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COEFFICIENT BOUNDS FOR A CERTAIN SUBCLASS OF CLOSE-TO-CONVEX FUNCTIONS ASSOCIATED WITH THE KOEBE FUNCTION

Sidik Rathi, Shaharuddin Cik Soh and Ajab Akbarally

Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia

sidik@tmsk.uitm.edu.my, shahar@tmsk.uitm.edu.my, ajab@tmsk.uitm.edu.my

ABSTRACT

We consider here the functions $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ *which are analytic and univalent in the open unit disc* $U=\{z\in\mathbb{C}:|z|<1\}$, *normalized by* $f(0)=0$ *and* $f'(0)=1$. *By* $\boldsymbol{C}(\lambda,\delta)$, *we denote a new subclass of close-to-convex function such that* $\operatorname{Re}\left\{e^{i\lambda}\frac{zf'(z)}{g(z)}\right\}>\delta$ *for which* $\cos\lambda>\delta$, $0\le\delta<1$, $|\lambda|<\frac{\pi}{2}$ *and* $g(z)=\frac{z}{(1-z)^2}$. *In this paper, we give the representation theorem and obtain the coefficient bounds for functions in* $\boldsymbol{C}(\lambda,\delta)$.

Keywords: *Close-to-convex functions, coefficient bounds, Koebe function, representation theorem, univalent functions.*

1. Introduction

Let ***A*** denote the class of functions *f* of the form

$$f(z)=z+\sum_{n=2}^{\infty}a_n z^n \tag{1}$$

analytic in the open unit disc U and normalized by $f(0)=0$ and $f'(0)=1$. Also, we denote S to be the class of functions in ***A*** containing univalent function of the form (1). According to Duren (1983), the function $f(z)\in S$ is called a convex function and starlike function if it satisfies $\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\}>0$ and $\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\}>0$ respectively. Alexander in 1915 defined $f\in S$ to be close-to-convex if it satisfy $\operatorname{Re}\left\{\frac{zf'(z)}{h(z)}\right\}>0$ with $h(z)\in S^*$ (Goodman, 1983). Later, Kaplan (1952) stated that $f\in S$ is said to be close-to-convex if and only if $\operatorname{Re}\left\{\frac{f'(z)}{g'(z)}\right\}>0$ for $z\in U$ and $g\in K$. Silverman and Silvia (1996) studied the class C_α of $\alpha-$close-to-convex functions where functions in this class satisfy $\operatorname{Re}\left\{e^{i\alpha}\frac{f'(z)}{g'(z)}\right\}>0$,

$|\lambda|<\frac{\pi}{2}$, $z\in U$, $g(z)=z$. Then, with the same $g(z)$, Mohamad (1998) defined $G(\alpha,\delta)$ as the generalized class of $\alpha-$close-to-convex functions satisfying $\mathrm{Re}\{e^{i\alpha}f'(z)\}>\delta$, $z\in U$, $|\alpha|\le\pi$ and $\cos\alpha>\delta$. Soh and Mohamad (2012) and Akbarally et al. (2011) also studied the class of $\alpha-$close-to-convex functions but with $g'(z)=\frac{1+z}{1-z}$ and $g'(z)=\frac{1}{1-z}$ respectively. Later, Yahya et al. (2012) studied the class $G_{St}(\alpha,\delta)$ such that $\mathrm{Re}\left\{e^{i\alpha}\frac{zf'(z)}{g(z)}\right\}>\delta$, $|\alpha|<\pi$, $\cos\alpha>\delta$ and $g(z)=\frac{z}{1-z^2}$.

In this paper, we investigate a similar class of function with $g\in S^*$. We define $\boldsymbol{C}(\lambda,\delta)$ as the class of functions that satisfy $\mathrm{Re}\left\{e^{i\lambda}\frac{zf'(z)}{g(z)}\right\}>\delta$, $|\lambda|<\frac{\pi}{2}$, $\cos\lambda>\delta$, $0\le\delta<1$ and $g(z)=\frac{z}{(1-z)^2}$, where $g(z)$ is the well-known Koebe function.

