Derivation of Three-Dimensional Heat Transfer Mathematical Equations to Predict Conductivity of Insulation Systems for Liquid Nitrogen Pipe Flow

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ABSTRACT

Cryogenics is dealing with very low temperatures of less than 120 K. Thermal insulation is a key to minimize heat leak in cryogenics during transportation in transfer line. The present study is conducted to derive three-dimensional heat transfer mathematical equations to predict conductivity of insulation systems for liquid nitrogen pipe flow. Elliptic differential equations on special domains were derived based on Fourier method for three-dimensional heat equations. As a result from the study, three-dimensional heat conduction equation for a cylindrical problem was successfully derived. The derived equation provides a mathematical solution to determine the heat transfer rate for a system of cryogenic pipe insulated with multiple layers.

Keywords: *Cryogenic, Thermal Insulation System, Liquid Nitrogen, Pipe Flow, Three-Dimensional Heat Conductivity*

ISSN 1823-5514, eISSN 2550-164X

Received for review: 2016-05-16
Accepted for publication: 2017-02-01
Published: 2017-05-15

Introduction

Cryogenics is a branch of physics dealing with the production and effects of very low temperatures of less than 120 K (−153 °C) [1], although this historical summary does not adhere to a strict 120 K definition [2]. Applications of cryogenic can be found in variety of fields such as physics, chemistry, biology, medicine, engineering and industry. Cryogenic technology gained widespread recognition during the 1960s with emphasis on cryogenic techniques, cooling system installation configurations, and applications [3].

Thermal insulation is a key to minimize heat leak in cryogenics during transportation of transfer line, which is extremely important in cryogenics that deal with very low temperature apparatuses and experiments. Heat leak in cryogenic insulated transfer line that caused temperature raise is a big concern. Efficient thermal insulation determines the cost effectiveness in cryogenics transfer line. Hence, effective thermal insulation is required to minimize heat leak.

The most important advancement in cryogenic insulation over the past 50 years has been the development of multilayer insulation (MLI) [4]. Based on the study done by Chorowski and Polinski [5], the most efficient and best matched to cryogenic conditions is a multilayer vacuum insulation (MLI). Designing an effective thermal insulation for cryogenic transfer line is a complex study. Studies on cryogenic transfer lines have regained importance in recent times with the growth of large-scale applications of superconductivity and the needs of space programmes [6, 7].

Liquid nitrogen

Liquid nitrogen (LN2) is obtained from air in large liquefaction and separation plants. LN2 is a cheap and safe source of cold that is able to maintain temperatures far below the freezing point of water. LN2 is commonly used in the pre-cooling of cryogenic equipment due to its high latent heat of evaporation [8].

LN2 boils at 77 K (−196 °C) and freezes at 63 K (−210 °C). To maintain LN2 in the liquefied form is difficult as there is a continuous boil off due to heat in leaks, especially in the ambient temperature (300 K) that is relatively high compared to LN2 boiling temperature (77 K). The liquid nitrogen boils off continuously due to various modes of heat transfer. LN2 is a very efficient coolant but limited by the Leidenfrost effect. LN2 is a compact and readily transported source of nitrogen gas without pressurization. LN2 is a fairly inert gas medium and has unique properties that makes it the most ideal space simulation chamber with cold drawing of stainless steel used in specific industrial and scientific research applications [3].

Heat transfer in cryogenic

The heat transfer processes in cryogenics are basically the same for any engineering temperature range. According to T.M. Flynn [9], heat transfer at low temperatures is governed by the same three mechanisms present at ambient and elevated temperatures: conduction, convection, and radiation. Therefore, all the general equations are appropriate for low-temperature applications as long as they are adjusted for the property changes in both materials and fluids [9].

According to Jha A.R. [3], cryogenic heat transfer depends on the operating parameters, flow pipe cross-sectional geometry (circular or rectangular or square), and fluid flow types such as laminar, turbulent, or transient. Heat transfer properties for a linear flow are quite different from those for a turbulent flow or transient flow. Under the turbulent flow environment, the flow will be treated as a non-homogenous flow; therefore, nonlinear flow equations will be involved in the thermal analysis of heat exchangers [3]. The rotation of the tube carrying the fluid does not affect the laminar flow resistance once the established cooling flow exists. However, under the turbulent flow conditions, flow resistance undergoes radical change [3].

