force in solid mechanics, sudden heating or cooling in heat transfer, and variation of groundwater potentials.

Table 1 (Rao S.S., 1982) gives some important examples of engineering applications of Finite Element Method in the three categories of boundary value problems.

2.0 MATHEMATICAL MODEL OF A FIELD PROBLEM

Any field problem of engineering can be expressed mathematically in its most general form as,

$$\frac{\partial}{\partial x}\left(k_x \frac{\partial \phi}{\partial x}\right) + \frac{\partial}{\partial y}\left(k_y \frac{\partial \phi}{\partial y}\right) + \frac{\partial}{\partial z}\left(k_z \frac{\partial \phi}{\partial z}\right) + C = \alpha \frac{\partial \phi}{\partial t} \qquad (1)$$

in which ϕ is the unknown field variable (dependent variable)

k_x , k_y , k_z , C , and α are known functions of x, y, z; and their physical interpretation depends on the particular physical problem.
x, y, z are the space coordinates, and
t is the time coordinate.

Equation (1) is the governing equation of the field problem and represents a three dimensional transient problem of a non-homogeneous and anisotropic medium. For homogeneous and isotropic medium, eqn (1) simplifies to:

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} + \frac{C}{k} = \frac{\alpha}{k} \frac{\partial \phi}{\partial t} \qquad (1a)$$

Equations (1) & (1a) are quasi-harmonic type of partial differential equations representing transient case of boundary value problem. This governing partial differential equation (1 or 1a) is subject to the boundary and initial conditions as prescribed below.

Boundary conditions are:

$$\phi = \overline{\phi} \quad \text{for } t > 0 \text{ on part of the boundary} \qquad (2)$$

$$k_x \left(\frac{\partial \phi}{\partial x}\right) \ell_x + k_y \left(\frac{\partial \phi}{\partial y}\right) \ell_y + k_z \left(\frac{\partial \phi}{\partial z}\right) \ell_z + q + r\phi = 0 \qquad (3)$$

on the remaining part of the boundary

where $\overline{\phi}$ is the prescribed value of the field variable on part of boundary

TABLE 1 : Engineering Applications of Finite Element Method

Area of Study	Equilibrium problems	Eigen value problems	Propagation problems
Aircraft Structures	Static analysis of aircraft wings, fins, rockets, spacecraft & missile structures	Natural frequencies, flutter, & stability of aircraft, rocket, spacecraft & missiles	Response of aircraft structures to random loads, dynamic response to aperiodic loads
Civil engineering structures	Static analysis of trusses, frames, folded plates, shell roofs, shear walls, bridges & prestressed concrete structures	Natural frequencies and modes of structures, stability of structures	Propagation of stress waves, response of structures to aperiodic loads
Hydraulic & water resources engineering	Analysis of potential flows, free surface flows, boundary layer flows, viscous flows, transonic aerodynamic problems, hydraulic structures & dams	Natural periods & modes of shallow basins, lakes & harbours, rigid & flexible containers	Analysis of unsteady fluid flow & wave propagation problems, transient seepage in aquifers, rarefied gas dynamics, magneto-hydrodynamics
Geomechanics	Analysis of excavations, retaining walls, underground openings, soil structure interaction, stress analysis in soils, dams, piles & machine foundations	Natural frequencies and modes of dam reservoir systems & soil-structure interaction problems	Time dependent soil-structure interaction problems, transient seepage in soils and rocks, stress wave propagation in soils & rocks
Heat transfer	Steady state temperature distribution in solids and fluids	—	Transient heat flow in rocket nozzles, I.C. engines, turbine blades

Area of Study	Equilibrium problems	Eigen value problems	Propagation problems
Mechanical design	Stress concentration & analysis of pressure vessels, composite materials, linkages & gears	Natural frequencies and stability of linkages, gears, and machine tools	Crack and fractur problems unde dynamic loads
Biomedical engineering	Stress analysis of eyeballs, bones & teeth, mechanics of heart valves	—	Impact analysis c skull, dynamics c anatomical structures
Nuclear engineering	Analysis of nuclear pressure vessels, steady state temperature distribution in reactors	Natural frequencies and stability of containment structures, neutron flux distribution	Response of reactors to dynamic loads unsteady temperature distribution in reactors, thermal & viscoelastic analysis

ℓ_x, ℓ_y, ℓ_z are the direction cosines in the three directions
q and r are the parameters of the boundary

Equations (2) & (3) are known as DIRICHLET and CAUCHY conditions respectively.