2. Representation Theorem

Let P be the class of functions analytic in U and having the form

$$p(z)=1+\sum_{n=1}^{\infty}p_nz^n \quad . \tag{2}$$

Since $p(z)$ in P satisfies $\mathrm{Re}\,p(z)>0$, for $z\in U$, we say that $p(z)$ is the Caratheodory function. We relate the functions in $\boldsymbol{C}(\lambda,\delta)$ to functions in P as given in the following theorem.

Theorem 2.1

Let $f\in S$ be given by (1). Then for $z\in U$, $f\in\boldsymbol{C}(\lambda,\delta)$ if and only if

$$e^{i\lambda}\left[zf'(z)/g(z)\right]-i\sin\lambda-\delta=A_{\lambda\delta}\,p(z)$$

where $A_{\lambda\delta}=\cos\lambda-\delta$.

Proof.

Let $f\in\boldsymbol{C}(\lambda,\delta)$, then $\frac{zf'(z)}{g(z)}=1+\sum_{n=1}^{\infty}b_nz^n$. Multiplying both sides of the equation with $e^{i\lambda}$ gives

$$\begin{aligned} e^{i\lambda}\frac{zf'(z)}{g(z)}&=e^{i\lambda}+e^{i\lambda}\sum_{n=1}^{\infty}b_nz^n,\\ &=\cos\lambda+i\sin\lambda+e^{i\lambda}\sum_{n=1}^{\infty}b_nz^n. \end{aligned}$$

Then

$$e^{i\lambda}\frac{zf'(z)}{g(z)} - i\sin\lambda - \delta = \cos\lambda - \delta + e^{i\lambda}\sum_{n=1}^{\infty} b_n z^n$$

and

$$\frac{e^{i\lambda}\dfrac{zf'(z)}{g(z)} - i\sin\lambda - \delta}{\cos\lambda - \delta} = 1 + \sum_{n=1}^{\infty} p_n z^n$$

with $p_n = \dfrac{e^{i\lambda}}{\cos\lambda - \delta} b_n$. Replacing $A_{\lambda\delta} = \cos\lambda - \delta$ gives

$$e^{i\lambda}\left[zf'(z)/g(z)\right] - i\sin\lambda - \delta = A_{\lambda\delta}\, p(z) \tag{3}$$

as required. Conversely, from (3) taking $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$ we can work backward to see that the conditions for functions to be in the class $\boldsymbol{C}(\lambda,\delta)$ is satisfied.

Next, we obtain the representation function for $f \in \boldsymbol{C}(\lambda,\delta)$ as given in the following theorem by using the Herglotz Representation Theorem for functions in *P*.

Theorem 2.2

Let $f \in \boldsymbol{C}(\lambda,\delta)$. Then for some probability measure μ on the unit circle X, f can be represented as

$$f(z) = \int_X \left[-e^{-i\lambda}\left(e^{-i\lambda} - 2\delta\right)\left(\frac{1}{(1-xz)} - 1\right) + A_{\lambda\delta} e^{-i\lambda}\left(\frac{1}{(1-xz)^2} - 1\right)\right]\frac{1}{x}\, d\mu(x)$$

for $|x| = 1$. Conversely, if f is given by the above equation, then $f \in \boldsymbol{C}(\lambda,\delta)$.

Proof.