Figure 1 shows the typical design of vacuum insulated pipe (VIP) recommended by the industry partner, Cryogas Tech [10]. The outer pipe temperature is equivalent to the ambient temperature, 300 K (27˚C) and the inner pipe temperature is equivalent to the liquid nitrogen temperature, 77 K (-196˚C). Since there is temperature difference between outer pipe and inner pipe, the heat transfer is in the direction of decreasing temperature which is from outer pipe to inner pipe. This means the heat flows into the pipe containing cryogenic fluid.

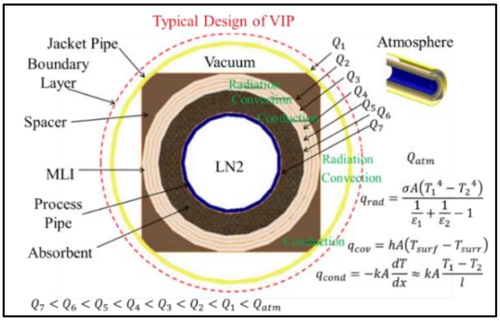


Figure 1: Typical Design of Vacuum Insulated Pipe (VIP) [10]

Concept of the heat flows into the pipe containing cryogenic fluid

Heat transfer (or heat) is thermal energy in transit due to a spatial temperature difference [11]. According to the second law of thermodynamics, heat flows from high temperature to lower temperature objects. If these objects are not thermal insulated and they are in proximity, heat flows between them can only be slowed instead of being stopped.

The Second Law of Thermodynamics states that "in all energy exchanges, if no energy enters or leaves the system, the potential energy of the state will always be less than that of the initial state." This is also commonly referred to as entropy.

Temperature and heat flow represents thermal energy available and movement of thermal energy from place to place respectively. Temperature and heat flow are the two things concerned with the discipline of heat transfer.

Heat transfer mechanisms can be grouped into conduction, convection, radiation or any combination of these. Heat conduction or thermal conduction is the transfer of energy from the more energetic (higher temperature) to the less energetic particles (lower temperature) of a substance due to interactions between the particles [11]. Based on Fourier Law of heat conduction, when there exists a temperature gradient within a body, heat energy will flow from the region of high temperature to the region of low temperature. This phenomenon is known as conduction heat transfer and is described by Fourier's Law (named after the French physicist Joseph Fourier). According to Fourier Law of heat conduction, thermal conductivity (k) is the property of a material that indicates its ability to conduct heat and it appears primarily in Fourier's Law for heat conduction.

Thermal resistance circuits in series and parallel for the thermal insulated cryogenic flow

For one-dimensional problem, where the temperature variation is the only variable (example, along the *x*-coordinate direction), Fourier's Law of heat conduction is simplified to the scalar equations as [9, 11, 12]:-

$$q'' = -k\frac{\partial T}{\partial x} \tag{1}$$

$$q = -kA\frac{\partial T}{\partial x} \tag{2}$$

where the heat flux, q'', dependent to the given temperature profile, T, and thermal conductivity, k. The negative sign is to indicate the heat flows down the temperature gradient.

From Equation (2), q represents the heat flow through a defined cross-sectional area A, measured in watts:-

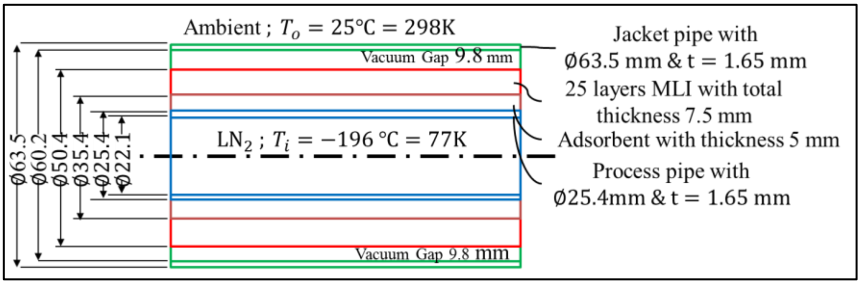
$$q = \int_A q'' \, dA \tag{3}$$

Integrating the one-dimensional heat flow equation through a material's thickness Δx gives: -

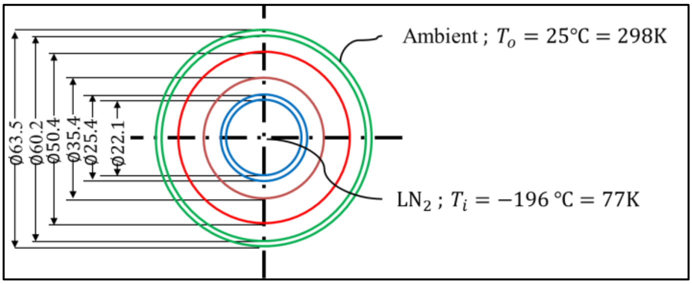
$$q = \frac{kA}{\Delta x}(T_1 - T_2) \tag{4}$$

where T_1 and T_2 are the temperatures at the two boundaries.