Initial conditions can be expressed as :

$\phi (x, y, z, t=0) = \phi_o(x, y, z)$ in part of the domain (4)

where ϕ_o is the prescribed value of the field variable in the initial state.

For homogeneous and isotropic case the eqn (1a) can be rewritten as

$$\nabla^2 \phi = - \frac{C}{k} + \frac{\alpha}{k} \frac{\partial \phi}{\partial t} \quad (5)$$

which is a Poisson equation.

For steady state case with C = 0 , eqn (5) simplifies to :

$$\nabla^2 \phi = o \quad (6)$$

which is a Laplace equation .

For homogeneous and isotropic medium with q = r = 0, the Cauchy condition (eqn 3) becomes

$$\frac{\partial \phi}{\partial n} = o \quad (7)$$

where n is the direction normal to the boundary.

Eqn (7) represents a non-conducting boundary and is known as NEUMAN condition.

The governing partial differential equation written in its most general form as eqn (1), along with its boundary and initial conditions, represents a variety of physical problems in engineering such as heat conduction, seepage flow, torsion of shafts, irrotational flow, fluid film lubrication, distribution of electric potential, electromagnetic waves, acoustic waves, groundwater pollutant dispersion, etc.

3.0 SOLUTION METHODOLOGY

The governing equation (eqn 1) along with its boundary and initial conditions, is a non-linear partial differential equation in which the dependent variable ϕ is varying with a second order differential whereas the parameter K is varying with a first order differential with respect to the space coordinates. Therefore the variables in the partial differential equation are in a coupled form making the equation highly complex. For such equations, closed form or exact solutions are not available in literature. Hence, many investigators tried different approximate solutions to this type of governing equation. Analytical solutions are among the earlier approximate solutions attempted by the investigators. In order to make the complex partial differential equation amenable to mathematical solution, several sweeping assumptions were made to find the analytical solutions. Such solutions were based on transformation techniques such as Laplace transforms, graphical transforms, Fourier analysis, Bessel functions etc. However, these solutions were not able to reflect the real field situations due to several assumptions made.

With the advent of fast computing systems, attempts have been made in recent times to obtain the solutions by employing numerical methods such as Finite Difference and Finite Element methods. This paper makes an attempt to explain the Finite Element application to the field problem in order to enthuse the engineers and scientists to take up the investigations of different engineering problems of practical importance.

Due to certain inherent advantages in simulating the non-linear boundary geometry and interpolating the variables, particularly for the nodes lying on the boundary of the domain, the Finite Element method is generally preferred over the traditional Finite Difference method for spatial discretisation (Jagmohan Das, 1984).

4.0 FINITE ELEMENT METHOD

Finite Element Method is basically a method employing an integral approach. Therefore, the governing partial differential equation is first converted into its equivalent integral form. Among the different methods available to write the equivalent integral equation, the variational approach and the approach of Galerkin technique are more popular (Pinder G.F & G.W. Gray, 1977). Once the corresponding integral equation is obtained, the solution methodology deals with this integral equation only.

In Finite Element Method, the region of interest or domain is discretised into a finite number of small subareas known as finite elements. These elements are assumed to be interconnected at discrete number of nodal points situated on the boundaries of each finite element. In literature, one finds different types of elements being used. Various elements that are in vogue can be distinguished by (i) geometries - one, two, and three dimensional, (ii) choice of interpolation functions such as Polynomial, Lagrange, Hermite Polynomial, (iii) choice of element coordinates-cartesian, natural etc. (Zienkiewicz, 1979)

After the region of interest is discretised into a finite number of suitable elements, the next step is the choice of interpolation function within the space of each finite element. These interpolation functions are generally in the form of polynomials of different degrees. They may also be the product of polynomials with trigonometric or exponential functions. In the polynomial expansion, the variation of a field variable within an element can be expressed in terms of unknown nodal variables.

Once the interpolation functions are selected, these functions are substituted into the integral equation and the integrations are performed to obtain a matrix equation at element level. By making use of this matrix equation, the coefficients of the matrices are computed for each of the finite elements of the domain. These computed values are then posted at their appropriate locations in the global framework of the domain to obtain a global matrix equation.

The next step is to perform the integration of the global matrix equation in the time domain. This integration is made by adopting the unconditionally stable Implicit scheme of finite difference method.