According to Herglotz formula, for some probability measure μ on the unit circle X,

$$p \in P \Leftrightarrow p(z) = \int_X \frac{1+xz}{1-xz}\, d\mu(x) .$$

Replacing $g(z) = \dfrac{z}{(1-z)^2}$ in (3) and with the aid of Herglotz formula, yields

$$f'(z) = \frac{e^{-i\lambda}}{(1-z)^2}\left(A_{\lambda\delta}\int_X \frac{1+xz}{1-xz}\, d\mu(x) + i\sin\lambda + \delta \right)$$

and upon simplification, we get

$$f'(z)=\int_X \frac{1}{(1-z)^2}\left(-e^{-i\lambda}(e^{-i\lambda}-2\delta)+\frac{2e^{-i\lambda}A_{\lambda\delta}}{1-xz}\right)d\mu(x). \tag{4}$$

Integrating (4) with respect to *z*, gives

$$f(z)=\int_0^z\left[\int_X \frac{1}{(1-x\phi)^2}\left(-e^{-i\lambda}(e^{-i\lambda}-2\delta)+\frac{2e^{-i\lambda}A_{\lambda\delta}}{1-x\phi}\right)d\mu(x)\right]d\phi,$$

$$=\int_X\left[-e^{-i\lambda}\left(e^{-i\lambda}-2\delta\right)\left(\frac{1}{(1-xz)}-1\right)+e^{-i\lambda}A_{\lambda\delta}\left(\frac{1}{(1-xz)^2}-1\right)\right]\frac{1}{x}d\mu(x), \tag{5}$$

which is the desired representation function. By this result, we note that the extreme points of $\boldsymbol{C}(\lambda,\delta)$ are the unit point masses

$$f_x(z)=\left[-e^{-i\lambda}\left(e^{-i\lambda}-2\delta\right)\left(\frac{1}{1-xz}-1\right)+e^{-i\lambda}A_{\lambda\delta}\left(\frac{1}{(1-xz)^2}-1\right)\right]\frac{1}{x}$$

with $|x|=1$ and the derivatives of the extreme points of $\boldsymbol{C}(\lambda,\delta)$ are the point masses

$$f'_x(z)=\left[\frac{1+(e^{-2i\lambda}-2\delta e^{-i\lambda})xz}{1-xz}\right]\left(\frac{1}{(1-xz)^2}\right),\ |x|=1.$$

3. Coefficient Bound

Using the representation formula established in Theorem 2.2, we now obtain the coefficient bound of functions in $\boldsymbol{C}(\lambda,\delta)$.

Theorem 3.1

Suppose that $f\in S$ *is given by (1), then the sharp inequality* $|a_n|\le 1+A_{\lambda\delta}(n-1)$ *holds for* $n=2,3,4,\ldots$. *Equality is attained for each n when f is an extreme point of* $\boldsymbol{C}(\lambda,\delta)$.

Proof.

From (5) we write

$$f(z)=\int_0^z\left[\int_X\left(\frac{-e^{-i\lambda}(e^{-i\lambda}-2\delta)}{(1-x\phi)^2}+\frac{2e^{-i\lambda}A_{\lambda\delta}}{(1-x\phi)^3}\right)d\mu(x)\right]d\phi$$

and since

$$\frac{1}{(1-x\phi)^2}=\sum_0^\infty(n+1)(x\phi)^n \text{ and } \frac{1}{(1-x\phi)^3}=\sum_0^\infty\frac{1}{2}(n+1)(n+2)(x\phi)^n,$$

then we have

$$f(z)=\int_0^z\left[\int_X\left(-e^{-i\lambda}(e^{-i\lambda}-2\delta)\sum_0^\infty(n+1)(x\phi)^n+2A_{\lambda\delta}e^{-i\lambda}\sum_0^\infty\frac{1}{2}(n+1)(n+2)(x\phi)^n\right)d\mu(x)\right]d\phi\,,$$

$$=\int_0^z\int_X\left[1+\left(-e^{-i\lambda}(e^{-i\lambda}-2\delta)\sum_1^\infty(n+1)x^n\right)\phi^n+\left(A_{\lambda\delta}e^{-i\lambda}\sum_1^\infty(n+1)(n+2)x^n\right)\phi^n\right]d\mu(x)d\phi\,,$$

$$=\int_0^z\left[1+\left(-e^{-i\lambda}(e^{-i\lambda}-2\delta)\right)\left(\int_X\sum_1^\infty(n+1)x^n d\mu(x)\right)\phi^n\right]d\phi$$

$$+\int_0^z A_{\lambda\delta}e^{-i\lambda}\left(\int_X\sum_1^\infty(n+1)(n+2)x^n d\mu(x)\right)\phi^n\,d\phi.$$