Figure 2 shows the side view and front view of typical design of vacuum insulated pipe for the present study.



(a) Side View



(b) Front View (Cross Sectional View)

Figure 2: Side View and Front View of Typical Design of Vacuum Insulated Pipe

Treatment of heat transfers analogous to electrical circuit

An analogy exists between the thermal resistance in a thermal circuit and electrical resistance in an electrical circuit [11]. Equivalent thermal circuit for composite slab or cylindrical shell system may be characterized by series–parallel configurations to determine the equivalent thermal conductivity for the system.

Figure 3 shows the electrical circuit or thermal resistance schematic of the insulation systems. The thermal resistances are connected in series from layer to layer of the insulation systems.

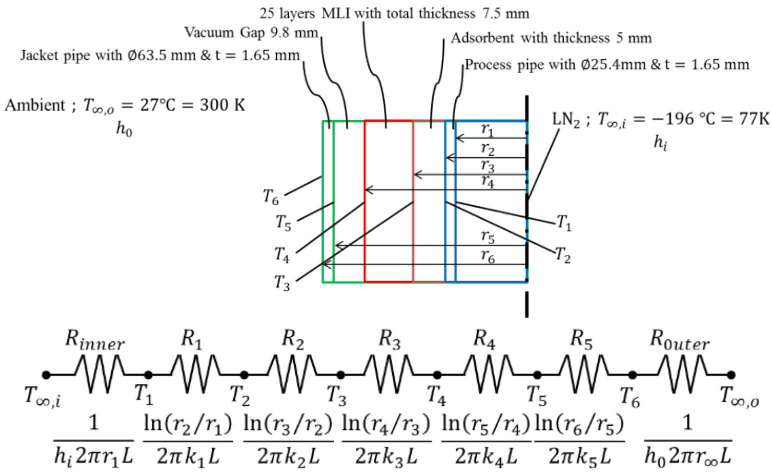


Figure 3: Thermal resistance schematic of the insulation systems

Mathematical Equation Result

Three-dimensional heat conduction

For one-dimensional problem, Fourier's Law of heat conduction is simplified to the scalar equations as Equation (1) and (2). For three-dimensional problem in Cartesian coordination (x, y, z) with temperature, $T(x, y, z)$, the conduction heat flux vector will be [9, 11, 12]:

$$q'' = -k\frac{\partial T}{\partial x}i - k\frac{\partial T}{\partial y}j - k\frac{\partial T}{\partial z}k \quad (5)$$

where: $q''_x = -k\frac{\partial T}{\partial x}$; $q''_y = -k\frac{\partial T}{\partial y}$; $q''_z = -k\frac{\partial T}{\partial z}$

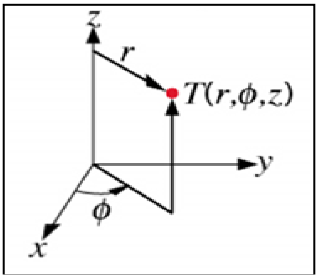


Figure 5: Cylindrical coordination systems

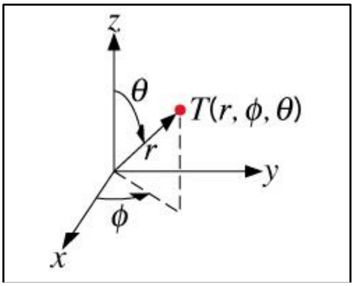


Figure 6: Spherical coordination systems

For three-dimensional problem in cylindrical coordination $(r, \emptyset, z)$ with temperature, $T(r, \emptyset, z)$, the conduction heat flux vector will be:

$$q'' = -k\frac{\partial T}{\partial r}i - k\frac{\partial T}{r\partial \emptyset}j - k\frac{\partial T}{\partial z}k \qquad (6)$$

where: $q''_r = -k\frac{\partial T}{\partial r}$; $q''_\emptyset = -k\frac{\partial T}{r\partial \emptyset}$; $q''_z = -k\frac{\partial T}{\partial z}$

