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Applications of leverenz theorem in univalent functions

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Abstract
Leverenz theorem is stated and equivalent finite form is then obtained. Examples are then illustrated to show some of the applications of the theorem. It was found that the results obtained are analogous to the some of the well-known inequalities for class P and class B.

Keywords: Univalent functions; Coefficients; Leverenz theorem; Starlike function

1. Introduction

Let U = $\{z : |z| < 1\}$ be the unit disc. Let H(U) be the space of functions analytic in U and GL(n) be the space of n x n matrices with complex entries. Let T_n : H(U) $\rightarrow$ GL(n) be defined such that T_n f is an n x n matrix where the ij - th entry of T_n f is given by

$$(T_n f)_{i,j} = \begin{cases} 0 & if \quad j < i \\ a_{j-i} & if \quad j \geq i \end{cases} \qquad i, j = 1,2,3, \ldots ,n$$

where $f(z) = \sum_{k=0}^{\infty} a_k z^k \in H(U)$.

Denote S* to be the class of analytic functions f in U normalized so that f(0) = f'(0) - 1 = 0 and such that $\mathrm{Re}\frac{zf'(z)}{f(z)} > 0, z \in U$. Thus a $f \in S^*$ maps U univalently onto a domain *starlike with respect to the origin.* Closely related to the classes S* is the class of all functions p analytic and having a positive real part in U with $p(0)=1$. If $p \in P$, Rep(z)>0 then $p(z) = 1 + \sum_{k=0}^{\infty} p_k z^k$ is called functions with *positive real part.* Let B denote the class of functions bounded functions w that are analytic in U and satisfies $|w(z)| \leq 1$. The class B and P are related through linear transformation $P(z) = \frac{1 - w(z)}{1 + w(z)}$.

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2. Some preliminaries

Definition 1

The *inner product* of any complex number x, y shall be denoted by $\langle x,y\rangle$ where $\langle x,y\rangle = \sum_{i=0}^{n-1} x_i \overline{y_i}$

Let A , D $\in$ GL(n). We say that A is *majorized* by D, denoted by A << D if $\|Ax\| \leq \|Dx\|$ for all $x \in C^n$ (Duren, 1983).

Theorem 1 (Leverenz Theorem)

Let $f(z) = \sum_{n=1}^{\infty} a_n z^n$ and $g(z) = \sum_{n=1}^{\infty} b_n z^n$ be analytic functions in U. Then $|f(z)| \leq |g(z)|$ in U if and only if

$$H(z) = H_f{}^*(z) - H_g{}^*(z)$$
$$= \sum_{j=0}^{\infty} \left\{ \left| \sum_{k=0}^{\infty} a_k z_{k+j} \right|^2 - \left| \sum_{k=0}^{\infty} b_k z_{k+j} \right|^2 \right\}$$

is positive semidefinite on the family of all sequences z.

We note that Leverenz theorem is in an infinite form. We next obtain the finite form of Leverenz theorem using Pick's theorem as stated below (Leverenz, 1984).

Theorem 2 (Pick's Theorem)

Let $f : U \rightarrow U$ be analytic and set $w_k = f(z_k)$, $k = 1,2,3,\ldots,n$. Then the Hermition form

$$Q_n(t) = \sum_{n,k=1}^{n} \frac{1 - w_n \overline{w_k}}{1 - z_k \overline{z_k}} \, t_n \overline{t_k}$$

is positive definite (or semi definite) (Pick, 1915).

Corollory 1

Let w be analytic in U and $w(z) = \sum_{n=0}^{\infty} c_n z^n$. Then $|w(z)| \leq 1$, if and only if for $n \in N$, and all z_0, z_1, z_2, ..., $z_n \in C$, we have

$$\sum_{k=0}^{n} \left| \sum_{m=k}^{n} c_{m-k} z_m \right|^2 \leq \sum_{k=0}^{n} |z_k|^2 .$$

Theorem 2 is anologous to the following result, that is if,

$$f(z) = \sum_{k=0}^{\infty} a_k z^k \text{ and } g(z) = \sum_{k=0}^{\infty} b_k z^k, \; w = \frac{f}{g}, \; g(z) \neq 0 \quad \text{is analytic in U,}$$

$|W| \leq 1$ if and only if, for all in,

$$\sum_{k=0}^{n} \left| \sum_{m=k}^{n} a_{m-k} z_m \right|^2 \leq \sum_{k=0}^{n} \left| \sum_{m=k}^{n} b_{m-k} z_m \right|^2$$

then $T_n f << T_n g$

3. Some applications

Examples are now given to illustrate how Leverenz theorem in the above form together with majorization results are used to give well known coefficient results in the class P and class of bounded function W.