After the time integration, the boundary conditions of the field problem are implemented on the global matrix equation. This results into a system of linear algebraic equations, which can be solved either by a direct method like Gaussian elimination or any other iterative method like Cholesky.

5.0 STEPS OF SOLUTION METHODOLOGY

The essential steps required in the implementation of Finite element solution are more or less identical for any field problem. The various steps involved (Sreenivasulu & Jagmohan Das, 1982) can be summarised as follows:

1. Convert the field problem into a mathematical model by writing the governing partial differential equation for the field problem together with its initial and boundary conditions.

2. Develop the equivalent integral equation either through variational formulation based on variational calculus or by Galerkin Technique based on the method of weighted residuals.

3. Select the appropriate type of elements together with their interpolation functions for substitution into the integral equation.

4. Discretise the domain into a suitable number of finite elements and identify the coordinates of all the nodal points.

5. Develop the element matrices for each of the finite elements.

6. Post the coefficients of the element matrices into global framework, at their appropriate locations, resulting in a global matrix equation.

7. Implement the initial conditions by adopting Implicit scheme of Finite difference method.
8. Implement the boundary conditions.
9. Solve the global system of equations either by direct or iterative method.
10. Repeat steps 7 to 9 to march the solution forward in time until the prescribed time level is reached.

The above steps of algorithm can be implemented on any field problem with the help of a computer program written in any scientific computer language. The steps of computation are illustrated in a Flow Chart in Fig 1 .

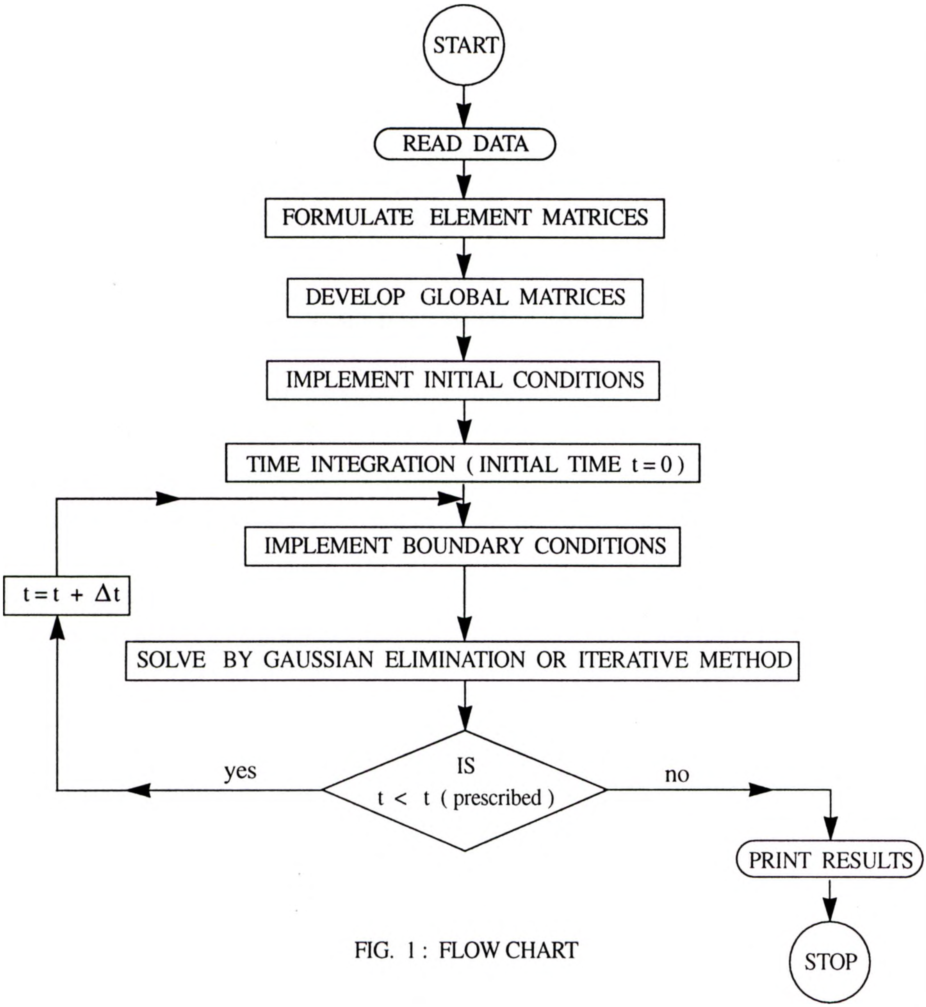
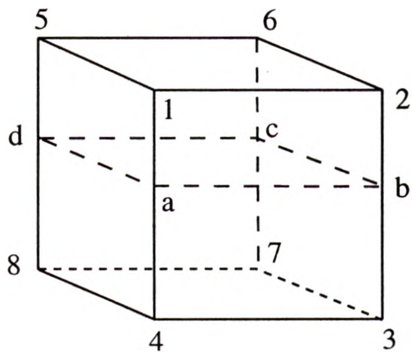