For 3-dimensional problem in spherical coordination $(r, \emptyset, \theta)$ with temperature, $T(r, \emptyset, \theta)$, the conduction heat flux vector will be:

$$q'' = -k\frac{\partial T}{\partial r}i - k\frac{\partial T}{r\partial \theta}j - k\frac{\partial T}{r\,sin\,\theta\,\partial \emptyset}k \qquad (7)$$

where: $q''_r = -k\frac{\partial T}{\partial r}$; $q''_\theta = -k\frac{\partial T}{r\partial \theta}$; $q''_\emptyset = -k\frac{\partial T}{r\,sin\,\theta\,\partial \emptyset}$

As shown in Figure 7, three-dimensional problem in cylindrical coordination $(r, \emptyset, z)$ with temperature, $T(r, \emptyset, z)$ are proposed to be used for the present study.

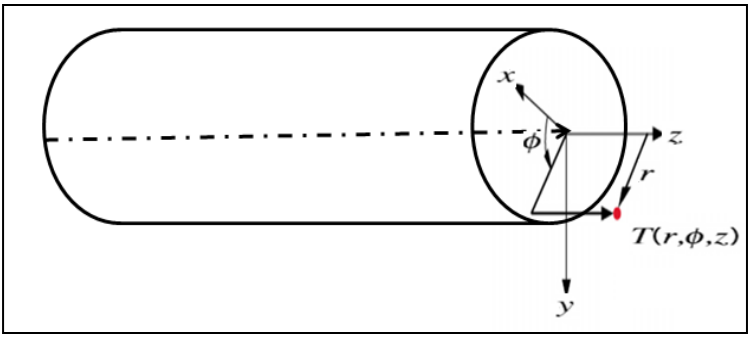


Figure 7: Cylindrical coordination systems proposed for the present study

Three-dimensional heat equation

By letting $u(x, y, z, t)$ as the temperature at a point (x, y, z) and instant time t, the differential equation as heat equation will be:

$$\frac{\partial u}{\partial t} = c^2 \nabla^2 u = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) \tag{8}$$

where the constant $c^2 = k/(\sigma\rho)$ is thermal diffusivity, k is the thermal conductivity, σ is the specific heat, and ρ is the density of the material.

Let the initial condition as $u(x, y, z, 0) = f(x, y, z)$, where $f(x, y, z)$ is the initial temperature distribution.

For the cylindrical problem, $S = \{x^2 + y^2 \leq r^2; 0 \leq z \leq Z_0\}$ with boundary curves $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$. (describe the boundary Γ_1, Γ_2 and Γ_3. For example $\Gamma_1 = \{(x, y, z): x^2 + y^2 = r^2, 0 \leq z \leq Z_0\}, \Gamma_2 = \{(x, y, z): x^2 + y^2 \backslash\ r^2, z = 0\}, \Gamma_3 = \{(x, y, z): x^2 + y^2 \leq r^2, z = Z_0\},$)

The boundary conditions as $u|_{\Gamma_i} = g\_i\ (x, y, z)$; for $i = 1,2,3$. We define

$$g(x, y, z) = \begin{cases} g_1(x, y, z), & (x, y, z) \in \Gamma_1; \\ g_2(x, y, z), & (x, y, z) \in \Gamma_2; \\ g_3(x, y, z), & (x, y, z) \in \Gamma_3. \end{cases}$$

This function is piecewise continuous function. Many engineering problems can be reduced to the heat equation with piecewise smooth function on the boundary. We can make the latter continuous by letting

$$(g_1 - g_2)|_{\Gamma_1 \cap \Gamma_2} = 0, (g_1 - g_3)|_{\Gamma_1 \cap \Gamma_3} = 0, (g_2 - g_3)|_{\Gamma_2 \cap \Gamma_3} = 0.$$

In this work, we present the separation method for the case of steady heat transfer processes.