Result 1

Let $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in$ P. Then

(i) $|p_3 - p_2 p_1| \leq 2$

(ii) $|p_3 - 2p_1 p_2 + p_1^3| \leq 2$

Proof

Let

$$W(z) = \frac{p(z) - 1}{z(p(z)+1)} = \frac{\sum_{n=1}^{\infty} p_n z^{n-1}}{2 + \sum_{n=1}^{\infty} p_n z^n}$$

Let $W(z) = f(z) / g(z)$ where $f(z) = [p(z) - 1]/z$ and $g(z) = p(z) + 1$. Hence,

by Leverenz theorem and $|W(z)| \leq 1$,

$$T_n f << T_n g$$

which implies that $\|(T_n f)x\| \leq \|(T_n g)x\|$, where $x \in C^n$.

For $n = 3$, we have that is $T_3\left(\frac{p(z)-1}{z}\right) << T_3(p+1)$

$$\begin{pmatrix} p_1 & p_2 & p_3 \\ 0 & p_1 & p_2 \\ 0 & 0 & p_1 \end{pmatrix} << \begin{pmatrix} 2 & p_1 & p_2 \\ 0 & 2 & p_1 \\ 0 & 0 & 2 \end{pmatrix}$$

Choosing $x = \begin{pmatrix} 0 \\ -p_1 \\ 1 \end{pmatrix}$ yields

$$|-p_2p_1 + p_3|^2 + |p_2 - p_1^2|^2 + |p_1|^2 \leq |p_2 - p_1^2|^2 + |-2p_1|^2 + 4 \tag{1}$$

which simplifies inequality 1.

Next let $x = \begin{pmatrix} p_1^2 - p_2 \\ -p_1 \\ 1 \end{pmatrix}$. Then

$$\left|p_1(p_1^2 - p_2) - p_2p_1 + p_3\right|^2 + \left|p_2 - p_1^2\right|^2 + |p_1|^2$$

$$\leq \left|2(p_1^2 - p_2) - p_1^2 + p_2\right|^2 + |-2p_1 + p_1|^2 + 4 \tag{2}$$

and this yields inequality 2.

A more general form of the inequalities above can also be derived as follows .

Result 2

Let $f(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \in$ P. Then

i) $\left|p_{n+k} - \frac{1}{2}p_n p_k\right|^2 \leq (2 - \frac{1}{2}|p_n|^2)(2 - \frac{1}{2}|p_k|^2) \leq \left(2 - \frac{|p_n p_k|}{2}\right)^2$

ii) $|p_{n+k} - p_n p_k| \leq 2$

for n, $k = 1, 2, 3, \ldots$

Proof

From the Example 1 , $T_{n+k}\left(\frac{p(z)-1}{z}\right) << T_{n+k}(p+1)$ where n , k positive integer. Let

$$A = \begin{pmatrix} p_1 & p_2 & p_3 & \cdot & p_{n+k} \\ 0 & p_1 & p_2 & \cdot & p_{n+k-1} \\ 0 & 0 & p_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & p_2 \\ 0 & 0 & \cdot & \cdot & p_1 \end{pmatrix}, \quad D = \begin{pmatrix} 2 & p_1 & p_2 & \cdot & p_{n+k-1} \\ 0 & 2 & p_1 & \cdot & p_{n+k-2} \\ 0 & 0 & 2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & p_1 \\ 0 & \cdot & \cdot & \cdot & 2 \end{pmatrix}$$

where $A = T_{n+k}\left(\frac{p(z)-1}{z}\right)$ and $D = T_{n+k}(p+1)$

Take u to be nth unit vector with 1 in the n-th position and v the (n+k)th unit vector with 1 in the (n+k)-th position , u and v $\in C^{n+k}$, (Siti Aishah, 1996).

$$Au = (p_n, p_{n-1}, \dots, p_1, 0, \dots, 0) \qquad Av = (p_{n+k}, p_{n+k-1}, \dots, p_1)$$

$$Du = (p_{n-1}, p_{n-2}, \dots, 2, 0, \dots, 0) \qquad Dv = (p_{n+k-1}, p_{n+k-2}, \dots, p_1, 2)ß$$