FIG. 1 : FLOW CHART

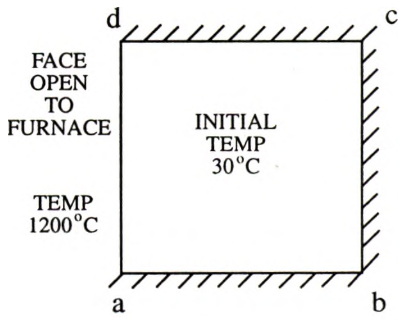
6.0 CASE STUDY

A case study of a heat conduction problem is presented to illustrate the solution by Finite element method.

A steel block of 50 cm x 50 cm x 50 cm is insulated on all sides except the one open to the furnace (Fig 2). The side open to the furnace is heated at a constant temperature of 1200 degrees centigrade. It is required to find the time of heating the block in order to attain about 800 degrees centigrade (annealing temperature) in the entire block (Myers G.E, 1971). The steel has the material property of diffusivity of 0.1736 cm2 /s, uniformly throughout the block. Thus the steel block is homogeneous and isotropic.



(A) STEEL BLOCK (50 x 50 x 50 cm)



(B) PLANE abcd (50 x 50 cm)

FIG. 2 CASE STUDY

As the medium is homogeneous and isotropic, the problem is reduced to be two-dimensional. Therefore, analysis is made on a two dimensional plane of steel block. The domain of the rectangular plane is discretised into 32 triangular elements having 25 nodal points on the global frame Work (Fig 3).

The data of this physical problem, including the local and global coordinates of all the nodal points, are fed into the computer program already developed in Fortran 77 language (Jagmohan Das, 1996) and the solution is obtained. The distribution of temperatures at all the nodal points, obtained at the time steps of $\Delta t = 900$ seconds are presented in Table 2 .

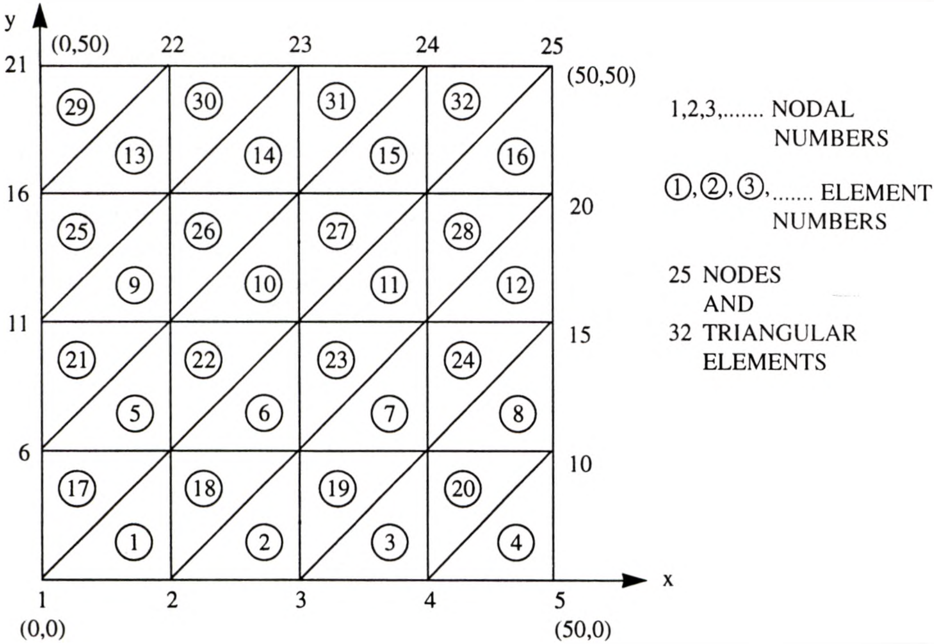


FIG 3 RECTANGULAR PLANE DISCRETISATION

It can be observed from the results presented in Table 2 that the furnace is required to heat the block of steel (with a constant temperature of 1200 degrees C) for a period of 8100 seconds (2 hours and 15 minutes) in order to attain the prescribed annealing temperature of about 800 degrees centigrade in the entire block of steel (last column of Table 2). This information helps the engineers of the industrial processing to continue the heating of the block up to that time.