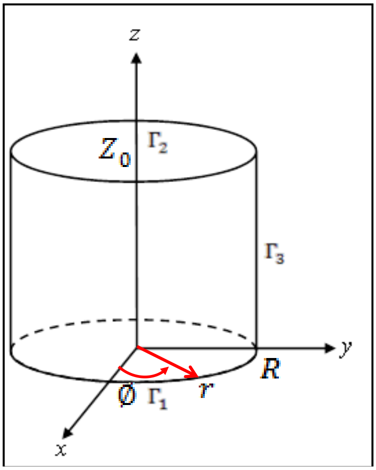


Figure 8: Cylindrical problem

For the steady-state, the heat does not depend on the time changes:

$$\frac{\partial u}{\partial t} = 0$$

In the case of steady state, the boundary problem for the heat equation becomes:

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0, (x, y, z) \in S, \tag{9}$$

$$u(x, y, z)|_{\Gamma} = g(x, y, z). \tag{10}$$

Shifting to the cylindrical coordinates

$$\begin{array}{l} x = r\cos\emptyset\,, \\ y = r\sin\emptyset, \\ z = z \end{array} \qquad 0 \le \emptyset \le 2\pi, 0 \le r \le R, 0 \le z \le Z_0, \tag{11}$$

We obtain the following forms for the boundary conditions:

$$\begin{array}{ll} u(r,\emptyset,0) = g_1(r,\emptyset) & 0 \le r \le R, 0 \le \emptyset \le 2\pi \\ u(r,\emptyset,Z_0) = g_2(r,\emptyset) & 0 \le r \le R, 0 \le \emptyset \le 2\pi \\ u(R,\emptyset,z) = g_3(\emptyset,z) & 0 \le \emptyset \le 2\pi, 0 \le z \le Z_0 \end{array}$$

The heat equation will have the following form:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \emptyset^2} + \frac{\partial^2 u}{\partial z^2} = 0. \tag{12}$$

To solve Equation (12), three-dimensional Laplace's equation, the separation-of-variables method is used. By assuming the solution to the Equation (12) as a product of functions of the independent variables$(r,\emptyset,z)$:

$$u = R\Phi Z \quad where\ R = R(r), \Phi = \Phi(\emptyset), and\ \ Z = Z(z) \tag{13}$$

Substituting the product solution into the Equation (12), the derivatives are now total derivatives:

$$\frac{d^2R}{dr^2}\Phi Z + \frac{1}{r}\frac{dR}{dr}\Phi Z + \frac{1}{r^2}R\frac{d^2\Phi}{d\emptyset^2}Z + R\Phi\frac{d^2Z}{dz^2} = 0 \tag{14}$$

Then divide Equation (14) by the product expression for the solution to obtain:

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{r}\frac{1}{R}\frac{dR}{dr} + \frac{1}{r^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\emptyset^2} + \frac{1}{Z}\frac{d^2Z}{dz^2} = 0 \tag{15}$$

The last term of Equation (15) is a function of z only, so it must be a constant. Let's set this constant equal to λ. Then an ordinary differential equation for Z is obtained as:

$$\frac{d^2Z}{dz^2} - \lambda Z = 0 \tag{16}$$

Where λ is called separation constant.
With assumption: (if you take boundary function 0 then the solution is zero)

$$u(r,\emptyset,0) = g_1(r,\emptyset) = 0 \; and \; u(r,\emptyset,Z_0) = g_2(r,\emptyset) = 0$$
$$\text{for} \quad 0 \le r \le R, 0 \le \emptyset \le 2\pi$$

At $z = 0$

$$u(r,\emptyset,0) = R(r)\cdot\Phi(\emptyset)\cdot Z(0) = 0$$

Since $R(r) \neq 0 \; and \; \Phi(\emptyset) \neq 0, then \; Z(0) = 0$

At $z = Z_0$

$$u(r,\emptyset,Z_0) = R(r)\cdot\Phi(\emptyset)\cdot Z(Z_0) = 0$$

Since $R(r) \neq 0 \; and \; \Phi(\emptyset) \neq 0, then \; Z(Z_0) = 0$
Solution to Differential Equation (16) is shown in Table 3.

$$\frac{d^2Z}{dz^2} - \lambda Z = 0; Z(0) = 0, Z(Z_0) = 0$$

Table 3: Solution to Differential Equation (16)