Applying Theorem 1 we then get the following

$$\left|\sum_{i=1}^{n} p_i \overline{p_{k+i}} - \sum_{i=0}^{n-1} p_i \overline{p_{k+i}}\right|^2$$

$$\leq \left(\sum_{i=0}^{n-1}|p_i|^2 - \sum_{i=1}^{n}|p_i|^2\right)\left(\sum_{i=0}^{n+k-1}|p_i|^2 - \sum_{i=1}^{n+k}|p_i|^2\right)$$

and this simplifies to

$$\left|p_n \overline{p}_{n+k} - 2\overline{p_k}\right|^2 \leq (4-|p_n|^2)(4-|p_{n+k}|^2)$$

$$= |p_n p_{n+k}|^2 + 4|p_k|^2 - 4\,\mathrm{Re}(p_n \overline{p}_{n+k} p_k) \leq 16 - 4|p_n|^2 - 4|p_{n+k}|^2 + |p_n|^2|p_{n+k}|^2$$

$$= 4|p_{n+k}|^2 - 4\,\mathrm{Re}(p_n \overline{p}_{n+k} p_k) + |p_n p_k|^2$$

$$\leq 16 - 4|p_n|^2 - 4|p_k|^2 + |p_n p_k|^2$$

$$= |2p_{n+k} - p_n p_k|^2 \leq (4-|p_n|^2)(4-|p_k|^2)$$

This establishes inequality (i) . Inequality (ii) can be derived from the result in (i), since

$$4|p_{n+k}|^2 - 4\,\mathrm{Re}\left\{ p_n p_k \overline{p_{n+k}} \right\} \le 16 - 4|p_n|^2 - 4|p_k|^2$$

$$|p_{n+k}|^2 + |p_n|^2 + |p_k|^2 + \mathrm{Re}\left\{ p_n p_k \overline{p_{n+k}} \right\} \le 2$$

The results are sharp for $p(z) = \dfrac{1+z^k}{1-z^k}$

Result 3

Let $w = \sum_{n=0}^{\infty} c_n z^n, \in$ B . Then

$$|c_n|^2 \le \left(1-|c_0|^2\right)\left(1-\sum_{k=0}^{n-1}|c_k|^2\right), \quad n = 1,2,3, \ldots$$

Proof

Let $w = \sum_{n=0}^{\infty} c_n z^n$, and consider $|w(z)| \le 1$

Denote u and v as a unit vector, that is

$u = (1,0,0, \ldots ,0,0)$, $v = (0,0,0, \ldots ,0,1)$ where $u,v \in C^{n+1}$

$$A = T_n w = \begin{pmatrix} c_0 & c_1 & c_2 & . & c_n \\ 0 & c_0 & c_1 & . & c_{n-1} \\ . & . & c_0 & 0 & . \\ . & . & . & . & c_1 \\ . & . & . & . & c_0 \end{pmatrix}$$

Hence $Au = (T_n w)\, u = (c_o , 0 , 0 , \ldots , 0)$; $Av = (T_n w)v = (c_n , c_{n-1} , \ldots ,c_0)$

$\langle Au, Av \rangle = c_0 \overline{c_n}$. By Leverenz theorem $T_n w << I_n$.

Let $H = I_n$, $Hu = (1, 0 , 0 , \ldots , 0)$ and $Hv = (0 ,0 , 0 , \ldots , 1)$.

Applying Theorem 1, $\left|\rangle Au, Av\langle - \rangle Hu, Hv\langle\right|^2 = |c_0 \overline{c_n}|^2$

$$\|Hu\|^2 - \|Au\|^2 = \|I_n u\|^2 - \|(T_n w)u\|^2 = 1 - |c_o|^2, \quad \|Hv\|^2 - \|Av\|^2 = 1 - \sum_{k=0}^{n} |c_k|^2$$

The above yield the following

$$\left|c_0\overline{c_n}\right|^2 \leq \left(1-|c_0|^2\right)\left(1-\sum_{k=0}^{n}|c_k|^2\right) - |c_n|^2\left(1-|c_0|^2\right) + \left(1-|c_0|^2\right)\left(1-\sum_{k=0}^{n-1}|c_k|^2\right)$$

and the result follows.

Examples above illustrate how the application of Leverenz theorem can be used to obtain well known coeficient results. A similar method can be apply to obtain a new coefficients results of other class of univalent functions such as obtain by Rosihan & Halim (4) for class P.

References

Duren P.L 1983. Univalent Functions ,Vol 1, Springer Verlag, New York.

Leverenz C.R 1984. "Hermitian forms in function Theory", *Trans Amer. Math. Soc.*286 , 675-688

Pick .G. 1915. "On Hermitian form" , *Annals of maths*,Vol. 179,1-6,9-23

Rosihan Md. Ali & Halim 1996. "Pekali Fungsi yang Terbatas dan Tak Lenyap", *Simposium Matematik kebangsaan ke-1V*, Shah Alam

Siti Aishah Sheikh Abdullah 1996. "Functions Starlike With respect to a boundary Point" , Unpublished Master Thesis, USM

Biography

Siti Aishah Sheikh Abdullah is a Mathematics Lecturer at UiTM Penang Campus. She graduated with BSc(Hons)Leeds, UK, MSc in (USA) in Pure Mathematics and currently pursuing her PhD in Mathematics Education at USM.