Integrating $f(z)$ with respect to ϕ gives

$$f(z)=z+\left[\left(-e^{-i\lambda}(e^{-i\lambda}-2\delta)\right)\int_X\sum_{n=1}^\infty(n+1)x^n d\mu(x)\frac{z^{n+1}}{n+1}\right]+\left[A_{\lambda\delta}e^{-i\lambda}\int_X\sum_{n=1}^\infty(n+1)(n+2)x^n d\mu(x)\frac{z^{n+1}}{n+1}\right],$$

$$=z+\left[\left(-e^{-i\lambda}(e^{-i\lambda}-2\delta)\right)\int_X\sum_{n=2}^\infty nx^{n-1}d\mu(x)\frac{z^n}{n}\right]+\left[A_{\lambda\delta}e^{-i\lambda}\int_X\sum_{n=2}^\infty n(n+1)x^{n-1}d\mu(x)\frac{z^n}{n}\right]$$

and upon factorization we obtain

$$f(z)=z+\left[\left(-e^{-i\lambda}(e^{-i\lambda}-2\delta)+A_{\lambda\delta}e^{-i\lambda}(n+1)\right)\int_X\sum_{n=2}^\infty nx^{n-1}d\mu(x)\right]\frac{z^n}{n}\,. \tag{6}$$

Now, comparing (6) and (1) gives

$$a_n=\left[-e^{-i\lambda}(e^{-i\lambda}-2\delta)+A_{\lambda\delta}e^{-i\lambda}(n+1)\right]\int_X x^{n-1}d\mu(x),$$

$$=\left[1-2Ae^{-i\lambda}+A_{\lambda\delta}e^{-i\lambda}(n+1)\right]\int_X x^{n-1}d\mu(x),$$

$$=\left[1+A_{\lambda\delta}e^{-i\lambda}(n-1)\right]\int_X x^{n-1}d\mu(x).$$

Then,

$$|a_n|=\left|1+A_{\lambda\delta}e^{-i\lambda}(n-1)\int_X x^{n-1}d\mu(x)\right|\le 1+\left|A_{\lambda\delta}e^{-i\lambda}(n-1)\right|\int_X\left|x^{n-1}\right|d\mu(x)=1+A_{\lambda\delta}(n-1)$$

and for $n\ge 2$, we have $|a_n|\le 1+A_{\lambda\delta}(n-1)$ as required. This completes the proof of Theorem 3.1 where the equality holds when f is an extreme point of $\boldsymbol{C}(\lambda,\delta)$.

References

Akbarally, A., Mohamad, D., Soh, S. C. & Kaharudin, N. (2011). On the properties of a new class of α -close-to-convex functions. *Int. Journal of Math. Analysis,* vol 5, no. 8, 387-396.

Duren, P.L. (1983). *Univalent Function*. Springer Verlag New York Inc.

Goodman, A. W. (1983). *Univalent Function*. Mariner Publishing Co. Inc. Tampa, University of south Florida.

Kaplan, W. (1952). Close-to-convex schlicht function. *Michigan Mathematical Journal, 1*. 169-185.

Mohamad, D. (1998). Function whose derivative maps the unit disc into a half plane. PhD Thesis. University of Wales Swansea, UK.

Silverman, H., & Silvia, E. M. (1996). On α-close-to-convex Functions. *Publicationes Mathematicae Debrecen,* 49, 532-537.

Soh, S. C., & Mohamad. D. (2008). Coefficient bounds for certain classes of close-to-convex functions. *International Journal of Mathematical Analysis, Vol. 2, No. 27,* 1343-1351.

Yahya, A., Soh, S. C. & Mohamad, D. (2012). Some extremal properties of a generalised Sakaguchi class of cnalytic functions. *Proceedings of the 20th National Symposium on Mathematical Sciences*.