Case I: $\lambda > 0$	Case II: $\lambda = 0$	Case III: $\lambda < 0$
G.S: $Z(z) = Ae^{\sqrt{\lambda}z} + Be^{-\sqrt{\lambda}z}$ When $Z(0) = 0$ $A + B = 0$ $\Longrightarrow A = -B$ When $Z(Z_0) = 0$ $Ae^{\sqrt{\lambda}Z_0} + Be^{-\sqrt{\lambda}Z_0} = 0$ $Ae^{\sqrt{\lambda}Z_0} - Ae^{-\sqrt{\lambda}Z_0} = 0$ Multiply the equation by $e^{\sqrt{\lambda}Z_0}$ $Ae^{2\sqrt{\lambda}Z_0} - A = 0$ $A(e^{2\sqrt{\lambda}Z_0} - 1) = 0$ Since $\left(e^{2\sqrt{\lambda}Z_0} - 1\right) \neq 0$ $\Longrightarrow A = 0$	The Differential Equation (16) becomes: $\frac{d^2Z}{dz^2} = 0$ G.S: $Z(z) = az + b$ When $Z(0) = 0$ $a(0) + b = 0 \Longrightarrow b = 0$ When $Z(Z_0) = 0$ $aZ_0 = 0$ Since $Z_0 > 0 \Longrightarrow a = 0$ Hence, no solution for case II.	Let $\lambda = -\sigma^2$, the Differential Equation (16) becomes: $\frac{d^2Z}{dz^2} + \sigma^2 Z = 0$ G.S: $Z(z) = A\,cos(\sigma z) + B\,sin(\sigma z)$ When $Z(0) = 0$ $A\,cos(0) + B\,sin(0) = 0$ $A + 0 = 0 \Longrightarrow A = 0$ When $Z(Z_0) = 0$ $B\,sin(\sigma Z_0) = 0$ Taking B = 1 $sin(\sigma Z_0) = 0$ $\Longrightarrow \sigma Z_0 = n\pi$

Hence, no solution for case I.		$\sigma = \frac{n\pi}{Z_0}, n = 1,2, \ldots$ Hence, solution for case III: $Z_n(z) = sin\left(\frac{n\pi}{Z_0}z\right)$ (17) For $n = 1,2, \ldots$

Equation (15) becomes:

$$\frac{1}{R}\frac{d^2R}{dr^2} + \frac{1}{r}\frac{1}{R}\frac{dR}{dr} + \frac{1}{r^2}\frac{1}{\Phi}\frac{d^2\Phi}{d\emptyset^2} - \sigma^2 = 0 \tag{18}$$

Multiply the Equation (18) with r^2, giving:

$$r^2\frac{1}{R}\frac{d^2R}{dr^2} + r\frac{1}{R}\frac{dR}{dr} - \sigma^2 r^2 + \frac{1}{\Phi}\frac{d^2\Phi}{d\emptyset^2} = 0 \tag{19}$$

Separate the last term of Equation (19) by taking the separation constant as $-m^2$. Then an ordinary differential equation for Φ is obtained as:

$$\frac{d^2\Phi}{d\emptyset^2} + m^2\Phi = 0 \tag{20}$$

Solution to Differential Equation (20) is

$$\Phi(\emptyset) = c_m\, cos(m\emptyset) + d_m\, sin(m\emptyset)\ for\ m = 0,1,2, \ldots \tag{21}$$

Then the ODE for the radial function is obtained which is written in standard form as:

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left(-\sigma^2 - \frac{m^2}{r^2}\right)R = 0$$

$$\frac{d^2R}{dr^2} + \frac{1}{r}\frac{dR}{dr} + \left[-\left(\frac{n\pi}{Z_0}\right)^2 - \frac{m^2}{r^2}\right]R = 0 \tag{22}$$
$$\text{for}\ \ n = 1,2, \ldots\ \text{and}\ \ m = 0,1,2, \ldots$$

Equation (22) is Bessel's equation of order n and its solutions are called as Bessel functions. It is known that one of the useful general forms for Bessel's equation is

$$\frac{d^2y}{dx^2} + \frac{(1-2a)}{x}\frac{dy}{dx} + \left[b^2c^2x^{2(c-1)} + \frac{(a^2-n^2c^2)}{x^2}\right]y = 0 \tag{23}$$

The quantities $a, b, c,$ and n are constants. The general solution of Equation (23) is

$$y(x) = x^a\{c_1J_n(bx^c) + c_1Y_n(bx^c)\} \tag{24}$$

By comparing Equation (22) to Equation (23) and taking $y = R, x = r$:
The numerical coefficient multiplying the first derivative is 1:

$$(1-2a) = 1 \quad \Rightarrow a = 0$$

The exponent on the term multiplying y and containing powers other than r^{-2} is zero:

$$2(c-1) = 0 \quad \Rightarrow c = 1$$

The numerical coefficient of the term multiplying y and the term r^{-2} is $-m^2$:

$$(a^2 - n^2c^2) = -m^2 \ \Rightarrow \ n = m$$

where $m = 0,1,2, \ldots$
The numerical coefficient of the term containing $r^{2(c-1)}$ is $-\sigma^2$

$$b^2c^2 = -\sigma^2 = -\left(\frac{n\pi}{Z_0}\right)^2 \quad \Rightarrow \quad b = i\frac{n\pi}{Z_0}$$