7.0 DISCUSSIONS AND GENERAL CONCLUSIONS

The case study presented is just an example of the solution of an engineering problem with the application of Finite Element Method. In order to explain the applicability, a simple case of homogeneous and isotropic medium having a rectangular shape of the domain has been selected. However, the computer program developed has been successfully implemented on more complex cases of medium having inhomogeneity and medium properties varying in all directions. The author has also the experience of testing the different types of elements and different coordinate systems. It has been observed that the triangular elements with natural coordinate system are giving better results with faster convergence, though the computational effort is slightly more (Sreenivasulu & Jagmohan Das, 1985). The

TABLE 2 SOLUTION OF TEMPERATURES AT THE NODAL POINTS

Temperatures in degrees centigrade at the end of time in seconds									
Time in sec Nodal No	900	1800	2700	3600	4500	5400	6300	7200	8100
1	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0
2	522.8	710.4	809.1	873.8	922.5	961.7	994.6	1022.7	1046.7
3	215.9	384.1	511.9	612.3	694.7	764.2	823.6	874.7	918.7
4	105.4	219.5	338.9	450.1	549.4	636.6	712.4	778.2	935.2
5	81.0	177.0	290.6	403.2	506.8	598.3	679.3	749.6	812.4
6	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0
7	524.3	712.6	811.1	875.6	924.0	963.1	995.8	1023.7	1047.6
8	208.5	381.6	511.5	612.7	695.4	764.9	824.2	875.2	919.2
9	100.6	215.2	335.8	447.6	547.6	638.1	711.2	777.2	834.4
10	75.1	167.6	280.5	393.7	498.2	591.3	672.9	743.9	805.5
11	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0
12	524.1	712.5	811.3	875.9	924.3	963.4	996.1	1024.0	1047.6
13	206.0	379.4	510.1	611.8	694.7	784.4	523.9	875.0	919.0
14	98.4	211.7	332.3	444.8	545.1	633.0	709.4	775.7	833.1
15	72.7	162.8	275.0	388.4	493.5	587.2	669.4	740.9	802.9
16	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0
17	524.0	712.5	811.4	876.1	924.6	963.7	996.4	1024.2	1048.1
18	203.6	377.2	508.6	610.8	694.1	764.0	823.6	874.7	918.8
19	19.1	208.8	328.9	441.9	542.6	630.8	707.5	774.1	831.7
20	70.3	158.2	269.5	683.1	488.7	583.1	665.8	737.8	800.2
21	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0	1200.0
22	525.2	714.7	813.5	878.0	926.2	965.1	997.6	1025.3	1049.0
23	195.9	374.6	508.3	611.3	694.8	764.7	824.3	875.4	919.4
24	91.5	203.8	325.7	439.5	540.8	629.4	706.4	773.1	830.9
25	65.3	149.4	259.6	673.5	480.0	575.4	659.2	732.0	798.3

versatality of the Finite Element Method has been established in solving variety of problems like Heat conduction (Jagmohan Das, 1996), Groundwater pollutant dispersion (Jagmohan Das, 1997), Transient groundwater flow (Sreenivasulu & Jagmohan Das, 1982), Inverse problems of geohydrology (Sreenivasulu & Jagmohan Das, 1985) etc.

The following are the general conclusions drawn , based on the investigations carried out on different engineering problems.

1. Finite Element Method is a powerful numerical technique that can be used for the solution of boundary value problem having complex partial differential equation as its governing equation.

2. The algorithm oriented procedural steps presented can handle all types of field complexities such as in homogeneous and anisotropic properties of medium, nonlinear boundary geometry, complex boundary conditions, etc.

3. Triangular elements with natural coordinate system are more advantageous and yield better solutions.

4. The computer program developed in Fortran 77 is capable of handling almost all types of field problems.

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