Therefore, the general solution of Equation (22) is

$$R(r) = c_1J_m\left(i\frac{n\pi}{Z_0}r\right) + c_2Y_m\left(i\frac{n\pi}{Z_0}r\right) \tag{25}$$

where $m = 0,1,2, \ldots$

$$\text{Since } r \to 0, Y_m(r) \to +\infty, \text{therefore } \ c_2 = 0$$
$$R(r) = c_1J_m\left(i\frac{n\pi}{Z_0}r\right)$$

The general solution to the Heat Equation (12) will become:

$$u(r,\emptyset,z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_1 J_m\left(i\frac{n\pi}{Z_0}r\right)\cdot[c_m\, cos(m\emptyset) + d_m\, sin(m\emptyset)]\cdot sin\left(\frac{n\pi}{Z_0}z\right) \quad (26)$$

At $r = R$,

$$u(R,\emptyset,z) = g_3(\emptyset,z) \qquad 0 \le \emptyset \le 2\pi, 0 \le z \le \mathrm{Z}_0$$

$$g_3(\emptyset,z) = \sum_{n=1}^{\infty}\sum_{m=0}^{\infty} c_1 J_m\left(i\frac{n\pi}{Z_0}R\right)\cdot[c_m\, cos(m\emptyset) + d_m\, sin(m\emptyset)]\cdot sin\left(\frac{n\pi}{Z_0}z\right) \quad (27)$$

By taking $c_1 = 1$ and letting

$$C_m(n) = J_m\left(i\frac{n\pi}{Z_0}r\right)\cdot c_m \text{ and } D_m(n) = J_m\left(i\frac{n\pi}{Z_0}r\right)\cdot d_m$$

Equation (27) becomes

$$g_3(\emptyset,z) = \sum_{n=1}^{\infty}\left(\sum_{m=0}^{\infty}[C_m\, cos(m\emptyset) + D_m\, sin(m\emptyset)]\right)\cdot sin\left(\frac{n\pi}{Z_0}z\right) \quad (28)$$

Let

$$\alpha_n(\emptyset) = \sum_{m=0}^{\infty}[C_m\, cos(m\emptyset) + D_m\, sin(m\emptyset)] \quad (29)$$

Equation (28) becomes

$$g_3(\emptyset,z) = \sum_{n=1}^{\infty}\alpha_n \cdot sin\left(\frac{n\pi}{Z_0}z\right)$$

By solving

$$\alpha_n(\emptyset) = \frac{z}{Z_0}\int_0^{Z_0} g_3(\emptyset,z)\cdot sin\left(\frac{n\pi}{Z_0}z\right)\, dz \quad (30)$$

Taking example of

$$g_3(\emptyset,z) = z^3\cdot cos\left(\frac{\emptyset}{4}\right)$$

Equation (30) becomes

$$\alpha_n(\emptyset) = \frac{z}{Z_0}\int_0^{Z_0} z^3 \cdot cos\left(\frac{\emptyset}{4}\right) \cdot sin\left(\frac{n\pi}{Z_0}z\right)\ dz$$

$$\alpha_n(\emptyset) = \frac{z}{Z_0} cos\left(\frac{\emptyset}{4}\right)\int_0^{Z_0} z^3 \cdot sin\left(\frac{n\pi}{Z_0}z\right)\ dz \tag{31}$$

Integration by part for Equation (31) yields

$$\alpha_n(\emptyset)$$
$$= \frac{z}{Z_0} cos\left(\frac{\emptyset}{4}\right)\left[\frac{3\left[\left(\frac{n\pi}{Z_0}\right)^2 z^2 - 2\right] sin\left(\frac{n\pi}{Z_0}z\right) - \left(\frac{n\pi}{Z_0}\right) z \left[\left(\frac{n\pi}{Z_0}\right)^2 z^2 - 6\right] cos\left(\frac{n\pi}{Z_0}z\right)}{\left(\frac{n\pi}{Z_0}\right)^4}\right]_0^{Z_0}$$

When $z = 0$

$$\frac{3[-2]\, sin(0) - \left(\frac{n\pi}{Z_0}\right)(0)\left[\left(\frac{n\pi}{Z_0}\right)^2 z^2 - 6\right] cos(0)}{\left(\frac{n\pi}{Z_0}\right)^4} = 0$$

When $z = Z_0$

$$\frac{3\left[\left(\frac{n\pi}{Z_0}\right)^2 {Z_0}^2 - 2\right] sin\left(\frac{n\pi}{Z_0}Z_0\right) - \left(\frac{n\pi}{Z_0}\right) Z_0 \left[\left(\frac{n\pi}{Z_0}\right)^2 {Z_0}^2 - 6\right] cos\left(\frac{n\pi}{Z_0}Z_0\right)}{\left(\frac{n\pi}{Z_0}\right)^4}$$
$$= \frac{3[(n\pi)^2 - 2]\, sin(n\pi) - (n\pi)[(n\pi)^2 - 6]\, cos(n\pi)}{\left(\frac{n\pi}{Z_0}\right)^4}$$
$$= \frac{-(n\pi)[(n\pi)^2 - 6]\, cos(n\pi)}{\left(\frac{n\pi}{Z_0}\right)^4}$$

Then Equation (31) becomes

$$\alpha_n(\emptyset) = \frac{z{Z_0}^3}{n^3\pi^3} cos\left(\frac{\emptyset}{4}\right)[(n^2\pi^2 - 6)\, cos(n\pi)] \tag{32}$$

$$\alpha_n(\emptyset) = \begin{cases} \dfrac{zZ_0{}^3}{n^3\pi^3} cos\left(\dfrac{\emptyset}{4}\right)(n^2\pi^2 - 6) & \text{for even integer } n \\ \dfrac{zZ_0{}^3}{n^3\pi^3} cos\left(\dfrac{\emptyset}{4}\right)(6 - n^2\pi^2) & \text{for odd integer } n \end{cases} \tag{33}$$

By substituting Equation (33) into Equation (29)

$$\sum_{m=0}^{\infty} [C_m\, cos(m\emptyset) + D_m\, sin(m\emptyset)]$$

$$= \begin{cases} \dfrac{zZ_0{}^3}{n^3\pi^3} cos\left(\dfrac{\emptyset}{4}\right)(n^2\pi^2 - 6) & \text{for even integer } n \\ \dfrac{zZ_0{}^3}{n^3\pi^3} cos\left(\dfrac{\emptyset}{4}\right)(6 - n^2\pi^2) & \text{for odd integer } n \end{cases}$$

By comparing the coefficient of *sin* term,

$$D_m = 0 \tag{34}$$

$$D_m(n) = J_m\left(i\frac{n\pi}{Z_0}r\right) \cdot d_m = 0 \;\Rightarrow\; \quad d_m = 0$$

When $\emptyset = 0$

$$\sum_{m=0}^{\infty} [C_m] = \begin{cases} \dfrac{zZ_0{}^3}{n^3\pi^3}(n^2\pi^2 - 6) & \text{for even integer } n \\ \dfrac{zZ_0{}^3}{n^3\pi^3}(6 - n^2\pi^2) & \text{for odd integer } n \end{cases} \tag{35}$$

When $\emptyset = 2\pi$

$$\sum_{m=0}^{\infty} [C_m] = 0 \tag{36}$$

$$C_m(n) = J_m\left(i\frac{n\pi}{Z_0}r\right) \cdot c_m = 0 \;\;\Rightarrow\;\; c_m = 0$$

Conclusion

The mathematical equation of the 3-dimensional heat conduction equations for a cylindrical problem was successfully derived with 3 cases of differential solutions for the separation constant, $\lambda > 0, \lambda = 0$, and $\lambda < 0$. The initial condition is taken as $u(x, y, z, 0) = f(x, y, z)$, where $f(x, y, z)$ is the initial temperature distribution and $S = \{x^2 + y^2 \leq r^2; 0 \leq z \leq Z_0\}$ with boundary curves $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$, where the boundary conditions are taken as $u|_{\Gamma_i} = g(x, y, z)$; for $i = 1,2,3$. The derived equation provides a mathematical solution to determine the heat transfer rate for a system with cryogenic pipe insulated with multiple layers.

Acknowledgement

I am grateful to the industry partner Cryogas Tech, which has been helpful and generous with their knowledge on cryogenics.

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